University of Basrah - College of Engineering

Department of Mechanical Engineering

Subject: *Engineering Analysis* **Stage:** *Third*

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Syllabus of Engineering Analysis

1. Complex Variables and Functions
2. Fourier Series and Transforms.
3. Laplace Transform of Special Functions and Applications
4. Solution of Ordinary Differential Equations
5. Bessel and Legendre Functions
6. Solution of Partial Differential Equations
7. Probability and Statistics

1. Complex Variables and Functions

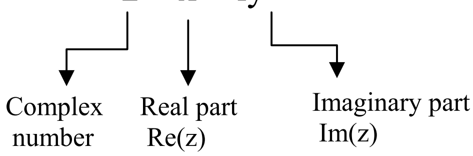
	-ve 0 +ve	
Real Numbers	$\cdots -3, -2, -1, 0, 1, 2, 3 \cdots$	Integers No.
	$\cdots -\frac{1}{4}, -\frac{1}{3}, -\frac{1}{2}, 0, \frac{1}{2}, \frac{1}{3}, \frac{1}{4} \cdots$	Rational No.
	$\cdots, \frac{-\sqrt{3}}{5}, -\sqrt{3}, 0, \sqrt{3}, \frac{\sqrt{3}}{5}, \sqrt{7} \cdots$	Irrational No.

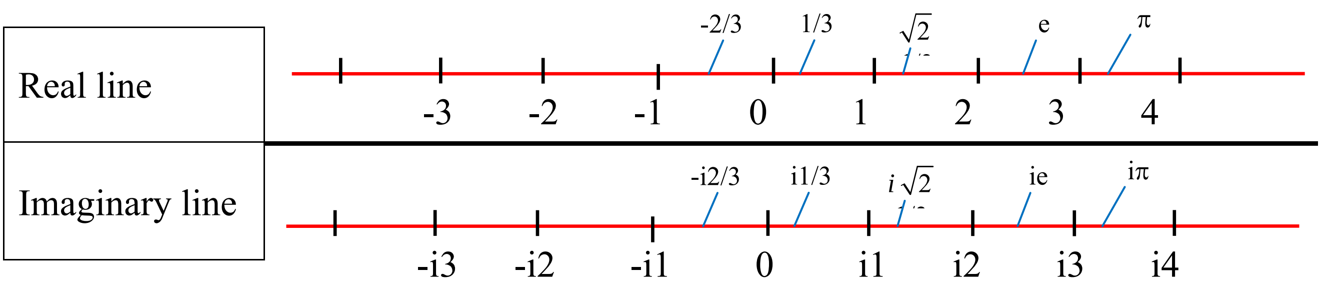
$\sqrt{-4} = \sqrt{-1}\sqrt{4} = \pm i\, 2$ Imaginary number

Complex number = Real number + Imaginary number

= Real + i Real

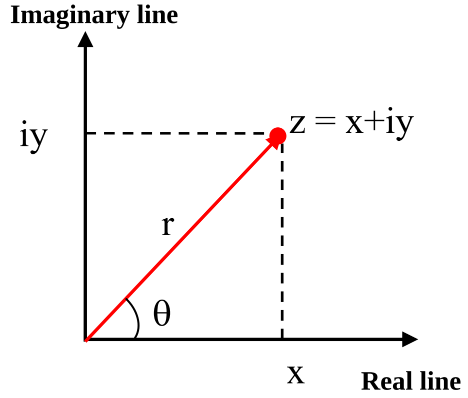
$z = x + iy$





∴ The complex number may be represented as a point on the complex plane.

$z = x + iy$, (x, iy), (x, y) **Cartesian Form**



Or

$z = r\underline{|\theta}$, (r, θ) **Polar form**

where;

x = Re (z)

y = Im (z)

r: Amplitude or modulus, $r = \sqrt{x^2 + y^2}$, $r = |z|$, r = Amp (z)

θ: Phase or Angle, $\theta = \tan^{-1}\dfrac{y}{x}$, θ = Arg (z)

$$\therefore x = r\cos\theta, \quad y = r\sin\theta$$

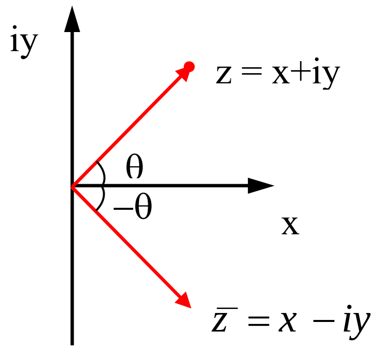
$$z = x + iy = r\cos\theta + ir\sin\theta$$

$$z = r(\cos\theta + i\sin\theta)$$

Definitions:

(1) Conjugate of z is denoted by $\bar{z}$

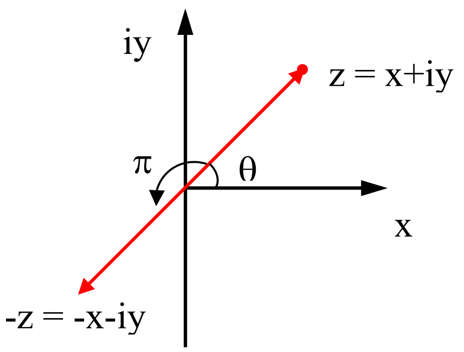
$$z = r\underline{|\theta} \rightarrow \bar{z} = r\underline{|-\theta}$$



(2) Reverse of z is written as -z

$$z = x + iy \rightarrow -z = -x - iy$$

$$z = r\underline{|\theta} \rightarrow -z = r\underline{|\theta + \pi}$$



Note: *All angles are measured in **radians** and positive in the counterclockwise sense.*

(3) Inverse of z. It is simply $\dfrac{1}{z} = ?$

Operations on Complex numbers

Let $z_1 = x_1 + iy_1 \quad and \quad z_2 = x_2 + iy_2$

(1) Addition and Subtraction

$$z_1 + z_2 = (x_1 + x_2) + i\,(y_1 + y_2)$$
$$z_1 - z_2 = (x_1 - x_2) + i\,(y_1 - y_2)$$

(2) Multiplication

$$z_1.z_2 = (x_1 + iy_1)(x_2 + iy_2)$$
$$= x_1x_2 + i^2 y_1y_2 + ix_1y_2 + iy_1x_2$$

$$\therefore z_1.z_2 = x_1x_2 - y_1y_2 + i\,(x_1y_2 + x_2y_1)$$

Note: $z\bar{z} = x^2 + y^2 = r^2 = |z|^2$

$$i^2 = i.i = -1$$
$$i^3 = i^2.i = -i$$
$$i^4 = i^2.i^2 = 1$$
$$\vdots \quad \vdots$$

(3) Division $\dfrac{z_1}{z_2}$

$$\frac{1}{z} = \frac{1}{z}\cdot\frac{\bar{z}}{\bar{z}} = \frac{x - iy}{(x + iy)(x - iy)} = \frac{x - iy}{x^2 + y^2}$$

$$\therefore \frac{1}{z} = \frac{x}{x^2 + y^2} - i\,\frac{y}{x^2 + y^2} \qquad ,r^2 = x^2 + y^2$$

$$\frac{z_1}{z_2} = z_1\left(\frac{1}{z_2}\right) = (x_1 + iy_1)\left(\frac{x_2 - iy_2}{x_2^2 + y_2^2}\right) = \frac{(x_1 + iy_1)(x_2 - iy_y)}{x_2^2 + y_2^2}$$

$$\therefore \frac{z_1}{z_2} = \frac{x_1x_2 + y_1y_2 + i\,(y_1x_2 - x_1y_2)}{x_2^2 + y_2^2}$$

*** In polar form:**

$$\therefore z_1.z_2 = x_1x_2 - y_1y_2 + i\,(x_1y_2 + x_2y_1)$$

$$= (r_1\cos\theta_1 r_2\cos\theta_2 - r_1\sin\theta_1 r_2\sin\theta_2) + i\,(r_1\cos\theta_1 r_2\sin\theta_2 + r_2\cos\theta_2 r_1\sin\theta_1)$$

$$= r_1r_2\left[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i\,(\cos\theta_1\sin\theta_2 + \cos\theta_2\sin\theta_1)\right]$$

$$= r_1r_2\left[\cos(\theta_1+\theta_2) + i\,\sin(\theta_1+\theta_2)\right]$$

$$\therefore z_1z_2 = r_1r_2\underline{|(\theta_1+\theta_2)}$$

$$\frac{1}{z} = \frac{1}{z}\cdot\frac{\bar{z}}{\bar{z}} = \frac{r\underline{|-\theta}}{r\underline{|\theta} * r\underline{|-\theta}} = \frac{\not{r}\underline{|-\theta}}{r^{\not{2}}}$$

$$\therefore \frac{1}{z} = \frac{1}{r}\underline{|-\theta}$$

$$\frac{z_1}{z_2} = z_1\left(\frac{1}{z_2}\right) = r_1\underline{|\theta_1}\left(\frac{1}{r_2}\underline{|-\theta_2}\right)$$

$$\therefore \frac{z_1}{z_2} = \frac{r_1}{r_2}\underline{|\theta_1-\theta_2}$$

Properties of Operations;

(1) $z_1 + z_2 = z_2 + z_1$, $z_1\cdot z_2 = z_2\cdot z_1$

(2) $z_1(z_2+z_3) = z_1z_2 + z_1z_3$

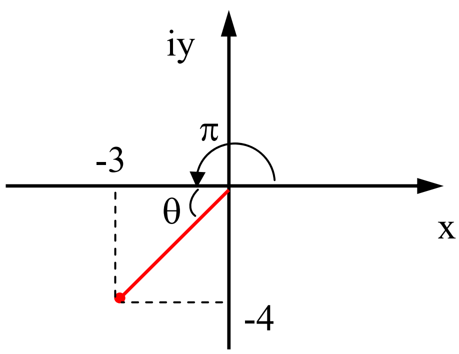
(3) $\bar{z}_1 + \bar{z}_2 = \overline{(z_1+z_2)}$, $\bar{z}_1\cdot\bar{z}_2 = \overline{z_1\cdot z_2}$

$$\frac{\bar{z}_1}{\bar{z}_2} = \overline{\left(\frac{z_1}{z_2}\right)}$$

Ex. Given $z_1 = -3 - i\,4$, $z_2 = 1 - i\,2$

Find $\bar{z}_1,\ z_2^{-1},\ z_1+z_2,\ z_2-z_1,\ z_1z_2,\ and\ \frac{z_1}{z_2}$

Sol.



$$z_1 = -3 - i\,4,\quad r = \sqrt{(-3)^2 + (-4)^2} = 5,$$

$$\theta = \tan^{-1}\frac{-4}{-3} = 0.927 + \pi = 4.06\ rad$$

$$\therefore z_1 = 5\underline{|4.06}$$

$$z_2 = 1 - i\,2,\quad r = \sqrt{1^2 + (-2)^2} = \sqrt{5},\quad \theta = \tan^{-1}\frac{-2}{1} = -1.107\ rad$$

(a) $\bar{z}_1 = -3 + i\,4$

(b) $z_2^{-1} = \dfrac{1}{z_2} = \dfrac{x_2 - iy_2}{x_2^2 + y_2^2} = \dfrac{1 + i\,2}{5} = \dfrac{1}{5} + i\,\dfrac{2}{5}$

(c) $z_1 + z_2 = (-3+1) + i\,(-4-2) = -2 - i\,6$

(d) $z_2 - z_1 = (1+3) + i\,(-2+4) = 4 + i\,2$

(e) $z_1 z_2 = (-3 - i\,4)(1 - i\,2) = (-3-8) + i\,(6-4) = -11 + i\,2$

Or Using Polar formula

$$z_1 z_2 = r_1 r_2 \underline{|\theta_1 + \theta_2} = 5\sqrt{5}\underline{|2.953}$$

(f) $\dfrac{z_1}{z_2} = z_1 . \dfrac{1}{z_2} = (-3 - i\,4)(0.2 + i\,0.4) = 2.2 - i\,2$

Or

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\underline{|\theta_1 - \theta_2} = \frac{5}{\sqrt{5}}\underline{|5.167}$$

H.W. Given $z_1 = -2 + i\,4$, $z_2 = 5 + i\,3$, $z_3 = 2 - 6$

find $(z_1 + z_3)\dfrac{z_1}{z_2 - z_3}$, z_1^2 , $\left(\dfrac{z_2}{z_3}\right)^2$

Note: ex. $z = -3 + i$

$*if \;\; x < 0 \;\; \rightarrow \theta = \theta_{calc.} + \pi$

$$** \text{ if } x = 0 \quad \begin{cases} y > 0 & \Rightarrow \theta = \dfrac{\pi}{2} \\ y < 0 & \Rightarrow \theta = \dfrac{3\pi}{2} \end{cases}$$

Simple function of complex variables;

(1) Power functions $z^n = ?$

when n : +ve integer

$$z = r\angle\theta$$
$$z^2 = (r\angle\theta)(r\angle\theta) = r^2\angle 2\theta$$
$$z^3 = z^2.z = (r^2\angle 2\theta)(r\angle\theta) = r^3\angle 3\theta$$
$$\vdots \qquad \vdots$$

$$z^n = r^n\angle n\pi \quad \textbf{or} \quad z^n = r^n[\cos n\theta + i\sin n\theta]$$

when n:-ve

$$z^{-n} = \frac{1}{z^n} = \frac{1}{r^n\angle n\theta} = \frac{1}{r^n}\angle -n\theta = r^{-n}\angle -n\theta$$

$$\therefore z^{-n} = r^{-n}\angle -n\theta$$

Ex. Find $(3-i4)^3$

Sol.

$$z = 3 - i4 = 5\angle -0.927$$
$$z^3 = 5^3\angle 3*(-0.927) = 125\angle -2.781$$

(2) Root function $z^{1/n} = ?$

let $w = z^{1/n}$, *and* $w = R\angle\Phi$

$$w^n = z \quad \Rightarrow \quad R^n\angle n\Phi = r\angle\theta$$
$$\therefore R^n = r \quad \Rightarrow \quad R = r^{1/n}$$
$$n\Phi = \theta \quad \Rightarrow \quad \Phi = \frac{\theta}{n}$$

$$w = R\angle\Phi = z^{1/n} = r^{1/n}\angle\frac{\theta}{n}$$

$$z^{1/n} = r^{1/n}\angle\frac{\theta}{n} \quad \text{" Principle value"}$$

Generally: $z^{1/n} = r^{1/n} \underline{\left| \frac{\theta + 2\pi k}{n} \right.}$ where k = 0, 1, 2, ... , n -1

Or $z^{1/n} = r^{1/n}\left[\cos\left(\frac{\theta+2\pi k}{n}\right) + i\sin\left(\frac{\theta+2\pi k}{n}\right)\right]$ n: numbers of roots

Ex. Find $\sqrt[3]{(5+i4)}$

Sol. $z = 5 + i4 = \sqrt{41}\underline{|0.67} = 6.4\angle 0.67$

$$z^{1/3} = (6.4)^{1/3} \underline{\left| \frac{0.67 + 2\pi k}{3} \right.}$$ **where k = 0, 1 , 2**

$$z^{1/3} = 1.85 \rightarrow \underline{|0.223} \qquad k = 0$$
$$\rightarrow \underline{|2.317} \qquad k = 1$$
$$\rightarrow \underline{|4.41} \qquad k = 2$$

H.W. Find $\sqrt{-1}$, *and* $\sqrt{1}$

(3) Exponential function: $w = e^z$

$$e^z = e^{(x+iy)} = e^x e^{iy}$$
$$= e^x \left[1 + \frac{iy}{1!} + \frac{(iy)^2}{2!} + \frac{(iy)^3}{3!} + \frac{(iy)^4}{4!} + \cdots\right]$$
$$= e^x \left[\left(1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \frac{y^6}{6!} + \cdots\right) + i\left(\frac{y}{1!} - \frac{y^3}{3!} + \frac{y^5}{5!} - \cdots\right)\right]$$

$$e^z = e^x(\cos y + i\sin y) = e^x \cos y + ie^x \sin y$$

Or

$e^z = e^x \underline{|y}$ $\qquad |e^z| = e^x$, $Arg(e^z) = y$

Generally:-

$e^{i\theta} = \cos\theta + i\sin\theta$ **"Euler's Formula"**

$z = r\underline{|\theta} = r(\cos\theta + i\sin\theta)$ $\Rightarrow$ $\therefore z = re^{i\theta}$ **"Exponential Form"**

Results:-

$$e^{i\theta} = \cos\theta + i\,\sin\theta$$
$$e^{-i\theta} = \cos\theta - i\,\sin\theta$$
$$\overline{\hspace{6cm}} \quad +$$
$$e^{i\theta} + e^{-i\theta} = 2\cos\theta$$

$$\therefore \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

Subtracting $\Rightarrow \; e^{i\theta} - e^{-i\theta} = i\,\sin\theta$

$$\therefore \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{i2}$$

$$\therefore \cos i\,\theta = \frac{e^{-\theta} + e^{\theta}}{2} = \cosh\theta$$
$$\sin i\,\theta = \frac{e^{-\theta} - e^{\theta}}{i2} = -i\,\frac{e^{-\theta} - e^{\theta}}{2} = i\,\frac{e^{\theta} - e^{-\theta}}{2} = i\,\sinh\theta$$

Notes:- $\frac{1}{i} = \frac{1}{i} \cdot \frac{i}{i} = -i$

$$\therefore \cosh i\,\theta = \frac{e^{-i\theta} + e^{i\theta}}{2} = \cos\theta$$
$$\sinh i\,\theta = \frac{e^{i\theta} - e^{-i\theta}}{2} = -i\,\frac{e^{-\theta} - e^{\theta}}{2} = i\,\frac{e^{\theta} - e^{-\theta}}{2} = i\,\sin\theta$$

Also:

$\cos iz = \cosh z$; $\cosh iz = \cos z$

$\sin iz = i\,\sinh z$; $\sinh iz = i\,\sin z$

Ex. Find $\sin(1 + i\,3)$

$$\sin z = \sin(x + iy) = \sin x\,\cos iy + \cos x\,\sin iy$$
$$\therefore \sin z = \sin x\,\cosh y + i\,\cos x\,\sinh y$$

$$\therefore \sin(1 + i\,3) = \sin 1 \cosh 3 + i\,\cos 1 \sinh 3$$

Ex. Find $\sin^{-1}(1-2i)$

$$z=\sin^{-1}(1-2i) \quad \Rightarrow \quad 1-2i=\sin z=\sin(x+iy)$$

$$1-2i=\sin x\cosh y+i\cos x\sinh y$$

$$1=\sin x\cosh y \qquad ...(1) \qquad \Rightarrow \sin x=\frac{1}{\cosh y} \quad (3)$$

$$-2=\cos x\sinh y \qquad ...(2)$$

$$\cos x=\sqrt{1-\sin^2 x}=\sqrt{1-\frac{1}{\cosh^2 y}} \qquad ...(4)$$

Substitute eq.(4) in eq.(2) we have

$$-2=\sqrt{1-\frac{1}{\cosh^2 y}}\ \sinh y \qquad \Rightarrow \qquad 4=\left(1-\frac{1}{\cosh^2 y}\right)\sinh^2 y$$

$$\therefore 4=\left(1-\frac{1}{\cosh^2 y}\right)\left(\cosh^2 y-1\right)$$

$$4=\cosh^2 y-1-1+\frac{1}{\cosh^2 y}=\cosh^2 y-2+\frac{1}{\cosh^2 y}$$

Let $t=\cosh^2 y$

$$4=t-2+\frac{1}{t} \qquad \Rightarrow \qquad t^2-6t+1=0$$

$$t=\frac{6\mp\sqrt{32}}{2}=\begin{cases}5.828 \\ 0.17 \qquad \text{Im}\,possible \quad because\ <1\end{cases}$$

$$\cosh y=\sqrt{t}=\sqrt{5.828}=2.414$$

$$\therefore y=\cosh^{-1}2.414=1.528$$

$$\sin x=\frac{1}{2.414}=0.414 \quad \Rightarrow \quad x=\sin^{-1}0.414$$

$$\therefore x=0.427$$

$$\therefore \sin^{-1}(1-2i)=0.427+i\,1.528$$

(4) Logarithmic function: $w = \ln z$

let $w = \ln z \quad and \quad w = u + iv \quad , z = re^{i\theta}$

$$e^{w} = z \quad \Rightarrow \quad e^{u+iv} = re^{i\theta}$$

$$\therefore e^{u} = r \quad \Rightarrow \quad u = \ln r$$

$$e^{iv} = e^{i\theta} \quad \Rightarrow \quad v = \theta$$

$\therefore w = \ln z = \ln r + i\theta$ **"Principle value"**

In general;

$\ln z = \ln r + i\left(\theta + 2\pi k\right)$ k = 0, 1, 2, ...∞

Ex. Find ln (1 +i)

$$\ln(1+i) = \ln\left(\sqrt{2}\underline{|\pi/4}\right)$$

$$= \ln\sqrt{2} + i\left(\frac{\pi}{4} + 2\pi k\right)$$

$$= \frac{1}{2}\ln 2 + i\left(\frac{\pi}{4} + 2\pi k\right) \qquad k = 0, 1, 2, \cdots \infty$$

$$= \frac{1}{2}\ln 2 + i\,\frac{\pi}{4}$$ **"principle value k=0"**

Ex. Find $z \quad for \quad \ln z = 1 - i\pi$

$$\therefore z = e^{1-i\pi} = e^{1}e^{-i\pi} = e^{1}\left(\cos^{-1}(-\pi) - i\sin(-\pi)\right)$$

$$\therefore z = -e^{1}$$

H.W.

(1) Evaluate the following functions

1. $\ln(3 - i6)$, 2. $e^{(3\underline{|0.52})}$

3. $\cos^{-1}(1 + i3)$, 4. $\tanh^{-1}(2 - i5)$

(2) Find all the value of $\sin^{-1} 2$

(3) Prove that

$$(a)\ \sin^2 z + \cos^2 z = 1 \quad , \quad (b)\ \sin(-z) = -\sin z$$

$$(c)\ \ln e^{z} = z \quad , \quad (d)\ \cos(-z) = \cos z$$

(4) Find all values of z for which;

$$(a)\ e^{3z} = 1 \quad ; \quad (b)\ e^{z} = 1 - i \quad ; \quad (c)\ e^{4z} = i$$

(5) Find all roots of

$$1.\ \sqrt[3]{1+i} \quad ; \quad 2.\ \sqrt[3]{8i} \quad ; \quad 3.\ \sqrt[8]{1} \quad ; \quad 4.\ \sqrt{-7+24i}$$

Limit of Complex function;

A function f(z) is said to have the limit ℓ as z approaches a point z_0, written

$$\lim_{z \to z_0} f(z) = \ell$$

Ex. Find $\lim_{z \to 0} \dfrac{\bar{z}}{z}$ (a) along the x-axis. (b) along the y-axis.

Sol. (a) along the x-axis, y = 0

$$\therefore z = x\ ,\ \bar{z} = x \ , so\ that\ \frac{\bar{z}}{z} = 1$$

$$\lim_{\substack{y=0 \\ x \to 0}} \frac{\bar{z}}{z} = 1$$

(b) along the y-axis, x = 0

$$\therefore z = iy\ ,\ \bar{z} = -iy$$

$$\lim_{\substack{x=0 \\ y \to 0}} \frac{\bar{z}}{z} = \lim_{\substack{x=0 \\ y \to 0}} \frac{-iy}{iy} = -1$$

Ex. Find $\lim_{z \to 3+i} \left(z^2 - z\right)$

$$\lim_{z \to 3+i} \left(z^2 - z\right) = \left((3+i)^2 - (3+i)\right) = 5 + i\,5$$

Continuity of Complex function;

The complex function f(z) is defined to be continuous at z_0 if

(a) $f(z_0)$ exists, and

(b) $\lim_{z \to z_0} f(z)$ exists and is equal to $f(z_0)$.

Ex (1). In the function $f(z) = z^2$ continuous every where?

assume the function is to be tested as z_0 where z_0 is an arbitrary point.

$$f(z) = z^2 = (x^2 - y^2) + i(2xy)$$
$$f(z_0) = (x_0^2 - y_0^2) + i(2x_0 y_0)$$

(a) A long horizontal lines pass through z_0

$$\lim_{z \to z_0} f(z) = \lim_{\substack{y = y_0 \\ x \to x_0}} \left[(x^2 - y^2) + i(2xy)\right] = \lim_{x \to x_0} \left[(x^2 - y_0^2) + i\,2xy_0\right]$$
$$= (x_0^2 - y_0^2) + i\,2x_o y_0 = f(z_0)$$

(b) A long vertical path , $x = x_0$, $y \to y_0$

$$\lim_{z \to z_0} f(z) = \lim_{\substack{x = x_0 \\ y \to y_0}} \left[(x^2 - y^2) + i(2xy)\right] = \lim_{y \to y_0} \left[(x_0^2 - y^2) + i\,2x_0 y\right]$$
$$= (x_0^2 - y_0^2) + i\,2x_o y_0 = f(z_0)$$

(c) A long and straight line passes through z_0 , $y = ax + b$

$$\lim_{z \to z_0} f(z) = \lim_{\substack{y = ax + b \\ x \to x_0}} \left[(x^2 - y^2) + i\,2xy\right] = \lim_{x \to x_0} \left[\left(x^2 - (ax + b)^2\right) + i\,2x(ax + b)\right]$$
$$= \left(x_0^2 - (ax_0 + b)\right) + i\,2x_0(ax_0 + b)$$
$$= \left(x_0^2 - y_0^2\right) + i\,2x_0 y_0 = f(z_0)$$

Since the limit exist along any path passes through z_0 and equal $f(z_0)$, then the function $f(z) = z^2$ is continuous everywhere.

Ex.(2) where does the function $f(z)=\frac{x}{z}$ fail to be continuous?

Sol. $f(z)=\frac{x}{x+iy}$

if we select a path on the line y=ax

$$\lim_{z\to z_0} f(z)=\lim_{\substack{y=ax\\ x\to x_0}} f(z)=\lim_{x\to x_0}\left[\frac{x}{x+iax}\right]=\frac{1}{1+ia}$$

Since the limit value depend on the path, then the function fail to be continuous on any line y = ax.

Ex.(3) where does the function $f(z)=\frac{x^2}{x^2+y}$ fail to be analytic?

Answer: where $y = ax^2$

Ex.(4) where does the function $f(z)=\frac{x^2}{x^2+2xy}$ fail to be analytic?

Answer: where y = ax

Note:

(1) Polynomials are continuous functions everywhere except at $\pm\infty$ perhaps, thus sin z, cos z, e^z, ... etc. are continuous functions.

(2) Addition, subtraction, multiplication and division of continuous functions are also continuous except the denominator is zero.

(3) The continuous function of a continuous function is also continuous.

Ex. tan z is continuous function, except when cos z = 0 ---> z = nπ/2 where n is odd.

Ex.

$$f(z)=z^2 \quad , \; g(z)=\cos z$$

$$f[g(z)]=(\cos z)^2=\cos^2 z \qquad \textit{continuous}$$

$$g[f(z)]=\cos z^2 \qquad \textit{continuous}$$

H.W. Solve 6 problems about continuity. In Wylie books

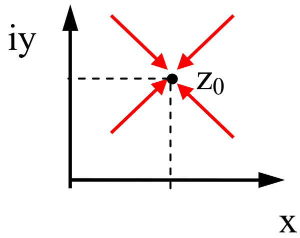
Differentiation of Complex Function;

The function f(z) is differentiable, analytic, holomorphic if the limit

$$\lim_{\Delta z \to 0} \frac{f(z+\Delta z)-f(z)}{\Delta z}$$

exist along all possible paths.

Ex. Show that the function f(z) = z is differentiable everywhere.



Sol.

$$\lim_{\Delta z \to 0} \frac{f(z+\Delta z)-f(z)}{\Delta z} = \lim_{\Delta z \to 0} \frac{(\cancel{z}+\Delta z)-\cancel{z}}{\Delta z} = \lim_{\Delta z \to 0} \frac{\Delta z}{\Delta z} = 1$$

since the limit has only one value that in not depend on the path, then the function is differentiable.

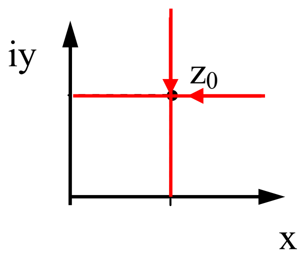
Ex. Is the function $f(z)=\bar{z}$ is differentiable everywhere?

$$f(z)=\bar{z}=x-iy \quad , \quad \Delta z = \Delta x + i\,\Delta y$$

$$\lim_{\Delta z \to 0} \frac{f(\bar{z}+\Delta z)-f(z)}{\Delta z} = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{(\bar{z}+\Delta z)-\bar{z}}{\Delta z}$$

$$\lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{(x+\Delta x)-i(y+\Delta y)-(x-iy)}{\Delta x + i\,\Delta y} = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\Delta x - i\,\Delta y}{\Delta x + i\,\Delta y}$$

(a) along horizontal path, $\Delta y = 0 \quad , \Delta x \to 0$



$$\lim_{\substack{\Delta x \to 0 \\ \Delta y = 0}} \frac{\Delta x - i\,\Delta y}{\Delta x + i\,\Delta y} = \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x} = 1$$

(a) along vertical path, $\Delta x = 0 \quad , \Delta y \to 0$

$$\lim_{\substack{\Delta x = 0 \\ \Delta y \to 0}} \frac{\Delta x - i\,\Delta y}{\Delta x + i\,\Delta y} = \lim_{\Delta y \to 0} \frac{-i\,\Delta y}{i\,\Delta y} = -1$$

Since the limit depends on the path, then the function $f(z)=\bar{z}$ is not analytic.

Ex. Is the function $f(z) = x^2 - y^2$ is differentiable everywhere?

$$f(z) = x^2 - y^2$$

$$f(z+\Delta z) = (x+\Delta x)^2 - (y+\Delta y)^2 = x^2 + 2x\Delta x + \cancel{(\Delta x)^2} - \left(y^2 + 2y\Delta y + \cancel{(\Delta y)^2}\right)$$

very small

$$\therefore f(z+\Delta z) = (x+\Delta x) = \left(x^2 - y^2\right) + 2\left(x\Delta x - y\Delta y\right)$$

$$\lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\cancel{\left(x^2 - y^2\right)} + 2\left(x\Delta x - y\Delta y\right) - \cancel{\left(x^2 - y^2\right)}}{\Delta x + i\Delta y}$$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{2\left(x\Delta x - y\Delta y\right)}{\Delta x + i\Delta y}$$

(a) along horizontal path, $\Delta y = 0 \quad, \Delta x \to 0$

$$\lim_{\substack{\Delta x \to 0 \\ \Delta y = 0}} \frac{2\left(x\Delta x - y\Delta y\right)}{\Delta x + i\Delta y} = \lim_{\Delta x \to 0} \frac{2x\Delta x}{\Delta x} = 2x$$

(a) along vertical path, $\Delta x = 0 \quad, \Delta y \to 0$

$$\lim_{\substack{\Delta x = 0 \\ \Delta y \to 0}} \frac{2\left(x\Delta x - y\Delta y\right)}{\Delta x + i\Delta y} = \lim_{\Delta y \to 0} \frac{-2y\Delta y}{i\Delta y} = i\,2y$$

Since the limit depends on the path, then the function $f(z) = x^2 - y^2$ is not analytic.

Notes:-

(1) Polynomials are analytic functions everywhere except at $\pm\infty$ possibly.

(2) Addition, subtraction, multiplication and division of analytic functions are also analytic except the denominator is zero.

(3) The analytic function of a analytic function is also analytic.

(4) If a function is analytic, then it is not a function of $\bar{z}$.

Ex. Show that the function $f(z)=x^2-y^2+i2xy$ is analytic.

Sol. Can be rewritten the above function for z *and* $\bar{z}$

$$\left.\begin{aligned} z&=x+iy \\ \bar{z}&=x-iy \end{aligned}\right\} \quad solve \qquad x=\frac{z+\bar{z}}{2} \quad , \quad y=\frac{z-\bar{z}}{i2}$$

$$f(z)=\left(\frac{z+\bar{z}}{2}\right)^2-\left(\frac{z-\bar{z}}{i2}\right)^2+i2\left(\frac{z+\bar{z}}{2}\right)\left(\frac{z-\bar{z}}{i2}\right)$$

$$=\frac{1}{4}\left(z^2+2z\bar{z}+\bar{z}^2\right)+\frac{1}{4}\left(z^2-2z\bar{z}+\bar{z}^2\right)+\frac{1}{2}\left(z^2-\bar{z}^2\right)$$

$$f(z)=z^2$$

Since f(z) is a function of z only, then it is analytic function.

Ex. Is the function $f(z)=x^2-y^2$ is analytic?

$$f(z)=\left(\frac{z+\bar{z}}{2}\right)^2-\left(\frac{z-\bar{z}}{i2}\right)^2$$

$$=\frac{1}{4}\left(z^2+2z\bar{z}+\bar{z}^2\right)+\frac{1}{4}\left(z^2-2z\bar{z}+\bar{z}^2\right)$$

$$f(z)=\frac{1}{2}\left(z^2+\bar{z}^2\right)$$

Since the function f(z) **contains** $\bar{z}$, then it is non analytic.

Note: All the rules of differentiation to real functions can be used for complex functions;-

$$\frac{d}{dz}\left(z^n\right)=nz^{n-1} \quad , \quad \frac{d}{dz}(\sin z)=\cos z$$

$$\frac{d}{dz}\left(g(z).f(z)\right)=g(z)f'(z)+f(z)g'(z)$$

$$\vdots \qquad \vdots$$

H.W. Problems P.649 continuity.

Problems P.656 differentiation.

Cauchy - Riemann Conditions (C.R.C)

To find a set of conditions on the component u and v so that w is analytic.

If $w = f(z) = u(x,y) + iv(x,y)$ is analytic then

$$f'(z) = \lim_{\Delta z \to 0} \left\{ \frac{f(z+\Delta z) - f(z)}{\Delta z} \right\}$$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \left\{ \frac{\left[u(x+\Delta x, y+\Delta y) + iv(x+\Delta x, y+\Delta y)\right] - \left[u(x,y) + iv(x,y)\right]}{\Delta x + i\Delta y} \right\}$$

Consider two particular paths.

(a) along horizontal path, $\Delta y = 0 \quad , \Delta x \to 0$

$$f'(z) = \lim_{\Delta x \to 0} \left\{ \frac{\left[u(x+\Delta x, y) + iv(x+\Delta x, y)\right]}{\Delta x} - \frac{u(x,y) + iv(x,y)}{\Delta x} \right\}$$

$$= \lim_{\Delta x \to 0} \left\{ \frac{u(x+\Delta x, y) - u(x,y)}{\Delta x} + i\frac{v(x+\Delta x, y) - v(x,y)}{\Delta x} \right\}$$

$$i.e. \quad f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial f}{\partial x} \qquad \cdots(1)$$

(a) along vertical path, $\Delta x = 0 \quad , \Delta y \to 0$. Leave it to you to show that

$$f'(z) = \frac{1}{i}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} = \frac{1}{i}\frac{\partial f}{\partial y} \qquad \cdots(2)$$

By comparing the real and Imaginary parts of Eqs. (1) & (2)

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \textbf{and} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \qquad \textbf{"CRC"}$$

Ex. Show that f(z) = z^2 is analytic using Cauchy-Riemann Conditions.

$$f(z) = z^2 = x^2 - y^2 + i\,2xy$$

$$\therefore u = x^2 - y^2 \qquad , v = 2xy$$

$$\frac{\partial u}{\partial x} = 2x \quad , \frac{\partial v}{\partial y} = 2x \quad \Rightarrow \quad \therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial y} = -2y \quad , \frac{\partial v}{\partial x} = 2y \quad \Rightarrow \quad \therefore \frac{\partial v}{\partial x} = \frac{-\partial u}{\partial y}$$

$\therefore$ f(z) is analytic.

Ex. Is the function $f\left(z\right)=x^{2}+y^{2}-i\,2xy$ analytic?

$$\therefore u=x^{2}+y^{2} \qquad ,v=-2xy$$

$$\frac{\partial u}{\partial x}=2x \quad ,\frac{\partial v}{\partial y}=-2x \qquad \Rightarrow \qquad \therefore \frac{\partial u}{\partial x}\neq\frac{\partial v}{\partial y}$$

$\therefore$ f(z) is not analytic.

Connection with Laplace's Equation in 2-D

Suppose f(z) is analytic is some region.

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y} \qquad \text{differentiate w. r. t. x}$$

$$\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 v}{\partial x\,\partial y} \qquad (1)$$

$$\frac{\partial v}{\partial x}=-\frac{\partial u}{\partial y} \qquad \text{differentiate w. r. t. y}$$

$$\frac{\partial^2 u}{\partial y^2}=-\frac{\partial^2 v}{\partial y\,\partial x} \qquad (2)$$

Adding eq.(1) to eq.(2)

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \qquad \text{Or} \qquad \nabla^2 u=0$$

In a similar way we find that

$$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=0 \qquad \text{Or} \qquad \nabla^2 v=0$$

We call **u** and **v** the harmonic functions.

Results:-

(1) If u and v satisfy C.R.C., then the function f(z) = u +iv is analytic.

(2) If u and v satisfy C.R.C., then both u and v satisfy Laplace equation in 2-D.

(3) If u and v satisfy C.R.C., then the lines u = c, v = k (c and k are constants) are orthonormal.

Proof of Results (3):

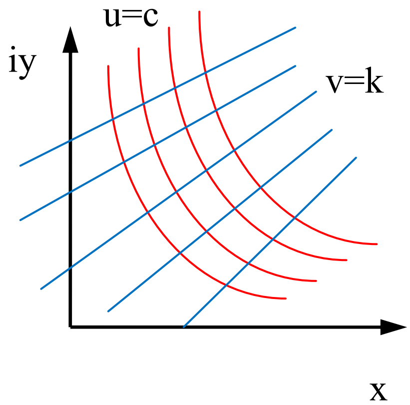
$$u = c,\ du = 0 = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$$

$$\frac{dy}{dx} = \frac{-\partial u/\partial x}{\partial u/\partial y} = slop|_u \qquad ...(1)$$

$$v = k,\ dv = 0 = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy$$

$$\frac{dy}{dx} = \frac{-\partial v/\partial x}{\partial v/\partial y} = slop|_v \qquad ...(2)$$

$$slop|_u \times slop|_v = \left(\frac{-\partial u/\partial x}{\partial u/\partial y} \times \frac{-\partial v/\partial x}{\partial v/\partial y}\right) = -1$$



$\therefore$ The lines u = c and v = k are orthonormal.

Ex. Show that both u and v $f(z) = z^2 + 2z$ satisfy Laplace eqs.

Sol. $f(z) = z^2 + 2z = (x^2 - y^2 + 2x) + i(2xy + 2y)$

$$u = x^2 - y^2 + 2x \rightarrow \frac{\partial u}{\partial x} = 2x + 2 \Rightarrow \frac{\partial^2 u}{\partial x^2} = 2, \quad \frac{\partial u}{\partial y} = -2y \Rightarrow \frac{\partial^2 u}{\partial y^2} = -2 \quad \Bigg\} \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$v = 2xy + 2y \rightarrow \frac{\partial v}{\partial x} = 2y \Rightarrow \frac{\partial^2 v}{\partial x^2} = 0, \quad \frac{\partial u}{\partial y} = 2x + 2 \Rightarrow \frac{\partial^2 v}{\partial y^2} = 0 \quad \Bigg\} \quad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

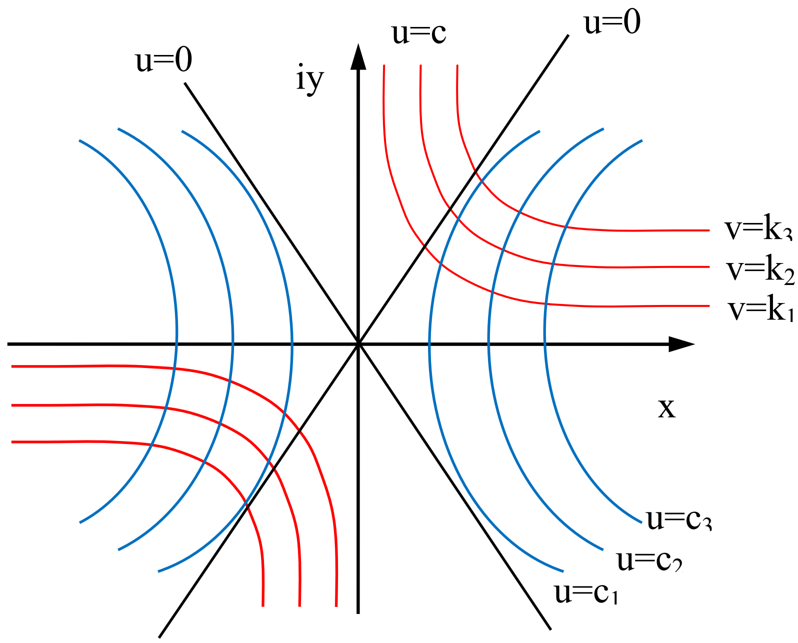
Note: If u and v satisfy Laplace eq. in 2-Dimension then $f(z) = u + iv$ **is not** necessarily analytic.

Ex. Draw the lines u = c, and v = k for the function $f(z) = z^2$.

Sol. $f(z) = z^2 = x^2 - y^2 + i\,2xy \quad \Rightarrow \quad \therefore u = x^2 - y^2 \quad , v = 2xy$

(1) $u = c \rightarrow x^2 - y^2 = c$

$y = \pm\sqrt{x^2 - c}$



(2) $v = k \rightarrow 2xy = y$

$y = \dfrac{k}{2x} = \dfrac{k}{2} \cdot \dfrac{1}{x}$

Ex. Verify that $u = x^2 - y^2 - y$ is harmonic and find a conjugate harmonic function v of u. Hence express u +iv as an analytic function of z.

Sol. $u = x^2 - y^2 - y$

$u_x = 2x \rightarrow u_{xx} = 2$

$u_y = -2y - 1 \rightarrow u_{yy} = -2$

$\because u_{xx} + u_{yy} = 0 \quad \therefore u \text{ is harmonic}$

To find v by using C.R.E.

$v_y = u_x = 2x \quad \cdots(1)$

$v_x = -u_y = 2y + 1 \quad \cdots(2)$

Integration eq.(1) with respect to y

$v = 2xy + h(x) \xrightarrow{Diff.\ w.r.t\ x} v_x = 2y + h'(x) \quad \cdots(3)$

Comparison eq.(3) with eq.(2)

$2y + h'(x) = 2y + 1 \rightarrow h'(x) = 1 \xrightarrow{Integral} \therefore h(x) = x + c$

$\therefore v = 2xy + x + c$

$f(z) = u + iv = x^2 - y^2 - y + i(2xy + x + c) = \left(x^2 - y^2 - i\,2xy\right) + (-y + ix) + ic$

$\therefore f(z) = z^2 + iz + ic$

C.R.E. in Polar form.

$$\frac{\partial u}{\partial r}=\frac{1}{r}\frac{\partial v}{\partial \theta} \quad \text{and} \quad \frac{1}{r}\frac{\partial v}{\partial \theta}=-\frac{\partial u}{\partial r}$$

$$f(z)=u(x,y)+iv(x,y)$$
$$=u(r,\theta)+iv(r,\theta)$$

$$\frac{\partial u}{\partial r}=\frac{\partial u}{\partial x}.\frac{\partial x}{\partial r} \quad , \quad \frac{\partial u}{\partial \theta}=\frac{\partial u}{\partial y}.\frac{\partial y}{\partial \theta}$$

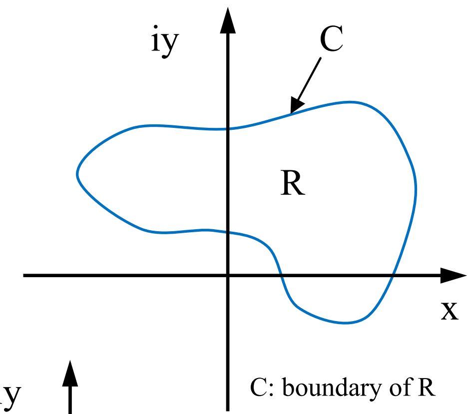
Hint:
$x=r\cos\theta$
$y=r\sin\theta$

H.W. Proof That.

H.W. Write Laplace equations for u and v in polar form.

H.W. Solve 8 problems about differentiation and analytic function.

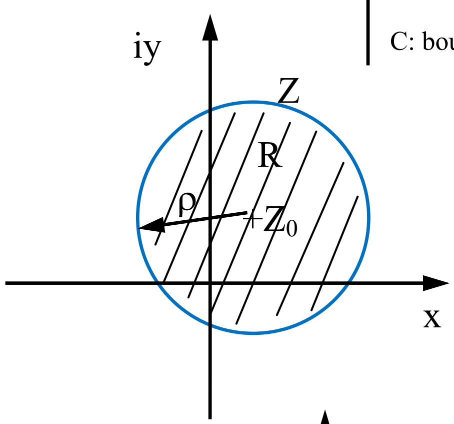
Regions in Complex Plane



(1) Open region:

In this region the boundaries don't belong to the region. $C \notin R$

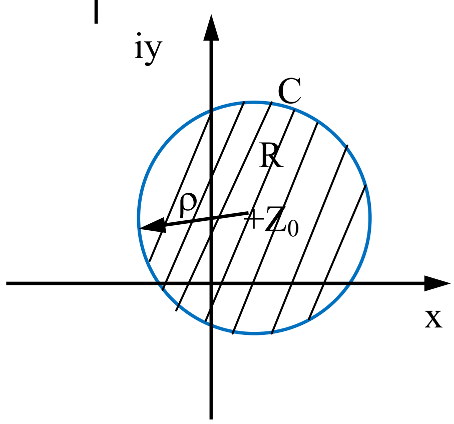
$$R=\{z:|z-z_0|<\rho\}$$



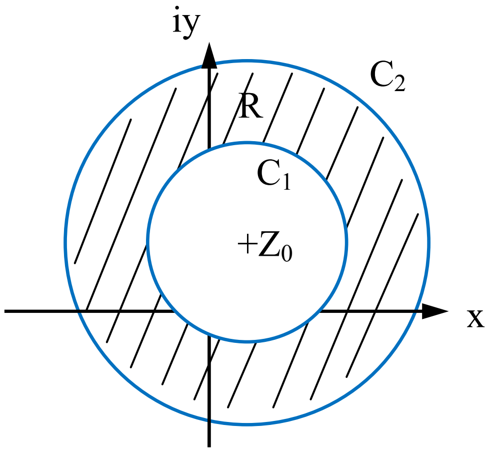
(2) Close region:

The boundaries points belong to the region. $C \in R$

$$R=\{z:|z-z_0|\le\rho\}$$



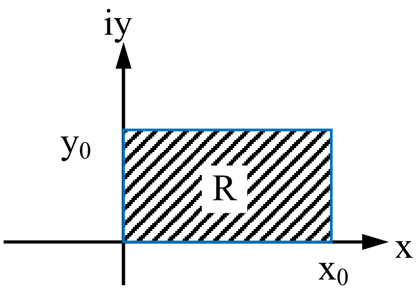
(3) Neither open nor closed:



In this region part of the boundary does not belong to the region.

$$R = \{z : C_1 \le |z - z_0| < C_2\}$$
$$C_1 \in R$$
$$C_2 \notin R$$

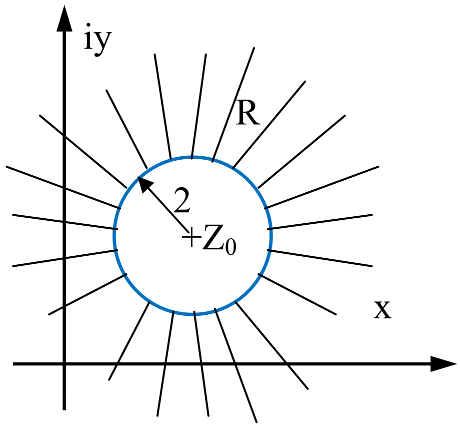
(4) Bounded region:



In this region, each two points can be connected by a line of finite length.

$$R = \{z : 0 \le \mathrm{Re}(z) \le x_o \ , 0 \le \mathrm{Im}(z) \le y_o\}$$

(5) Unbounded region:

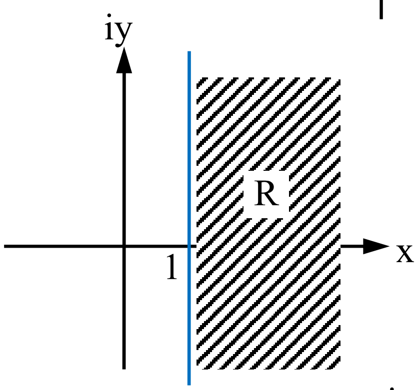


In this region, there exist at least two points that not be connected by a finite length.

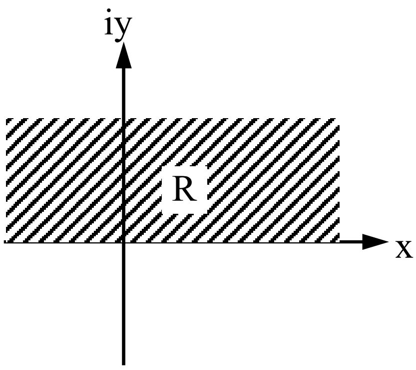
$$R = \{z : |z - z_0| \ge 2\}$$

or

$$R = \{z : \mathrm{Re}(z) > 1\}$$



(6) Connected region:



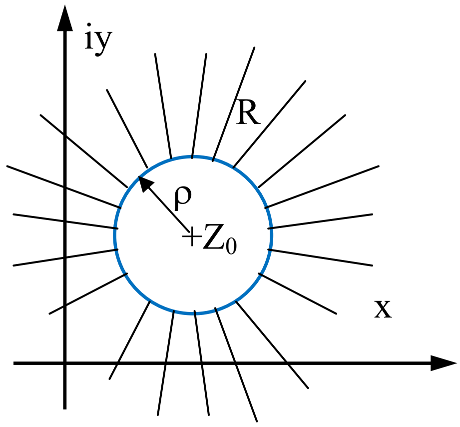
In that region, each two points can be connected by a path all of its points belong to region.

$$Ex. \ R = \{z : \mathrm{Im}(z) \ge 0\}$$

(a) Simple connected there is no holes.

(b) multiply connected there is one or more holes.

$$R = \{z : |z - z_0| \geq \rho\}$$

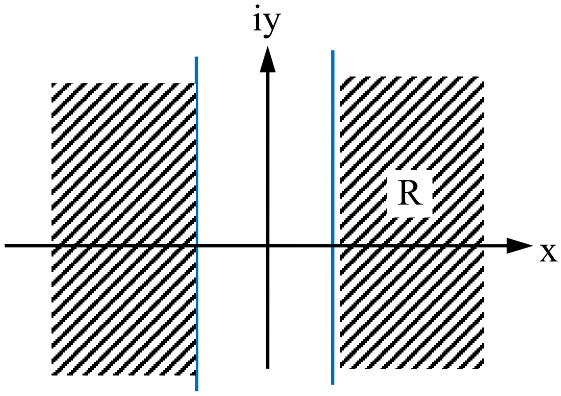


(7) Unconnected region:

A region in which there exist

two or more points that not can be

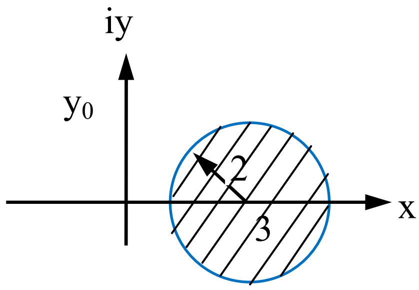
connected by a line all of its points belong to region.



Describe and draw the following regions:

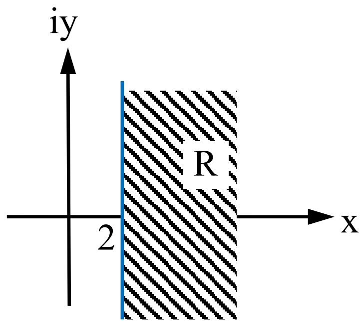
(1) $R = \{z : |z - 3| \leq 2\}$

close, bounded, simply connected region.



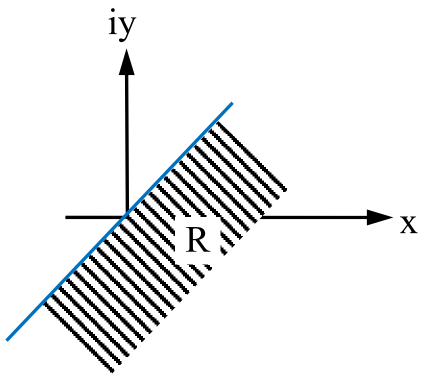
(2) $R = \{z : \text{Re}(z) \geq 2\}$

Neither open nor close, unbounded, simply connected region.



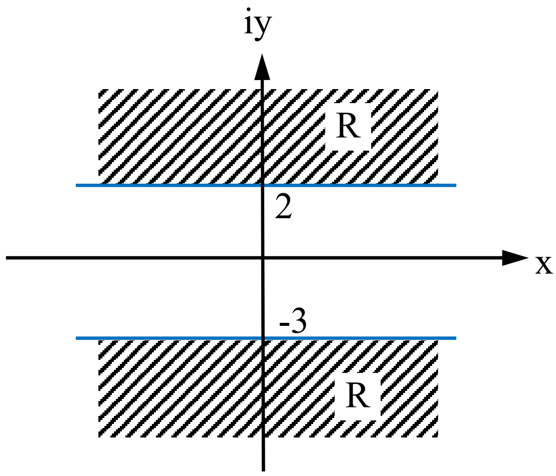
(3) $R = \{z : \text{Re}(z) \geq \text{Im}(z)\}$

Neither open nor close, unbounded, simply connected region.



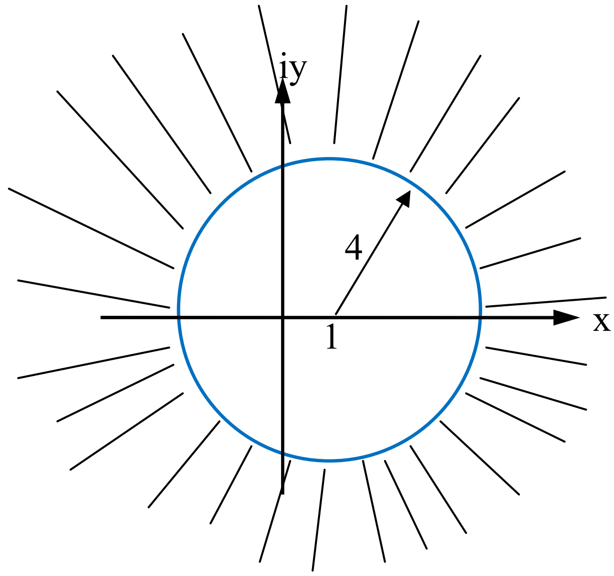
(4) $R = \{z : \mathrm{Im}(z) \geq 2, \mathrm{Im}(z) \leq -3\}$

Neither open nor close, unbounded, disconnected region.



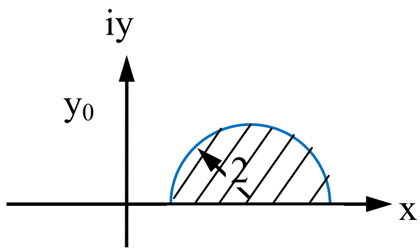
(5) $R = \{z : |z - 1| > 4\}$

open, unbounded, multiply connected region.



(6) $R = \{z : 0 \leq Arg(z) \leq \pi, \ |z| \leq 2\}$

close, bounded, simply connected region.



Complex Integrals:-

If f(z) is a single-value, continuous function in some region R, then we define the integral of f(z) along path C in R as;-

$$f(z) = u + iv \quad , dz = dx + idy$$

$$\int_c f(z)dz = \int_c (u + iv)(dx + idy)$$

$$\int_c f(z)dz = \int_c (udx - vdy) + i\int_c (vdx + udy)$$

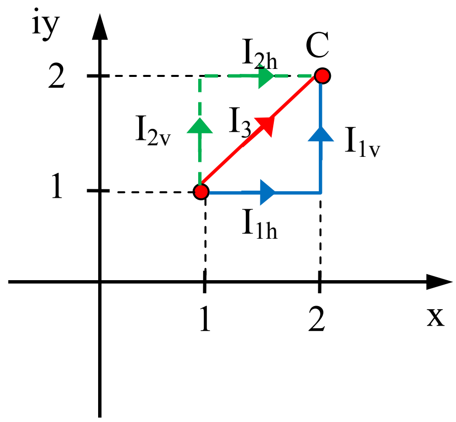
Ex. Find $\int_{1+i}^{2+i2} zdz$ along the following paths-

(a) horizontally from 1 + i to 2 + i then vertically to 2 + i2.

(b) vertically from 1 + i to 1 + i2 then horizontally to 2 + i2.

(c) along line from 1 + i directly to 2 + i2.

Sol. (a) $I_1 = I_{1h} + I_{1v}$



a long I_{1h} ; $y = 1, \quad dy = 0$
$dx : 1 \rightarrow 2$

$$I_{1h} = \int_{1+i}^{2+i} (udx - \cancel{vdy}) + i\int_{1+i}^{2+i} (vdx + \cancel{udy})$$

$$= \int_1^2 xdx + i\int_1^2 ydx = \frac{x^2}{2}\Big|_1^2 + iyx\Big|_1^2$$

$$= \frac{3}{2} + i\left[1*(2-1)\right] = \frac{3}{2} + i$$

a long I_{1v} ; $x = 2, \quad dx = 0$
$dy : 1 \rightarrow 2$

$$I_{1v} = \int_{1+i}^{2+i} (\cancel{udx} - vdy) + i\int_{1+i}^{2+i} (\cancel{vdx} + udy)$$

$$= \int_1^2 -ydy + i\int_1^2 xdy = -\frac{y^2}{2}\Big|_1^2 + i\,xy\Big|_1^2$$

$$= -\frac{3}{2} + i\left[2*(2-1)\right] = -\frac{3}{2} + i2$$

$$\therefore I_1 = \left(\frac{3}{2} + i\right) + \left(-\frac{3}{2} + i\,2\right) = i\,3$$

(b) $I_2 = I_{2v} + I_{2h}$

a long I_{2v} ; $\begin{array}{l} x = 1, \quad dx = 0 \\ dy : 1 \to 2 \end{array}$

$$I_{2v} = \int_1^2 -y\,dy + i\int_1^2 x\,dy = -\frac{y^2}{2}\bigg|_1^2 + i\,xy\big|_1^2 = -\frac{3}{2} + i\left[1*(2-1)\right] = -\frac{3}{2} + i\,1$$

a long I_{2h} ; $\begin{array}{l} y = 2, \quad dy = 0 \\ dx : 1 \to 2 \end{array}$

$$I_{2h} = \int_1^2 x\,dx + i\int_1^2 y\,dx = \frac{x^2}{2}\bigg|_1^2 + iyx\big|_1^2$$

$$= \frac{3}{2} + i\left[2*(2-1)\right] = \frac{3}{2} + i\,2$$

$$\therefore I_2 = \left(-\frac{3}{2} + i\right) + \left(\frac{3}{2} + i\,2\right) = i\,3$$

(c) Along the line (1, 1) ---> (2, 2) the line has the equation

$y = x$

$dy = dx$ <-----**using** $\dfrac{(y_2 - y_1)}{(x_2 - x_1)} = \dfrac{(y - y_1)}{(x - x_1)}$

$$I_3 = \int_{1+i}^{2+i} (u\,dx - v\,dy) + i\int_{1+i}^{2+i} (v\,dx + u\,dy)$$

$$= \int_{1+i}^{2+i} (x\,dx - y\,dy) + i\int_{1+i}^{2+i} (y\,dx + x\,dy)$$

$$= i\int_{1+i}^{2+i} 2x\,dx = ix^2\big|_1^2 = i\,3$$

$$\therefore I_1 = I_2 = I_3$$

Note: The integration of an analytic function does not depend on the path.

$$\therefore \int_{1+i}^{2+i\,2} z\,dz = \frac{z^2}{2}\bigg|_{1+i}^{2+i\,2} = \frac{(2+i\,2)^2}{2} - \frac{(1+i)^2}{2} = i\,3$$

This is coincidence if f (z) analytic only.

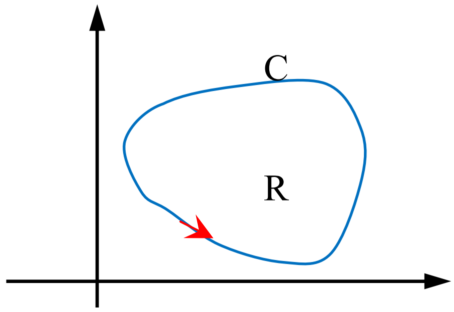
H.W. Evaluate $\int_{1+i2}^{2+i3} \bar{z}dz$ along the following paths-

(a) horizontally from 1 + i2 to 2 + i2 then vertically to 2 + i3.

(b) vertically from 1 + i to 1 + i3 then horizontally to 2 + i3.

Contour Integration

When the integration starts and end at the same point along a closed path "C", it is called "Contour Integration".



Positive direction: moving along C such that R is to your left.

It is denoted: $\oint$

Pole: The value of z which makes the combination $\dfrac{f(z)}{z-z_o}$ to not analytic; i.e. the pole is z_0. (the denominator is zero)

Theorems:-

(1) Cauchy-Goursat Theorem:- If f(z) is analytic in a simply connected bounded region R then $\oint f(z)dz = 0$ for every simple closed path C lying in the region R.

Proof:-

$$\oint f(z)dz = \oint (udx - vdy) + i\oint (vdx + udy)$$

Using Green's Theorem

$$\iint \left(\frac{\partial Q}{\partial x} + \frac{\partial P}{\partial y}\right)dxdy = \oint (Qdy - Pdx)$$

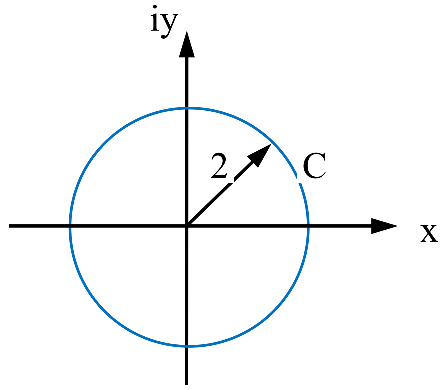
$$\oint f(z)dz = -\oint (vdy - udx) + i\oint (udy + vdx)$$

$$= -\iint \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)dxdy + i\iint \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)dxdy$$

Using C.R.E.

$$\therefore \oint f(z)dz = 0$$

Ex. Find $\oint f(z)dz$, where f(z) = z and C is the circle $|z| = 2$



$$z = re^{i\theta} \rightarrow dz = rie^{i\theta}d\theta$$

$$\oint_c z\,dz = \int_0^{2\pi} \left(re^{i\theta}\right)\left(rie^{i\theta}\right)d\theta$$

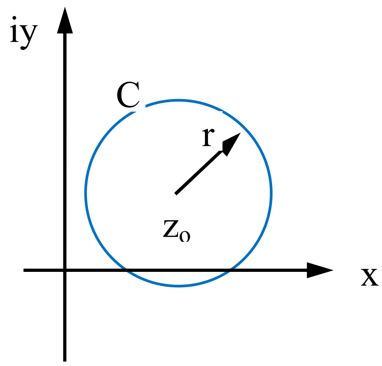
$$= ir^2 \int_0^{2\pi} e^{i2\theta}d\theta = \frac{r^2}{2}\left[e^{i2\theta}\right]_0^{2\pi}$$

$$= \frac{r^2}{2}\left[\cos 2\pi + i\sin 2\pi\right]_0^{2\pi} = 0$$

(2)

$$\oint \frac{dz}{(z-z_o)^{n+1}} = \begin{cases} i2\pi & \text{if } n = 0, \text{ and } z_o \text{ inside } C \\ 0 & \text{if } n \neq 0, \text{ or } z_o \text{ outside } C \end{cases}$$

Proof:



$$z - z_o = re^{i\theta}$$

$$dz = ire^{i\theta}d\theta$$

$$\oint_c \frac{dz}{(z-z_o)^{n+1}} = \int_0^{2\pi} \frac{ire^{i\theta}d\theta}{r^{n+1}e^{i(n+1)\theta}} = \frac{i}{r^n}\int_0^{2\pi} e^{-in\theta}d\theta$$

$$= \frac{i}{r^n}\int_0^{2\pi} (\cos n\theta - i\sin n\theta)d\theta$$

When n = 0

$$= \frac{i}{r^0}\int_0^{2\pi}(1-0)d\theta = i2\pi$$

if $n \neq 0$

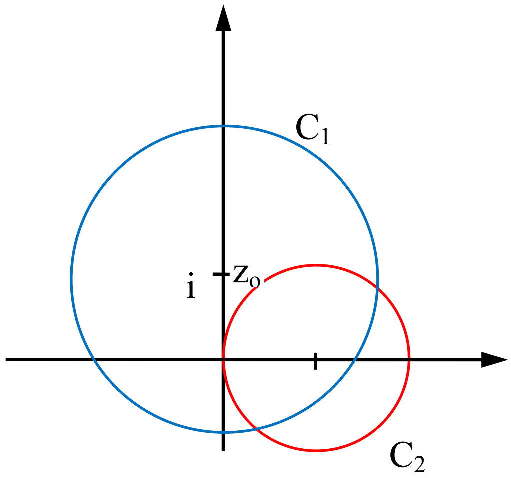
$$= \frac{i}{r^n}\int_0^{2\pi}(\cos n\theta - i\sin n\theta)d\theta = 0$$

If z_0 outside C, then $\dfrac{1}{(z-z_o)^{n+1}}$ will be analytic inside C hence as integral around C is zero.

Ex. Evaluate $\oint \frac{dz}{(z-i)}$ around the following paths:-

a. the circle $|z-i|=2$

b. the circle $|z-1|=1$



Sol. n = 0 , Pole = i = z_0

a. along C_1

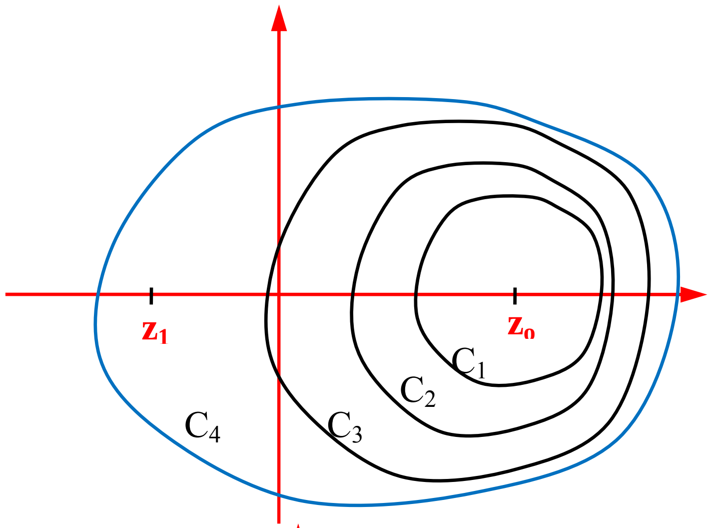
$$\oint \frac{dz}{(z-i)} = i\,2\pi \text{ because } z_o \text{ inside } C_1$$

b. along C_2

$$\oint \frac{dz}{(z-i)} = 0 \text{ because } z_o \text{ outside } C_2$$

Theorem (3):- The path of integration around z_0 can be deformed freely without affecting the value of integration given that the new path contains the same number of poles.

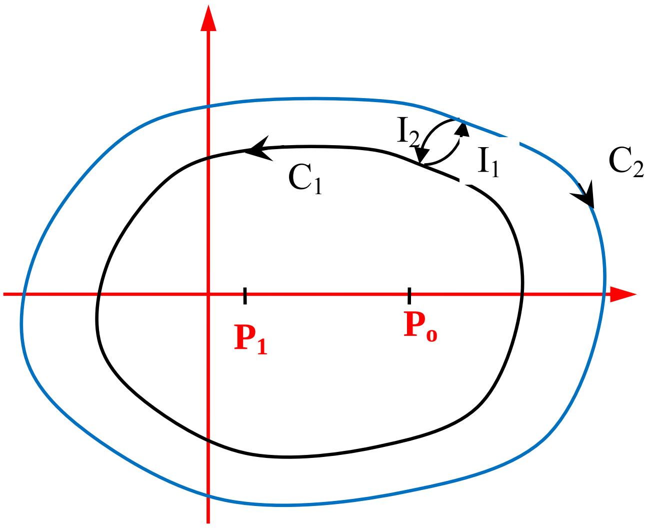
$$\oint_{c_1} = \oint_{c_2} = \oint_{c_3} \neq \oint_{c_4}$$

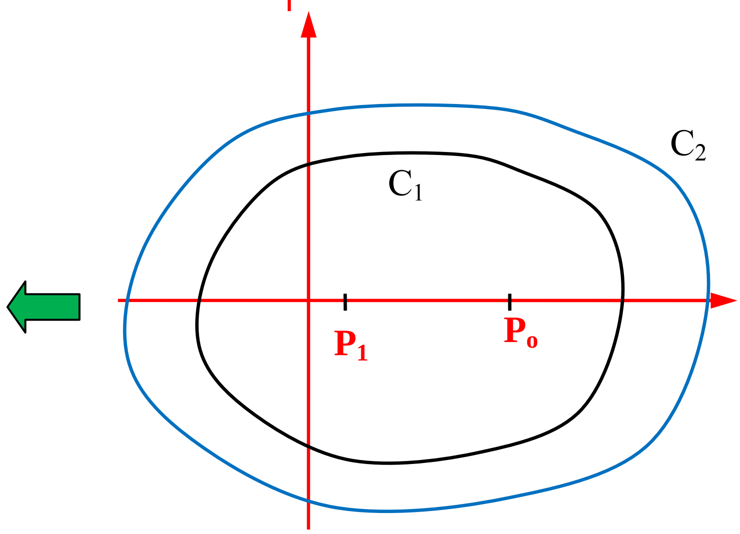


Since C_4 contains z_0, z_1

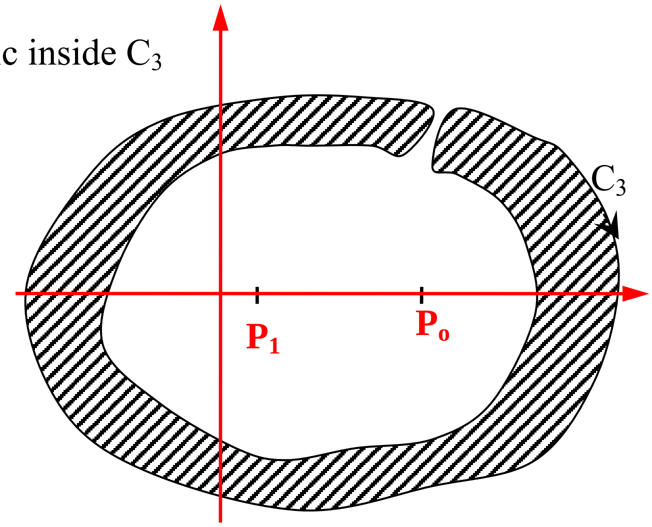
Proof:-

Assume P_1 and P_0 are poles.





C_3 does not contain any pole, thus f(z) is analytic inside C_3



$$\oint_{c_1} + \int_{I_1} + \int_{I_2} - \oint_{c_2} = 0 = \oint_{c_3}$$

$$\because \int_{I_1} = -\int_{I_2}$$

$$\therefore \oint_{c_1} = \oint_{c_2}$$

Theorem (4) Morera's Theorem

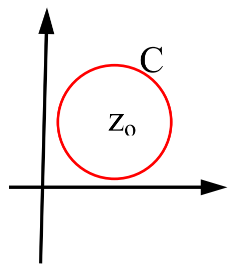
If f(z) is analytic inside and on C;

$$\oint \frac{f(z)}{(z-z_o)^{n+1}} dz = \begin{cases} \frac{i2\pi}{n!} f^{n}(z_o) & \text{if } z_o \text{ inside or on } C \\ 0 & \text{if } z_o \text{ outside } C \end{cases}$$

Proof:

Let n = 0 ---> **Cauchy's Integral Formula**

$$\oint \frac{f(z)}{(z-z_o)} dz = \oint \frac{f(z)-f(z_o)+f(z_o)}{z-z_o} dz$$



assuming C is very small closed path around z_o.

$$\lim_{z\to z_o} \oint \frac{f(z)-f(z_o)+f(z_o)}{z-z_o} dz$$

$$= \oint \left[\lim_{z\to z_o} \frac{f(z)-f(z_o)}{z-z_o} + \lim_{z\to z_o} \frac{f(z_o)}{z-z_o} \right] dz$$

$$= \oint \left[f'(z_o) + \lim_{z\to z_o} \frac{f(z_o)}{z-z_o} \right] dz$$

$$= \oint \cancel{f'(z_o)}^{=0} dz + \lim_{z\to z_o} \oint \frac{f(z_o)}{z-z_o} dz$$

$$= 0 + \lim_{z\to z_o} f(z_o) \oint \frac{dz}{z-z_o}$$

$$= i2\pi f(z_o)$$

$$\oint \frac{f(z)}{(z-z_o)} dz = i2\pi f(z_o)$$ differentiation with respect to z_o

$$\oint (-1)\frac{f(z)}{(z-z_o)^2}(-1) dz = i2\pi f'(z_o)$$

$$\oint \frac{f(z)}{(z-z_o)^2} dz = \frac{i2\pi}{1!} f'(z_o)$$ diff. w.r.t. z_o

$$\oint (-2)\frac{f(z)}{(z-z_o)^3}(-1) dz = i2\pi f''(z_o)$$

$$\oint \frac{f(z)}{(z-z_o)^3} dz = \frac{i2\pi}{2!} f''(z_o)$$

$$\vdots \qquad \vdots$$

$$\therefore \oint \frac{f(z)}{(z-z_o)^{n+1}} dz = \frac{i2\pi}{n!} f^{n}(z_o)$$

Ex. Evaluate $\oint_c \frac{e^z}{z^2-1} dz$ where C is

(1) the circle $|z| = 1/2$

(2) the circle $|z+1| = 1$

(3) the circle $|z-1| = 1$

(4) the rectangular from (-2, -2) to (2, 2)

(5) the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$

Sol. $\oint_c \frac{e^z}{z^2-1} dz = \oint_c \frac{e^z}{(z-1)(z+1)} dz$

$$\frac{1}{(z-1)(z+1)} = \frac{A}{z-1} + \frac{B}{z+1}$$

$$A(z+1) + B(z-1) = 1$$

$$(A+B)z + (A-B) = 1$$

$$\left.\begin{array}{l} A+B=0 \\ A-B=1 \end{array}\right\} \rightarrow \quad \begin{array}{l} A = 1/2 \\ B = -1/2 \end{array}$$

Or

$$A = \lim_{z\to 1} \frac{1}{z+1} = \frac{1}{2}$$

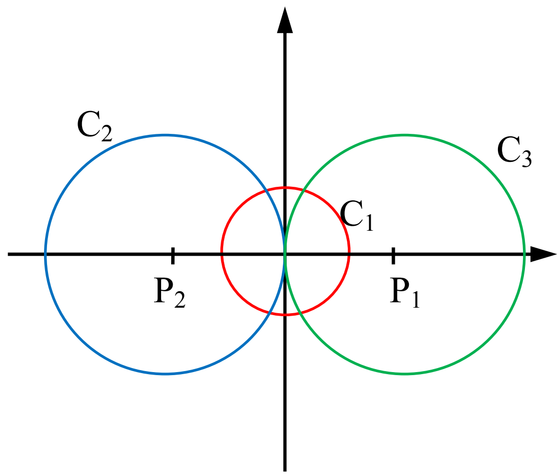
$$B = \lim_{z\to -1} \frac{1}{z-1} = -\frac{1}{2}$$

$$\oint_c \frac{e^z}{z^2-1}dz = \frac{1}{2}\oint_c \frac{e^z}{(z-1)}dz - \frac{1}{2}\oint_c \frac{e^z}{(z+1)}dz$$

Poles $P_1 = 1$, $P_2 = -1$

(1) C_1 the circle $|z| = 1/2$

$$\oint_{C_1} \frac{e^z}{z^2-1}dz = 0$$ since C_1 does not contains any poles



(2) C_2 the circle $|z+1| = 1$

Poles P_1 outside C_2, P_2 inside C_2

$$\oint_{C_2} \frac{e^z}{z^2-1}dz = \frac{1}{2}\oint_{C_2} \cancel{\frac{e^z}{(z-1)}}^{=0} dz - \frac{1}{2}\oint_{C_2} \frac{e^z}{(z+1)}dz$$

$$= 0 - \frac{1}{2}\left(i\,2\pi e^{-1}\right) = \frac{-i\pi}{e}$$

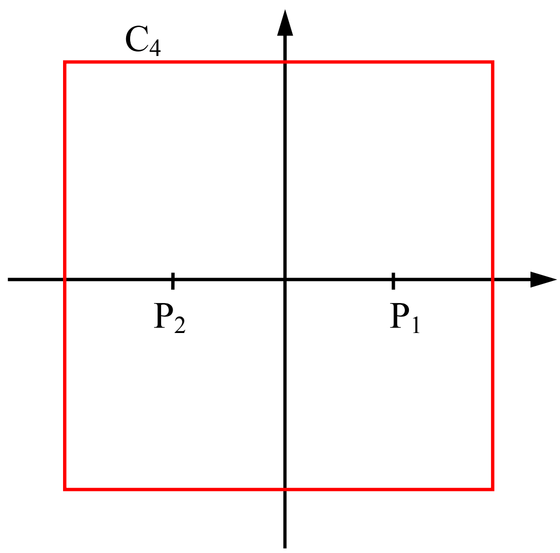
(3) C_3 the circle $|z-1| = 1$

Poles P_1 inside C_3, P_2 Outside C_3

$$\oint_{C_3} \frac{e^z}{z^2-1}dz = \frac{1}{2}\oint_{C_3} \frac{e^z}{(z-1)}dz - \frac{1}{2}\oint_{C_3} \cancel{\frac{e^z}{(z+1)}}^{=0} dz$$

$$= \frac{1}{2}\left(i\,2\pi e^{1}\right) - 0 = i\pi e$$

(4) C_4 the rectangular from (-2, -2) to (2, 2)



The two poles P_1 and P_2 inside C_4

$$\oint_{C_4} \frac{e^z}{z^2-1}dz = \frac{1}{2}\oint_{C_4} \frac{e^z}{(z-1)}dz - \frac{1}{2}\oint_{C_4} \frac{e^z}{(z+1)}dz$$

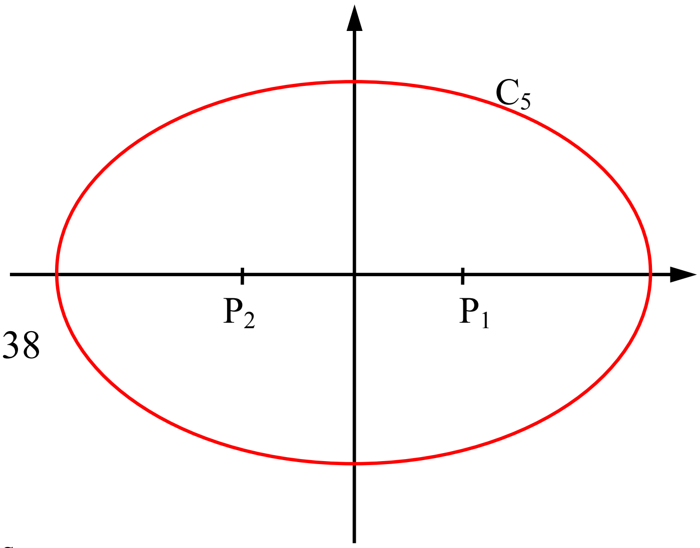
$$= \frac{1}{2}\left(i\,2\pi e^{1}\right) - \frac{1}{2}\left(i\,2\pi e^{-1}\right)$$

$$= i\pi e^{1} - i\pi e^{-1} = i\,2\pi \sinh 1 = i\,7.38$$

(5) the ellipse $\dfrac{x^2}{9}+\dfrac{y^2}{4}=1$

The two poles P_1 and P_2 inside C_4

$$\oint_{C_5}\frac{e^z}{z^2-1}dz=\oint_{C_4}\frac{e^z}{z^2-1}dz=i\,2\pi\sinh 1=i\,7.38$$



Ex. Evaluate $\displaystyle\oint_c \frac{z^2}{z^3+z^2-z-1}dz$ where C is

(1) the circle $|z-1|=1$

(2) the circle $|z+1-i|=2$

(3) the ellipse $\dfrac{x^2}{2}+y^2=2$

Sol.

$$f(z)=z^2$$

$$\therefore z^3+z^2-z-1=(z+1)(z^2-1)$$
$$=(z+1)(z+1)(z-1)$$
$$=(z+1)^2(z-1)$$

Synthetic division

-1	1	1	-1	-1
		-1	0	1
	1	0	-1	0

$(z^2-1)=0$

$$\therefore \frac{1}{z^3+z^2-z-1}=\frac{1}{(z+1)^2(z-1)}=\frac{Az+B}{(z+1)^2}+\frac{C}{(z-1)}$$

$$Az^2-Az+Bz-B+Cz^2+2Cz+C=1$$

$$\left.\begin{array}{ll} A+C=0 & \to A=-C \\ B-A+2C=0 & \to B=-3C \\ C-B=1 & \to C=1/4 \end{array}\right\} \Rightarrow A=\frac{-1}{4},\ B=\frac{-3}{4},\ C=\frac{1}{4}$$

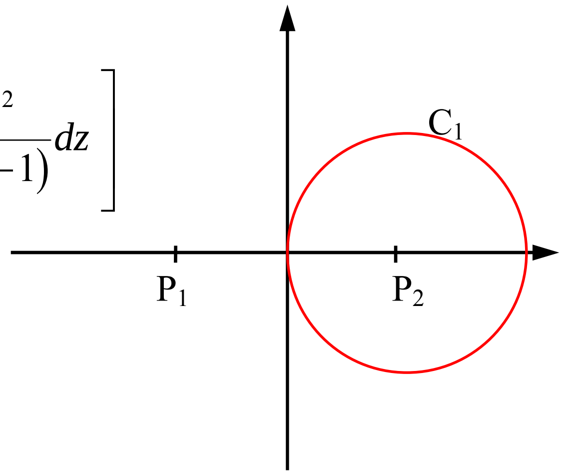
$$\oint_C \frac{z^2}{(z+1)^2(z-1)}dz=\frac{-1}{4}\left[\oint_C \frac{z^2(z+3)}{(z+1)^2}dz-\oint_C \frac{z^2}{(z-1)}dz\right]$$

Poles $P_1=-1$, $P_2=1$

(1) C_1 the circle $|z-1|=1$

$$\oint_{C_1}\frac{z^2}{(z+1)^2(z-1)}dz=\frac{-1}{4}\left[\cancelto{0}{\oint_{C_1}\frac{z^2(z+3)}{(z+1)^2}dz}-\oint_{C_1}\frac{z^2}{(z-1)}dz\right]$$

$$=\frac{-1}{4}\left[0-i\,2\pi(1)^2\right]=\frac{i\pi}{2}$$



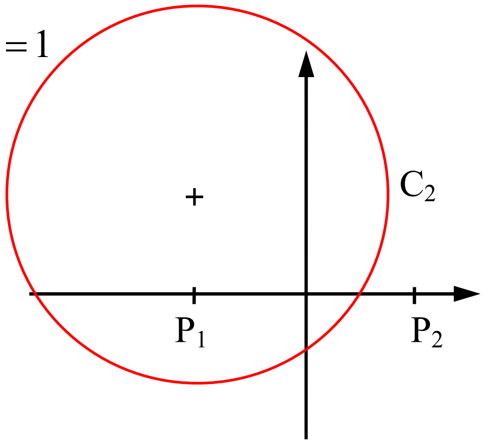
(2) C_2 the circle $|z+1-i|=2$

$$\oint_{C_2}\frac{z^2}{(z+1)^2(z-1)}dz=\frac{-1}{4}\left[\oint_{C_2}\frac{(z^3+3z^2)}{(z+1)^2}dz-0\right]\quad ,n=1$$

$$=\frac{-1}{4}\left[\frac{i\,2\pi}{1!}\frac{d}{dz}(z^3+3z^2)\Big|_{z=-1}\right]$$

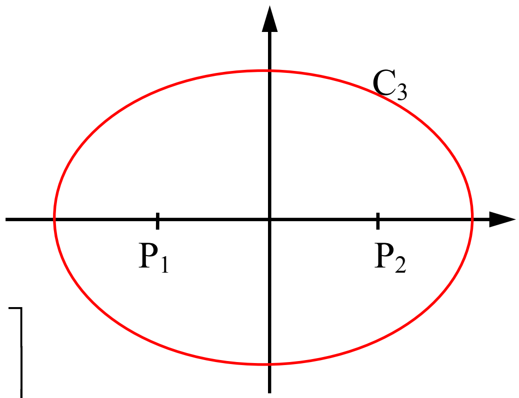
$$=\frac{-i\pi}{2}\left[3z^2+6z\right]_{z=-1}$$

$$=\frac{-i\pi}{2}(-3)=\frac{i\,3\pi}{2}$$



(3) C_3 the ellipse $\frac{x^2}{2}+y^2=2$

$$\frac{x^2}{4}+\frac{y^2}{2}=1\quad\rightarrow a=2,\ b=\sqrt{2}$$



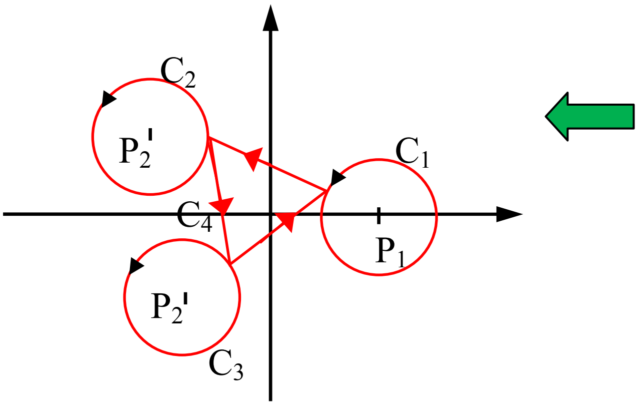
$$\oint_{C_3}\frac{z^2}{(z+1)^2(z-1)}dz=\frac{-1}{4}\left[\oint_{C_3}\frac{z^3+3z^2}{(z+1)^2}dz-\oint_{C_3}\frac{z^2}{(z-1)}dz\right]$$

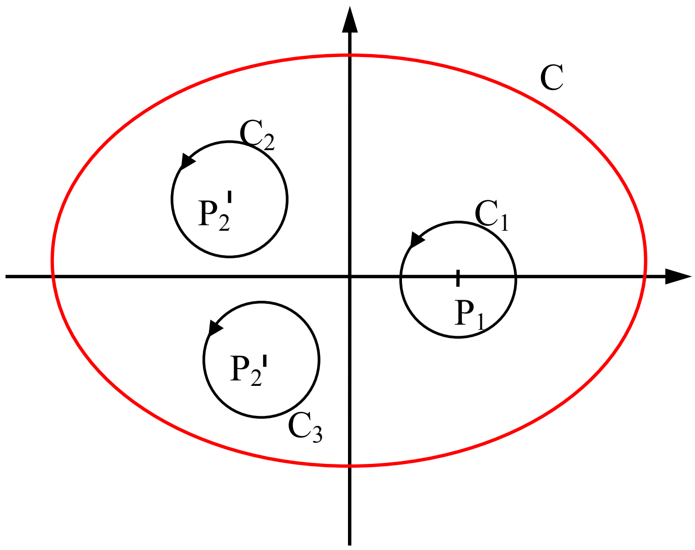
$$=\frac{i\,3\pi}{2}+\frac{i\pi}{2}=i\,2\pi$$

Residue Theorem:-

the contour integral around a path containing a number of poles equal to the sum of contour integration around paths of which contains a distinct pole.

$$\oint_C = \oint_{C_1} + \oint_{C_2} + \oint_{C_3}$$

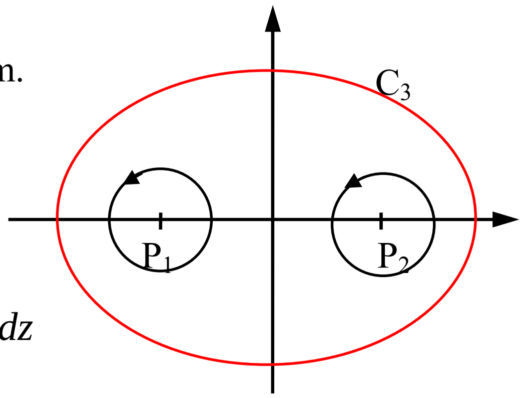




Proof:-

$$\oint_C = \oint_{C_1} + \oint_{C_2} + \oint_{C_3} + \underbrace{\oint_{C_4}}_{=0 \ \text{outside poles}}$$

Ex. Resolve the last example by using the residue theorem.



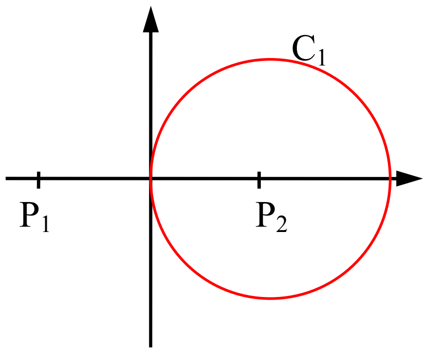
take path (3) the ellipse $\dfrac{x^2}{2} + y^2 = 2$

$$\oint_{C_3} \frac{z^2}{(z+1)^2(z-1)} dz = \oint_{P_1} \frac{z^2/(z-1)}{(z+1)^2} dz + \oint_{P_2} \frac{z^2/(z+1)^2}{(z-1)} dz$$

$$= \frac{i\,2\pi}{1!}\left[\frac{d}{dz}\left(\frac{z^2}{(z-1)}\right)\right]_{z=-1} + i\,2\pi\left[\frac{z^2}{(z+1)^2}\right]_{z=1}$$

$$= \frac{i\,2\pi}{1!}\left[\frac{2z(z-1)-z^2}{(z-1)^2}\right]_{z=-1} + \frac{i\pi}{2} = \frac{i\,3\pi}{2} + \frac{i\pi}{2} = i\,2\pi$$

(2) **(1)** C_1 the circle $|z-1| = 1$



$$\oint_{C_2} \frac{z^2}{(z+1)^2(z-1)} dz = \oint_{P_2} \frac{z^2/(z+1)^2}{(z-1)} dz$$

$$= \frac{i\,2\pi}{0!}\left[\frac{z^2}{(z+1)^2}\right]_{z=1} = \frac{i\pi}{2}$$

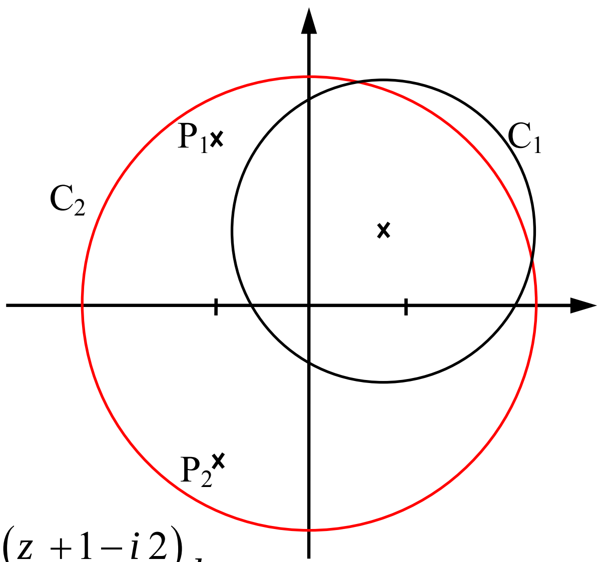
Ex. Evaluate $\oint_c \frac{\sin z}{z^2 2z+5} dz$ where C is

(1) the circle $|z-1-i|=2$

(2) the circle $|z|=3$

$$z=\frac{-2\mp\sqrt{4-20}}{2}=\begin{cases}-1+i\,2 & ,P_1\\ -1-i\,2 & ,P_2\end{cases}$$

Sol. $\frac{\sin z}{z^2 2z+5}=\frac{\sin z}{(z+1-i\,2)(z+1+i\,2)}$



(1) C_1 the circle $|z-1-i|=2$

center = 1 +i

radius = 2

The two poles outside C_1 $\therefore \oint_{C_1} = 0$

(2) C_2 the circle $|z|=3$

The two poles inside C_2

$$\oint_{C_2}\frac{\sin z}{z^2 2z+5}=\oint_{P_1}\frac{\sin z/(z+1+i\,2)}{(z+1-i\,2)}dz+\oint_{P_2}\frac{\sin z/(z+1-i\,2)}{(z+1+i\,2)}dz$$

$$=i\,2\pi\left[\frac{\sin z}{z+1+i\,2}\right]_{z=-1+i\,2}+i\,2\pi\left[\frac{\sin z}{z+1-i\,2}\right]_{z=-1-i\,2}$$

$$=i\,2\pi\left[\frac{\sin(-1+i\,2)}{i\,4}\right]+i\,2\pi\left[\frac{\sin(-1-i\,2)}{-i\,4}\right]$$

$$=\frac{\pi}{2}\sin(-1+i\,2)-\frac{\pi}{2}\sin(-1-i\,2)$$

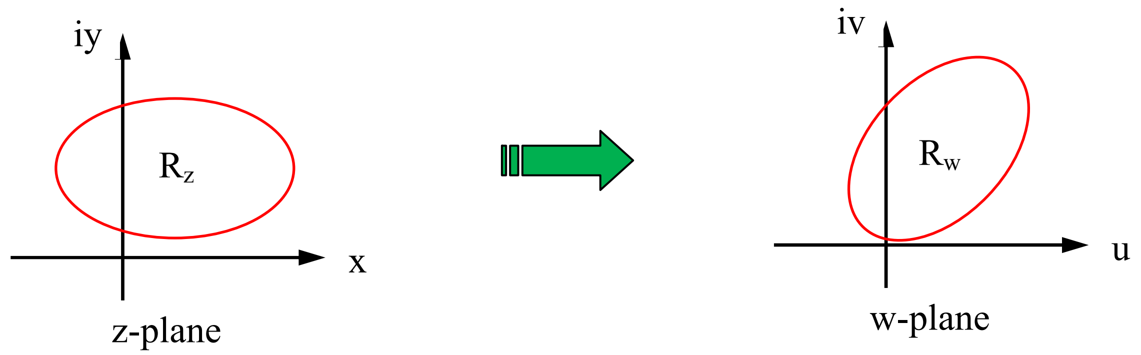
$$=\frac{\pi}{2}[2\cos 1\sin i\,2]=i\,\pi\cos 1\sinh 2$$

H.W. Solve problems about line and contour integration P.674 and P.703 in "Wylie".

Conformal Mapping

Mapping: is transformation from z-plane into w-plane using a function w=f(z).

If f(z) is analytic, then it is called "**conformal mapping**".

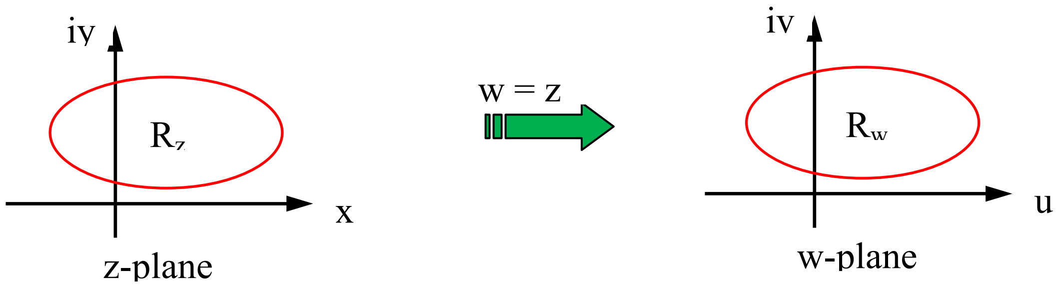


Types of Mapping:-

1- **Linear mapping** : w = az + b   "**a** and **b** are complex number"

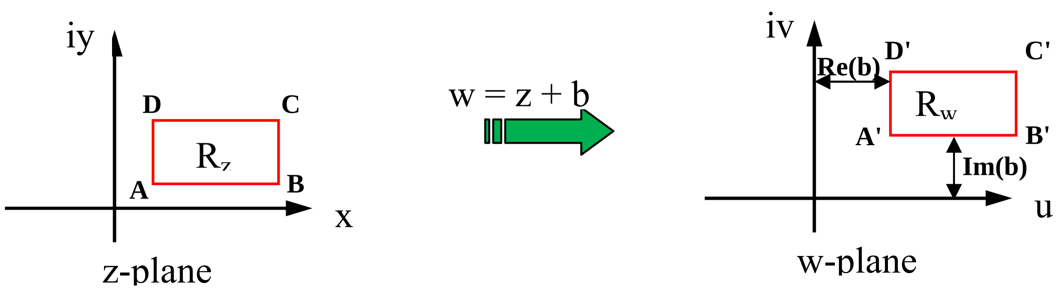
- If a = 1 , b = 0 then:-

  w = z "**Identity mapping**"



- If a = 1 , b ≠ 0, then;-

  w = z + b "**Shifting mapping**"

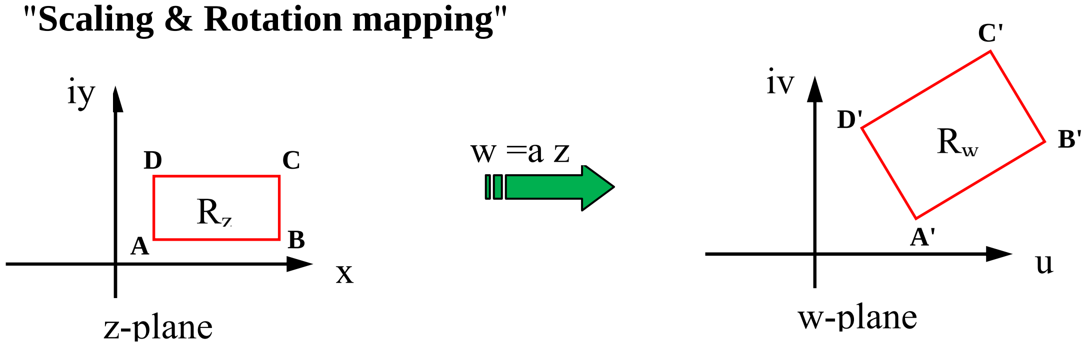


Re(A') = Re (A) + Re(b)

Im(A') = Im (A) + Im(b)

- If b = 0 , a ≠ 0 then:-

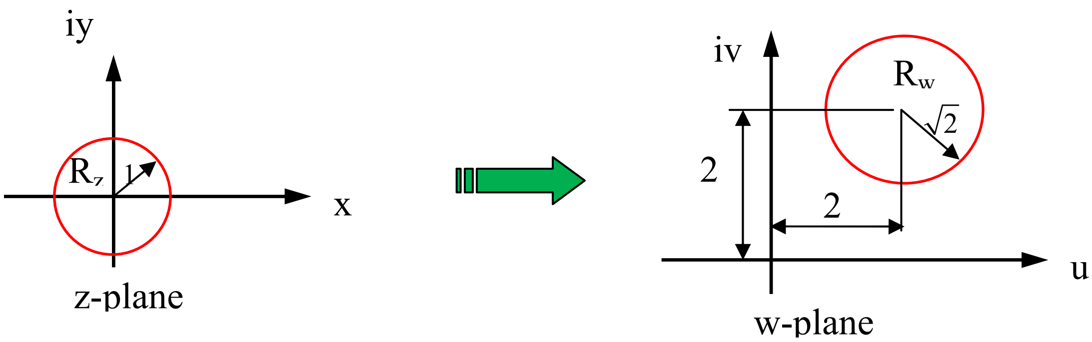
  w = az **"Scaling & Rotation mapping"**



$$|A'| = |A|.|a|$$
$$Arg(A') = Arg(A) + Arg(a)$$

Ex. Find R_w for the following transformation

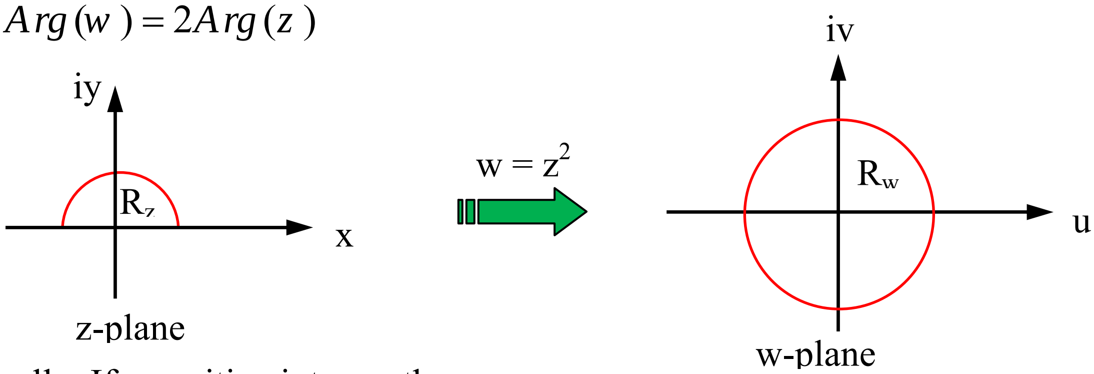
$$w = (1-i)z + (2+i2) \qquad where \quad Rz: \{z : |z| \le 1\}$$



2. **Power Mapping**: $w = z^n$

- If n = 1 "Identity mapping linear"
- If n = 2 ---> $w = z^2$ "squaring"

$$w = z^2 = r^2 \underline{|2\theta}$$
$$|w| = |z|^2, \; Arg(w) = 2Arg(z)$$



- Generally, If n positive integer, then:

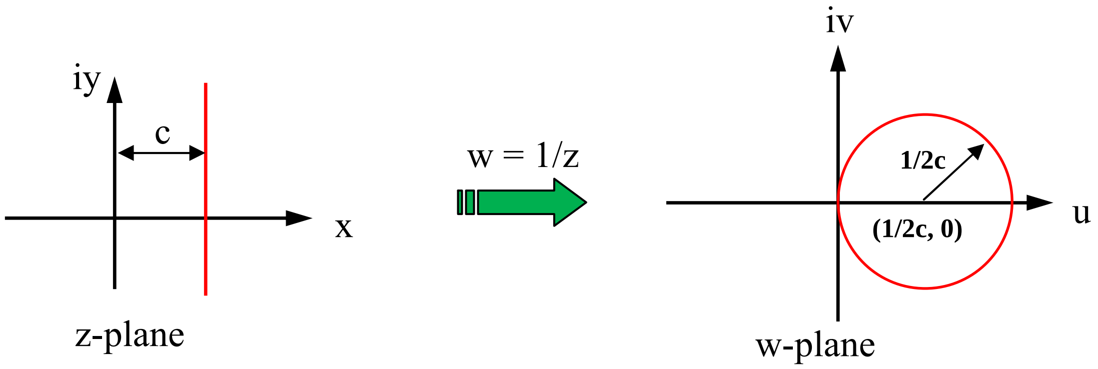
  $$w = z^n \quad \Rightarrow \quad |w| = |z|^n, \; Arg(w) = n.Arg(z)$$

(3) **Inversion Mapping**: $w = \dfrac{1}{z}$

$$w = \frac{1}{z} \quad \Rightarrow \quad |w| = \frac{1}{|z|}, \; Arg(w) = -Arg(z) \quad \to \phi = -\theta$$

This mapping translates the straight lines in the z-plane to circles in the w-plane and vise versa.

Ex. The line x = c is mapped into a circle of center $\left(\dfrac{1}{2c}, 0\right)$ and radius equal to $\dfrac{1}{2c}$



Sol. $$w = \frac{1}{z} = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}$$

$$u = \frac{x}{x^2+y^2}, \qquad v = -\frac{y}{x^2+y^2}$$

$$u = \frac{c}{c^2+y^2} \qquad \Rightarrow y^2 = \frac{c - c^2 u}{u} \qquad \cdots(1)$$

$$v = -\frac{y}{c^2+y^2} = \frac{\mp\sqrt{\dfrac{c-c^2u}{u}}}{c^2 + \dfrac{c-c^2u}{u}} = \mp\frac{\sqrt{\dfrac{c-c^2u}{u}}}{\dfrac{c}{u}} = \mp\sqrt{\frac{c-c^2u}{u}}\cdot\frac{u}{c}$$

$$v^2 = \frac{c-c^2u}{u}\cdot\frac{u^2}{c^2} \qquad \Rightarrow \; v^2 = \frac{u}{c} - u^2$$

$$\therefore v^2 + u^2 - \frac{u}{c} + \left(\frac{1}{2c}\right)^2 = \left(\frac{1}{2c}\right)^2$$

$$v^2 + \left(u - \frac{1}{2c}\right)^2 = \left(\frac{1}{2c}\right)^2$$ **center (1/2c, 0) and r = 1/2c**

(4) **Bilinear Mapping** : $w = \dfrac{az + b}{cz + d}$

In condition $\dfrac{a}{c} \neq \dfrac{b}{d}$

- If c = 0 , d = 1 ---> linear mapping
- If a = 0, b = 1, d = 0 ----> inversion mapping

Theorem of Bilinear mapping

* The bilinear mapping cannot contain more than two identical points if so, then it is identity mapping.

** If there are three points in z-plane z_1 , z_2 , z_3 and their images in w-plane w_1, w_2, w_3 then they can be characterize by the bilinear mapping.

$$\frac{w - w_1}{w - w_3} \cdot \frac{w_2 - w_3}{w_2 - w_1} = \frac{z - z_1}{z - z_3} \cdot \frac{z_2 - z_3}{z_2 - z_1}$$

Ex. Three points in z-plane i, 1, -1 and three images in w-plane 2, 1, 0. Find the bilinear mapping for f(z).

Sol. $$\frac{w - w_1}{w - w_3} \cdot \frac{w_2 - w_3}{w_2 - w_1} = \frac{z - z_1}{z - z_3} \cdot \frac{z_2 - z_3}{z_2 - z_1}$$

$$\frac{w - 2}{w - 0} \cdot \frac{1 - 0}{1 - 0} = \frac{z - i}{z + 1} \cdot \frac{1 + 1}{1 - i}$$

$$\frac{w - 2}{-w} = \frac{2z - i2}{(z + 1)(1 - i)}$$

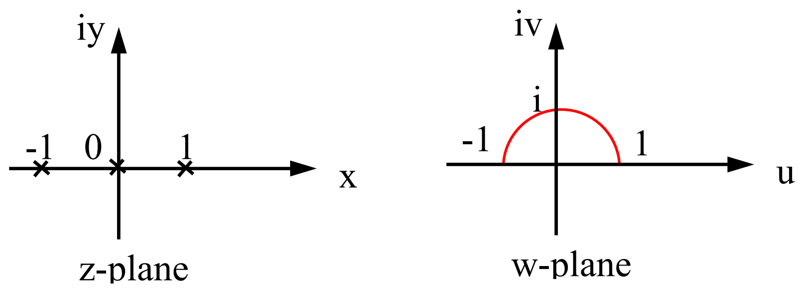
$$w(2z - i2) = (2 - w)(z + 1)(1 - i)$$

$$w\left[2z - i2 + (z + 1)(1 - i)\right] = (z + 1)(2 - i2)$$

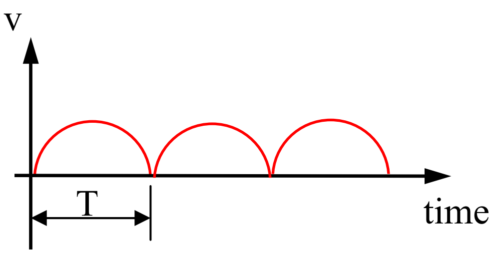
$$\therefore w = \frac{(2 - i2)z + (2 - i2)}{(3 - i)z + (1 - i3)}$$

H.W. Find the bilinear mapping maps the x-axis into a semicircle of radius unity as shown.

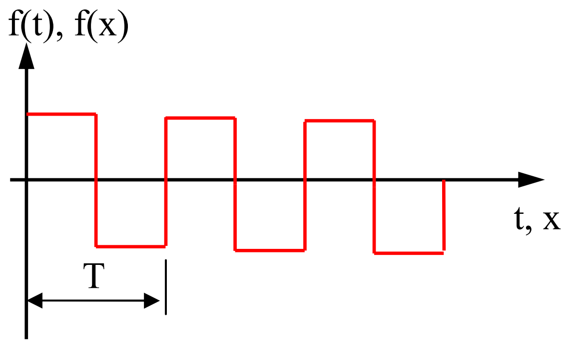
Ans. $w = \dfrac{z + i}{iz + 1}$



2. Fourier Series and Transform



Or



Periodic function: The function which repeats itself each "T" second, where "T" is called period.

Fourier Theorem: Any periodic function f(t) can be rewritten as a sum of sines and cosines components as follows:-

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos w_n t + b_n \sin w_n t$$

constant part mean, D.C. value — even part — odd part

$\frac{a_0}{2}$ **:** mean, average, DC-component**.**

a_n, b_n :- Coefficient of cosine and sine terms.

$\omega_n = \frac{2n\pi}{T}$: radian frequency (rad/s)

$f_n = \frac{n}{T} = \frac{\omega_n}{2\pi}$: frequency (Hz)

$f_1 = \frac{1}{T}$, first fundamental frequency.

$f_2 = \frac{2}{T} = 2f_1$, second fundamental frequency.

$f_3 = \frac{3}{T} = 3f_1$, Third fundamental frequency.

$p = \frac{T}{2}$, half period. $\therefore \omega_n = \frac{n\pi}{p}$

$$a_0 = \frac{1}{p}\int_{d}^{d+2p} f(t)dt$$

$$a_n = \frac{1}{p}\int_{d}^{d+2p} f(t)\cos\omega_n t dt$$

$$b_n = \frac{1}{p}\int_{d}^{d+2p} f(t)\sin\omega_n t dt$$

H.W. Proof Fourier theorem utilizing the following:-

$$\int_{d}^{d+T} \sin\omega_n t dt = 0 \qquad , \qquad \int_{d}^{d+T} \cos\omega_n t dt = 0$$

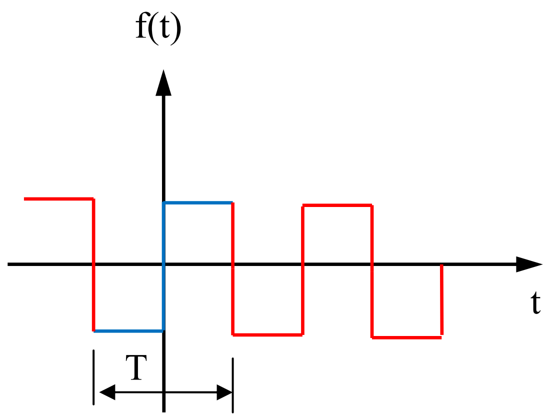
$$\int_{d}^{d+2p} \sin\omega_n t \cos\omega_m t dt = 0$$

$$\int_{d}^{d+2p} \sin\omega_n t \sin\omega_m t dt = \begin{cases} 0 & n \neq m \\ p & n = m \end{cases}$$

$$\int_{d}^{d+2p} \cos\omega_n t \cos\omega_m t dt = \begin{cases} 0 & n \neq m \\ p & n = m \end{cases}$$

Ex. Find the Fourier expansion of the periodic function whose definition in one period as:

$$f(t) = \begin{cases} -1 & -1 < t < 0 \\ 1 & 0 < t < 1 \end{cases}$$



$$\therefore T = 2, \ p = 1, \ \omega_n = \frac{n\pi}{p} = n\pi$$

$$a_0 = \frac{1}{p}\int_{d}^{d+2p} f(t)dt = \frac{1}{1}\int_{-1}^{1} f(t)dt$$

$$= \int_{-1}^{0}(-1)dt + \int_{0}^{1} 1dt = -t\Big|_{-1}^{0} + t\Big|_{0}^{1} = 0$$

$$a_n = \frac{1}{p}\int_{d}^{d+2p} f(t)\cos\omega_n t dt = \frac{1}{1}\int_{-1}^{1} f(t)\cos n\pi t dt$$

$$= \int_{-1}^{0} -\cos n\pi t dt + \int_{0}^{1}\cos n\pi t dt$$

$$= -\left[\frac{\sin n\pi t}{n\pi}\right]_{-1}^{0} + \left[\frac{\sin n\pi t}{n\pi}\right]_{0}^{1}$$

$$= -\left[0 + \frac{\sin n\pi}{n\pi}^{=0}\right] + \left[\frac{\sin n\pi}{n\pi}^{=0} - 0\right] = 0 \qquad \text{Note} : \sin n\pi = 0$$

$$b_n = \frac{1}{p}\int_{d}^{d+2p} f(t)\sin\omega_n t dt = \frac{1}{1}\int_{-1}^{1} f(t)\sin n\pi t dt$$

$$= \int_{-1}^{0} -\sin n\pi t dt + \int_{0}^{1}\sin n\pi t dt$$

$$= \left[\frac{\cos n\pi t}{n\pi}\right]_{-1}^{0} - \left[\frac{\cos n\pi t}{n\pi}\right]_{0}^{1}$$

$$= \left[\frac{1}{n\pi} - \frac{\cos n\pi}{n\pi}\right] - \left[\frac{\cos n\pi}{n\pi} - \frac{1}{n\pi}\right]$$

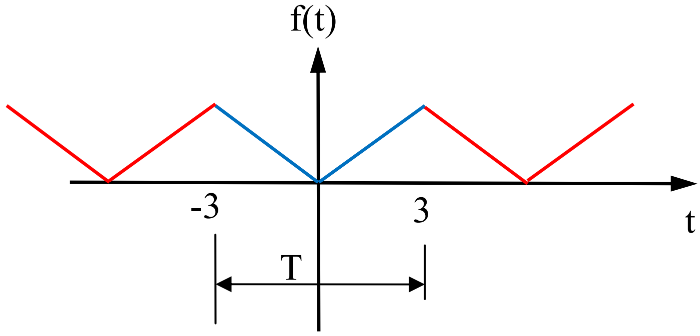
$$\therefore b_n = \frac{2}{n\pi}(1-\cos n\pi) = \begin{cases} 4/n\pi & n: odd \\ 0 & n: even \end{cases}$$

$$\therefore a_0 = 0, \quad and \quad a_n = 0$$

$$\therefore f(t) = \sum_{n=1}^{\infty}\frac{4}{n\pi}\sin n\pi t \qquad , \quad n: odd \ \ only$$

Ex. Find the Fourier expansion of the periodic function whose definition in one period is;

$$f(t) = \begin{cases} -t & -3 < t < 0 \\ t & 0 < t < 3 \end{cases}$$



$$\therefore T = 6, \ p = 3, \ \omega_n = \frac{n\pi}{p} = \frac{n\pi}{3}$$

$$a_0 = \frac{1}{p}\int_d^{d+2p} f(t)dt = \frac{1}{3}\left[\int_{-3}^{0} -tdt + \int_0^3 tdt\right] = \frac{1}{3}\left[\frac{-t^2}{2}\Big|_{-3}^{0} + \frac{t^2}{2}\Big|_0^3\right] = 3$$

$$a_n = \frac{1}{p}\int_d^{d+2p} f(t)\cos\omega_n t dt = \frac{1}{3}\int_{-3}^{0} -t\cos\frac{n\pi}{3}tdt + \frac{1}{3}\int_0^{-3} t\cos\frac{n\pi}{3}tdt$$

$$= \frac{-1}{3}\left[\frac{9}{n^2\pi^2}\cos\frac{n\pi}{3}t + \frac{3t}{n\pi}\sin\frac{n\pi}{3}t\right]_{-3}^{0} + \frac{1}{3}\left[\frac{9}{n^2\pi^2}\cos\frac{n\pi}{3}t + \frac{3t}{n\pi}\sin\frac{n\pi}{3}t\right]_0^3$$

$$= \frac{-3}{n^2\pi^2}(1-\cos n\pi) + \frac{3}{n^2\pi^2}(\cos n\pi - 1) = \frac{6}{n^2\pi^2}(\cos n\pi - 1) \qquad , n \neq 0$$

$$a_n = \frac{-12}{n^2\pi^2} \qquad , n: odd \ \ only \qquad n \neq 0$$

$$b_n = \frac{1}{p}\int_d^{d+2p} f(t)\sin\omega_n t dt = \frac{1}{3}\int_{-3}^{0} -t\sin\frac{n\pi}{3}tdt + \frac{1}{3}\int_0^{-3} t\sin\frac{n\pi}{3}tdt$$

$$= \frac{-1}{3}\left[\frac{9}{n^2\pi^2}\sin\frac{n\pi}{3}t - \frac{3t}{n\pi}\cos\frac{n\pi}{3}t\right]_{-3}^{0} + \frac{1}{3}\left[\frac{9}{n^2\pi^2}\sin\frac{n\pi}{3}t - \frac{3t}{n\pi}\cos\frac{n\pi}{3}t\right]_0^3$$

$$= \frac{3}{n\pi}\cos(-n\pi) - \frac{3}{n\pi}\cos(n\pi) = 0$$

Substitute in Fourier series

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos w_n t + b_n \sin w_n t$$

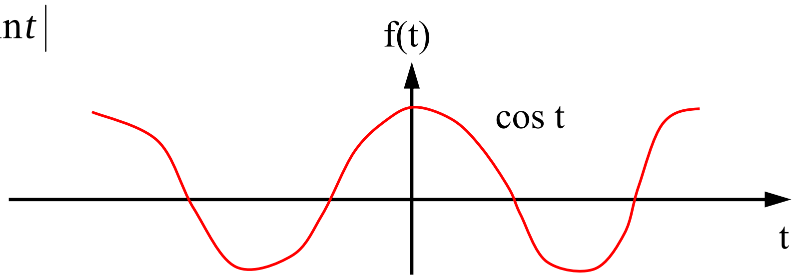
$$\therefore f(t) = \frac{3}{2} - \frac{12}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos\frac{n\pi}{3}t \qquad , n: odd \quad only$$

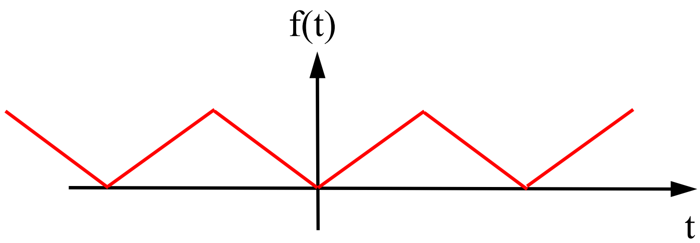
Or $$f(t) = \frac{3}{2} - \frac{12}{\pi^2}\left(\frac{1}{1}\cos\frac{\pi}{3}t + \frac{1}{9}\cos\frac{3\pi}{3}t + \frac{1}{25}\cos\frac{5\pi}{3}t + \cdots\cdots\right)$$

2.2 Even and Odd functions "Half Range Expansion"

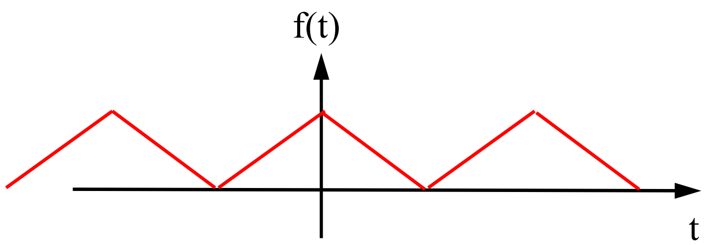
*** Even function**: a function which has $f(t) = f(-t)$

Ex. $\cos t,\ \sec t,\ \sin^2 t,\ f(t) = |\sin t|$





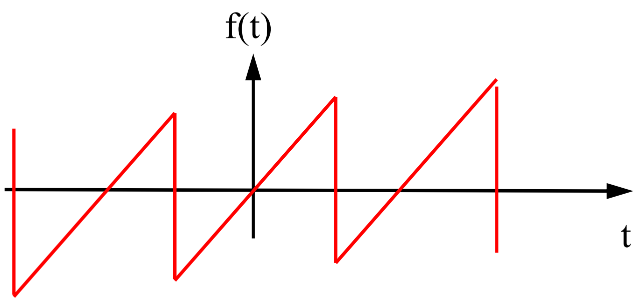
Ex. $f(t) = \begin{cases} -t & -3 < t < 0 \\ t & 0 < t < 3 \end{cases}$



Ex. $f(t) = \begin{cases} t+1 & -1 < t < 0 \\ 1-t & 0 < t < 1 \end{cases}$

*** Odd function**: a function which has $f(t) = -f(-t)$

Ex. $\sin t,\ \tan t,\ \cot t$



Ex. $f(t) = t \qquad -1 < t < 1$

Notes:

(1)

$$\begin{aligned} Even &\ {}^{\div}_{\times}\ Even = Even \\ Odd &\ {}^{\div}_{\times}\ Odd = Even \\ Even &\ {}^{\div}_{\times}\ Odd = Odd \end{aligned}$$

(2) Any function can be split into odd and even parts

$$f(t) = f_e(t) + f_o(t)$$

Where:

$$f_e(t) = \frac{f(t) + f(-t)}{2}$$

$$f_o(t) = \frac{f(t) - f(-t)}{2}$$

Example: Find even and odd part

$$f(t) = e^t$$

$$f_e(t) = \frac{f(t) + f(-t)}{2} = \frac{e^t + e^{-t}}{2} = \cosh t$$

$$f_o(t) = \frac{f(t) - f(-t)}{2} = \frac{e^t - e^{-t}}{2} = \sinh t \qquad \therefore e^t = \cosh t + \sinh t$$

(3) **For Even function**

$$b_n = o$$

$$a_0 = \frac{2}{p} \int_d^{d+p} f(t) dt$$

$$a_n = \frac{2}{p} \int_d^{d+p} f(t) \cos \omega_n t \, dt$$

(4) **For Odd function**

$$a_n = o$$

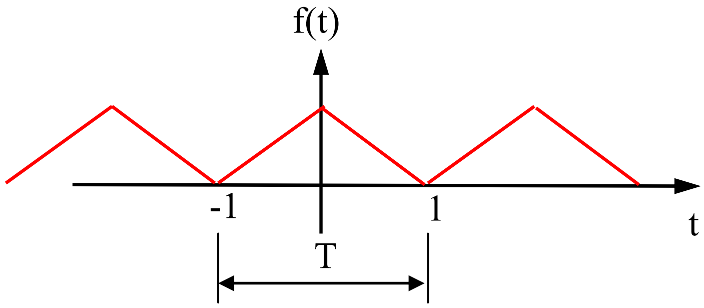
$$b_n = \frac{2}{p} \int_d^{d+p} f(t) \sin \omega_n t \, dt$$

H.W. Problems P. 188 and P. 195 "Wylie"

Ex. Find the Fourier series for the following function defined in one period as;

$$f(t) = |t| \qquad -1 < t < 1$$

$$T = 2, \; p = 1, \; \omega_n = \frac{n\pi}{p} = n\pi$$



$\therefore$ **Even function**

$$\therefore b_n = 0$$

$$a_n = \frac{2}{p}\int_{d}^{d+p} f(t)\cos\omega_n t\, dt = \frac{2}{1}\int_0^1 t\cos n\pi t\, dt$$

$$= 2\left[t\frac{\sin n\pi t}{n\pi} + \frac{\cos n\pi t}{n^2\pi^2}\right]_0^1 = 2\left[\frac{\cos n\pi}{n^2\pi^2} - \frac{1}{n^2\pi^2}\right]$$

$$\therefore a_n = \begin{cases} -4/n^2\pi^2 & n: odd\ only \quad n \neq 0 \\ 0 & n: even \end{cases}$$

$$a_0 = \frac{2}{p}\int_{d}^{d+p} f(t)dt = \frac{2}{1}\int_0^1 t\, dt = 1$$

$$\therefore f(t) = \frac{1}{2} + \sum_{n=1}^{\infty}\frac{4}{n^2\pi^2}\cos n\pi t \qquad , \ n: odd \quad only$$

$$= \frac{1}{2} - \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos n\pi t$$

H.W. Find the Fourier series for the following functions defined in one period as;

$$f(t) = \sin t \qquad 0 < t < \pi$$

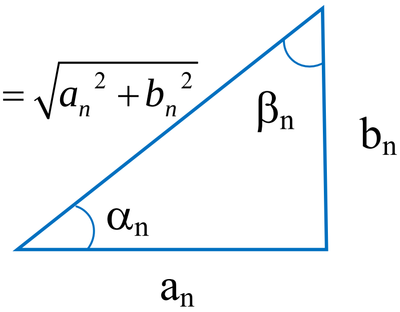
and $\quad f(t) = \sin^2 t \qquad 0 < t < \pi$

H.W. Problems P.188 and P. 195 "Wylie"

Alternative forms of Fourier series

$$f(t)=\frac{a_0}{2}+\sum_{n=1}^{\infty}A_n\left(\frac{a_n}{A_n}\cos\omega_n t+\frac{b_n}{A_n}\sin\omega_n t\right)$$

$$A_n=\sqrt{a_n^{\,2}+b_n^{\,2}}$$



$$=\frac{a_0}{2}+\sum_{n=1}^{\infty}A_n\left(\cos\alpha_n\cos\omega_n t+\sin\alpha_n\sin\omega_n t\right)$$

$$f(t)=A_0+\sum_{n=1}^{\infty}A_n\cos\left(\omega_n t-\alpha_n\right)$$ **"cosine series"**

Where $A_n=\sqrt{a_n^{\,2}+b_n^{\,2}}$, $\alpha_n=\tan^{-1}\frac{b_n}{a_n}$, $A_0=\frac{a_0}{2}$

A_n : Amplitude α_n : Phase angle for cosine series

Or. equally

$$f(t)=\frac{a_0}{2}+\sum_{n=1}^{\infty}A_n\left(\sin\beta_n\cos\omega_n t+\cos\beta_n\sin\omega_n t\right)$$

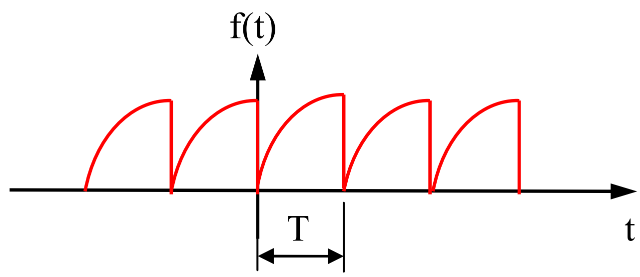
$$f(t)=A_0+\sum_{n=1}^{\infty}A_n\sin\left(\omega_n t+\beta_n\right)$$ **"sine series"**

Where : $A_n=\sqrt{a_n^{\,2}+b_n^{\,2}}$, $\beta_n=\tan^{-1}\frac{a_n}{b_n}\ \rightarrow\ Or\ \ \beta_n=\frac{\pi}{2}-\alpha_n$

β_n : Phase angle for sine series

Ex. Find the sine and cosine Fourier series for the following function:

$$f(t)=\sin t \qquad 0<t<\pi/2$$



$$a_0=\frac{1}{p}\int_{d}^{d+2p}f(t)dt=\frac{4}{\pi}\int_0^{\pi/2}\sin t\,dt$$

$$=\frac{-4}{\pi}\cos t\Big|_0^{\pi/2}=\frac{4}{\pi}$$

$$T=\frac{\pi}{2},\ p=\frac{\pi}{4},\ \omega_n=\frac{n\pi}{p}=4n$$

$$a_n = \frac{1}{p}\int_d^{d+2p} f(t)\cos\omega_n t dt = \frac{4}{\pi}\int_0^{\pi/2} \sin t \cos 4nt \ dt$$

$$= \frac{2}{\pi}\int_0^{\pi/2}\left[\sin(1+4n)t + \sin(1-4n)t\right]dt = \frac{2}{\pi}\left[\frac{-\cos(1+4n)t}{1+4n} - \frac{\cos(1-4n)t}{1-4n}\right]_0^{\pi/2}$$

$$= \frac{2}{\pi}\left\{\left[\frac{-\cancel{\cos(\pi/2)}^{=0}\cos 2n\pi}{1+4n} - 0\right] - \left[\frac{-1}{1+4n} - \frac{1}{1+4n}\right]\right\} = \frac{4}{\pi\left(1-16n^2\right)}$$

$$b_n = \frac{1}{p}\int_d^{d+2p} f(t)\sin 4nt \, dt = \frac{4}{\pi}\int_0^{\pi/2} \sin t \sin 4nt \ dt$$

$$= \frac{2}{\pi}\int_0^{\pi/2}\left[\cos(1-4n)t - \cos(1+4n)t\right]dt = \frac{2}{\pi}\left[\frac{\sin(1-4n)t}{1-4n} - \frac{\sin(1+4n)t}{1+4n}\right]_0^{\pi/2}$$

$$= \frac{2}{\pi}\left\{\left[\frac{\cos 2n\pi}{1-4n} - \frac{\cos 2n\pi}{1+4n}\right] - [0-0]\right\} = \frac{2}{\pi}\left[\frac{1}{1-4n} - \frac{1}{1+4n}\right]$$

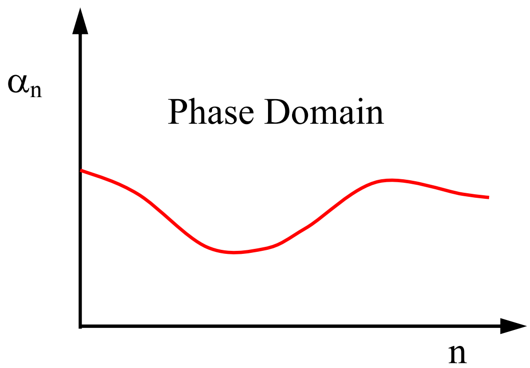
$$\therefore b_n = \frac{16n}{\pi\left(1-16n^2\right)}$$

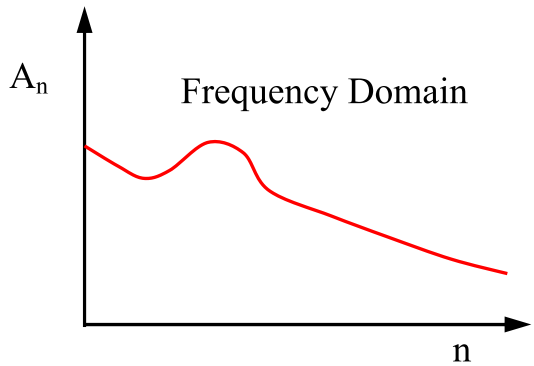
$$A_n = \sqrt{a_n^{\,2} + b_n^{\,2}} = \sqrt{\left(\frac{4}{\pi\left(1-16n^2\right)}\right)^2 + \left(\frac{16n}{\pi\left(1-16n^2\right)}\right)^2}$$

$$\therefore A_n = \frac{4}{\pi\left(1-16n^2\right)}\sqrt{1+16n^2}$$

$$\alpha_n = \tan^{-1}\frac{b_n}{a_n} = \tan^{-1}\frac{\dfrac{16n}{\pi\left(1-16n^2\right)}}{\dfrac{4}{\pi\left(1-16n^2\right)}} = \tan^{-1}4n$$

$$\therefore \beta_n = \tan^{-1}\frac{1}{4n}$$





Complex Fourier Series

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\omega_n t + b_n \sin\omega_n t$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \left(\frac{e^{i\omega_n t} + e^{-i\omega_n t}}{2} \right) + b_n \left(\frac{e^{i\omega_n t} - e^{-i\omega_n t}}{2i} \right)$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(\frac{a_n - ib_n}{2} \right) e^{i\omega_n t} + \left(\frac{a_n - ib_n}{2} \right) e^{-i\omega_n t}$$

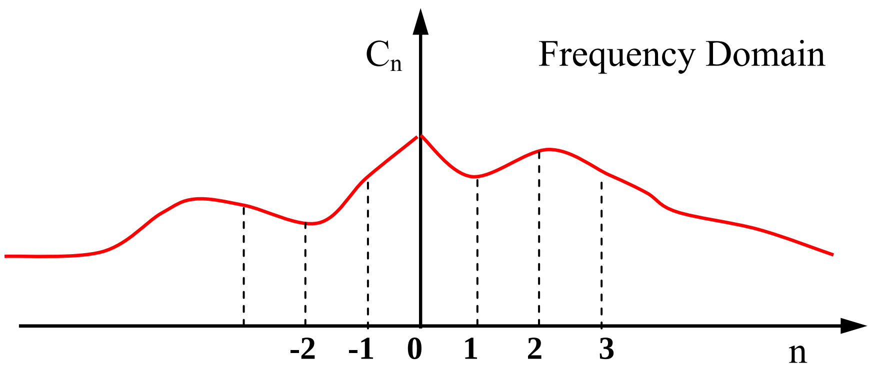
$$\therefore f(t) = C_0 + \sum_{n=1}^{\infty} C_n e^{i\omega_n t} + C_{-n} e^{-i\omega_n t}$$

where: $C_0 = \frac{a_0}{2}$, $C_n = \frac{a_n - ib_n}{2}$, $C_{-n} = \frac{a_n + ib_n}{2}$, $C_n = \overline{C_{-n}}$

We can rewritten as:

$$f(t) = \sum_{n=\infty}^{n=-\infty} C_n e^{-i\omega_n t}$$

where: $$C_n = \frac{1}{2p} \int_{d}^{d+2p} f(t) e^{-i\omega_n t} dt$$



Proof:

- when n = 0

$$C_0 = \frac{a_0}{2} = \frac{1}{2p} \int_{d}^{d+2p} f(t) dt$$

- when n is positive

$$C_n = \frac{a_n - ib_n}{2} = \frac{1}{2}\left[\frac{1}{p}\int_d^{d+2p} f(t)\cos\omega_n t\,dt - i\,\frac{1}{p}\int_d^{d+2p} f(t)\sin\omega_n t\,dt\right]$$

$$= \frac{1}{2p}\int_d^{d+2p} f(t)\left(\cos\omega_n t - i\sin\omega_n t\right)dt$$

$$\therefore C_n = \frac{1}{2p}\int_d^{d+2p} f(t)e^{-i\omega_n t}\,dt$$

- when n is negative

$$C_{-n} = \frac{a_n + ib_n}{2} = \frac{1}{2}\left[\frac{1}{p}\int_d^{d+2p} f(t)\cos\omega_n t\,dt + i\,\frac{1}{p}\int_d^{d+2p} f(t)\sin\omega_n t\,dt\right]$$

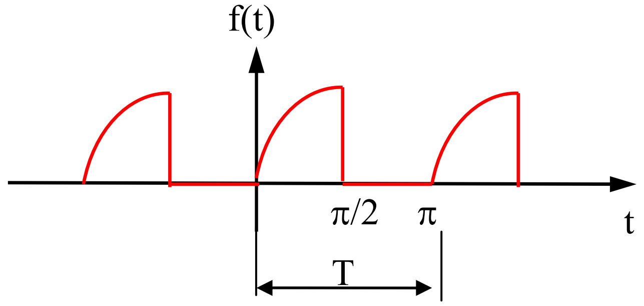
$$= \frac{1}{2p}\int_d^{d+2p} f(t)\left(\cos\omega_n t + i\sin\omega_n t\right)dt$$

$$\therefore C_{-n} = \frac{1}{2p}\int_d^{d+2p} f(t)e^{i\omega_n t}\,dt$$

Note: $A_n = 2|C_n|$, $\alpha_n = -Arg\left(C_n\right)$

Ex. Find the complex Fourier series for the following function defined in one period:

$$f(t) = \begin{cases} \sin t & 0 < t < \pi/2 \\ 0 & \pi/2 < t < \pi \end{cases}$$



$$T = \pi,\; p = \frac{\pi}{2},\; \omega_n = \frac{n\pi}{p} = 2n$$

$$C_n = \frac{1}{2p}\int_d^{d+T} f(t)e^{-i\omega_n t}dt$$

$$= \frac{1}{\pi}\int_0^{\pi/2} \sin t\, e^{-i\omega_n t}dt + \frac{1}{\pi}\int_{\pi/2}^{\pi} 0 * e^{-i\omega_n t}dt$$

$$= \frac{1}{\pi}\int_0^{\pi/2}\left(\frac{e^{it} - e^{-it}}{2i}\right)e^{-i\omega_n t}dt = \frac{1}{2\pi i}\int_0^{\pi/2}\left(e^{i(1-2n)t} - e^{-i(1-2n)t}\right)dt$$

$$C_n = \frac{1}{2\pi i}\left[\frac{e^{i(1-2n)t}}{i(1-2n)} + \frac{e^{-i(1+2n)t}}{i(1+2n)}\right]_0^{\pi/2}$$

$$= \frac{-1}{2\pi}\left\{\left[\frac{e^{i(1-2n)\pi/2}}{(1-2n)} + \frac{e^{-i(1+2n)\pi/2}}{(1+2n)}\right] - \left[\frac{1}{(1-2n)} + \frac{1}{(1+2n)}\right]\right\}$$

$$= \frac{-1}{2\pi}\left[\frac{-i\cos n\pi}{(1-2n)} + \frac{-i\cos n\pi}{(1+2n)} - \frac{2}{(1-4n^2)}\right]$$

$$= \frac{-1}{2\pi}\left[\frac{i\,4n\cos n\pi}{1-4n^2} - \frac{2}{1-4n^2}\right]$$

$$\therefore C_n = \frac{-1}{2\pi}\left[\frac{i\,4n(-1)^n - 2}{1-4n^2}\right]$$

Note:

$$e^{i(1-2n)\pi/2} = e^{i\left(\frac{\pi}{2}-n\pi\right)} = e^{i\frac{\pi}{2}}.e^{-in\pi}$$

$$= \Big(\underbrace{\cos(\pi/2)}_{=0} + i\underbrace{\sin(\pi/2)}_{=1}\Big)\Big(\cos n\pi - i\underbrace{\sin n\pi}_{=0}\Big)$$

$$= i\cos n\pi$$

$$\therefore f(t) = \sum_{n=\infty}^{n=-\infty} C_n e^{-i\omega_n t} = \sum_{n=\infty}^{n=-\infty} \frac{-1}{2\pi}\left[\frac{i\,4n(-1)^n - 2}{1-4n^2}\right] e^{-i\omega_n t}$$

For Check at n = 0

$$C_0 = \frac{1}{\pi} \qquad , C_0 = \frac{a_0}{2} = \frac{1}{2p}\int_d^{d+T} f(t)dt$$

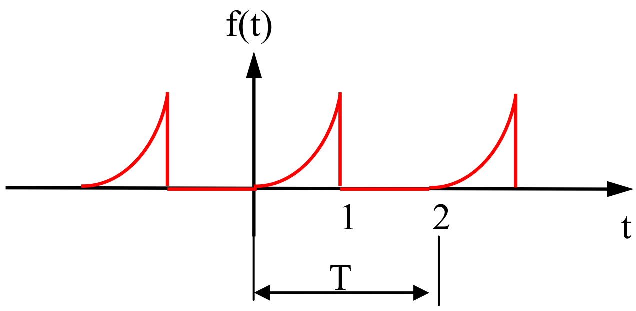
$$C_0 = \frac{1}{\pi}\int_0^{\pi/2} \sin t\, dt = \frac{-1}{\pi}[\cos t]_0^{\pi/2} = \frac{1}{\pi}$$

هناك حالات تحدث في المقام $(1\text{-}n^2)$ فلا يمكن حساب C_{-1} , C_1 نذهب الى التعريف الاساسي وهكذا

$$C_1 = \frac{1}{2\pi}\int f(t)e^{-i\omega_1 . t}dt$$

Ex. Find the complex Fourier series for the following function defined in one period:

$$f(t) = \begin{cases} t^2 & 0 < t < 1 \\ 0 & 1 < t < 2 \end{cases}$$



$$T = 2,\ p = 1,\ \omega_n = \frac{n\pi}{p} = n\pi$$

$$C_n = \frac{1}{2p}\int_{d}^{d+T} f(t)e^{-i\omega_n t}dt = \frac{1}{2}\int_0^1 t^2 e^{-in\pi t}dt + 0$$

$$= \frac{1}{2}\left[t^2 \frac{e^{-in\pi t}}{-in\pi} - 2t\frac{e^{-in\pi t}}{(-in\pi)^2} + 2\frac{e^{-in\pi t}}{(-in\pi)^3}\right]_0^1 = \frac{1}{2}\left[t^2 \frac{e^{-in\pi t}}{-in\pi} + 2t\frac{e^{-in\pi t}}{n^2\pi^2} + 2\frac{e^{-in\pi t}}{in^3\pi^3}\right]_0^1$$

$$= \frac{1}{2n\pi}\left\{\left[ie^{-in\pi} + 2\frac{e^{-in\pi}}{n\pi} - 2i\frac{e^{-in\pi}}{n^2\pi^2}\right] - \left(0 + 0 - \frac{i2}{n^2\pi^2}\right)\right\}$$

$$= \frac{1}{2n\pi}\left\{\left[i\cos n\pi + \frac{2\cos n\pi}{n\pi} - \frac{i2\cos n\pi}{n^2\pi^2}\right] + \frac{i2}{n^2\pi^2}\right\}$$

$$\therefore C_n = \frac{1}{2n\pi}\left\{\cos n\pi\left(i + \frac{2}{n\pi} - \frac{2i}{n^2\pi^2}\right) + \frac{i2}{n^2\pi^2}\right\}$$

Note:
$e^{-in\pi} = \left(\cos n\pi - i\,\overset{=0}{\sin n\pi}\right) = \cos n\pi$

$$\therefore f(t) = \sum_{n=\infty}^{n=-\infty} C_n e^{-i\omega_n t} = \sum_{n=\infty}^{n=-\infty} \frac{1}{2n\pi}\left\{\cos\left(i + \frac{2}{n\pi} - \frac{2i}{n^2\pi^2}\right) + \frac{i2}{n^2\pi^2}\right\}e^{-i\omega_n t}$$

For Check at n = 0

$$C_0 = \frac{1}{2p}\int_{d}^{d+T} f(t)dt\ \frac{1}{2}\int_0^1 t^2 dt = \frac{1}{6}$$

Fourier Integral (Transform)

تستخدم للدوال الغير دورية

Fourier series;

$$f(t)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n \cos w_n t+b_n \sin w_n t$$

Complex Fourier series;

$$f(t)=\sum_{n=\infty}^{n=-\infty}C_n e^{-i\omega_n t} \qquad \text{where:} \quad C_n=\frac{1}{2p}\int_{d}^{d+2p} f(t)e^{-i\omega_n t}dt$$

$$f(t)=\sum_{n=\infty}^{n=-\infty}\left[\frac{1}{2p}\int_{d}^{d+2p} f(t)e^{-i\omega_n t}dt\right]e^{-i\omega_n t} \qquad \textbf{... (1)}$$

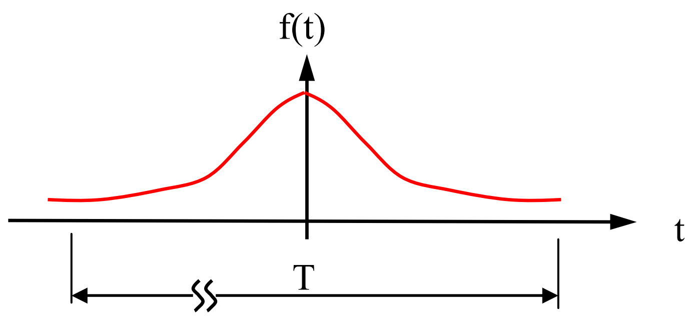
$$At \quad T=-\infty\rightarrow\infty$$

$$p\rightarrow\infty$$

$$d+2p\rightarrow\infty$$

$$\Delta\omega_n=\omega_{n+1}+\omega_n$$

$$=\frac{\pi(n+1)}{p}+\frac{\pi n}{p}=\frac{\pi}{p}$$



$$T=\infty,\ p=\infty,\ \omega_n=\frac{n\pi}{p}$$

* If times & divided eq.(1) in $\Delta\omega_n$

$$f(t)=\sum_{-\infty}^{\infty}\left[\frac{1}{2p}\int_{d}^{d+2p} f(t)e^{-i\omega_n t}dt\right]e^{-i\omega_n t} \cdot\frac{\Delta\omega_n}{\Delta\omega_n}$$

$$=\sum_{-\infty}^{\infty}\frac{\not p}{2\not p\pi}\int_{d}^{d+2p} f(t)e^{-i\omega_n t}dt\,.e^{-i\omega_n t}\,.\Delta\omega_n$$

$$as:\ \begin{matrix}p\rightarrow\infty\\ \Delta\omega\rightarrow 0\end{matrix} \quad then \quad \Delta\omega\rightarrow dw \ \ \&\sum\ \rightarrow\int \quad \&\ \omega_n=\omega$$

$$f(t)=\int_{-\infty}^{\infty}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)e^{-i\omega_n t}dt\right].e^{-i\omega_n t}\,.d\omega$$

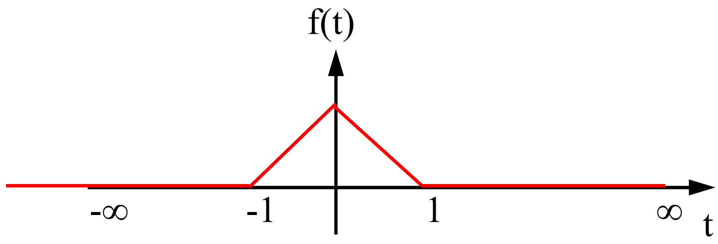
$$f(t)=\int_{-\infty}^{\infty}G(\omega)e^{-i\omega t}\,d\omega \qquad , \qquad G(\omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt$$

"Fourier Integral Pair"

بالنسبة الى (1/2π) يمكن ان تقسم الى جزئين او تكتب مع G(ω) او f(t).

Ex. Find Fourier integral for the following function:

$$f(t)=\begin{cases}1+t & -1<t<0\\ 0 & otherwise\\ 1-t & 0<t<1\end{cases}$$



$$G(\omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt$$

$$=\frac{1}{2\pi}\left[\int_{-\infty}^{-1}0.dt+\int_{-1}^{0}(1+t)e^{-i\omega t}dt+\int_{0}^{1}(1-t)e^{-i\omega t}dt+\int_{1}^{\infty}0.dt+\right]$$

$$=\frac{1}{2\pi}\left[\left(\frac{e^{-i\omega t}}{-i\omega}+\frac{te^{-i\omega t}}{-i\omega}-\frac{e^{-i\omega t}}{i^2\omega^2}\right)_{-1}^{0}+\left(\frac{e^{-i\omega t}}{-i\omega}-\frac{te^{-i\omega t}}{-i\omega}-\frac{e^{-i\omega t}}{-i^2\omega^2}\right)_{0}^{1}\right]$$

$$=\frac{1}{2\pi}\left[\left(\frac{1}{-i\omega}+\frac{0}{-i\omega}+\frac{1}{\omega^2}\right)-\left(\frac{e^{i\omega}}{-i\omega}+\frac{e^{i\omega}}{i\omega}+\frac{e^{i\omega}}{\omega^2}\right)+\left(\frac{e^{-i\omega}}{-i\omega}+\frac{e^{-i\omega}}{i\omega}-\frac{e^{i\omega}}{\omega^2}\right)-\left(\frac{1}{-i\omega}+\frac{0}{i\omega}-\frac{1}{\omega^2}\right)\right]$$

$$=\frac{1}{2\pi}\left[\frac{2}{\omega^2}+\frac{e^{i\omega}}{i\omega}-\frac{e^{i\omega}}{i\omega}-\frac{e^{i\omega}}{\omega^2}-\frac{e^{-i\omega}}{i\omega}+\frac{e^{-i\omega}}{\omega}-\frac{e^{-i\omega}}{\omega^2}\right]$$

$$G(\omega)=\frac{1}{2\pi}\left[\frac{2}{\omega}-\frac{2\cos\omega}{\omega^2}\right]=\frac{1}{\pi\omega^2}(1-\cos\omega)$$

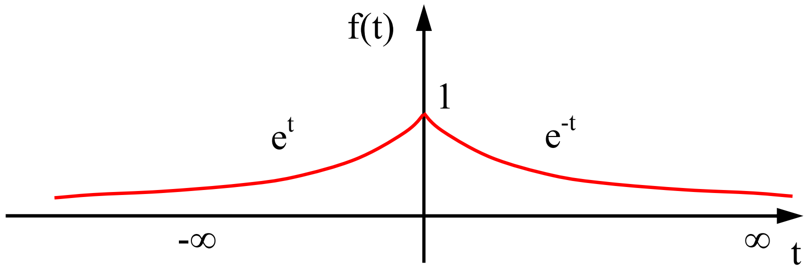
Note:
$e^{i\omega}+e^{-i\omega}$
$(\cos\omega+i\sin\omega)+(\cos\omega-i\sin\omega)=2\cos\omega$

$$f(t)=\int_{-\infty}^{\infty}G(\omega).e^{i\omega t}d\omega=\int_{-\infty}^{\infty}\frac{1}{\pi\omega^2}(1-\cos\omega).e^{i\omega t}d\omega$$

Ex. Find Fourier integral for the following function: $f(t) = e^{-|t|}$

Sol.

$$\therefore f(t) = \begin{cases} e^{t} & t < 0 \\ e^{-t} & t > 0 \end{cases}$$



$$f(t) = \int_{-\infty}^{\infty} G(\omega).e^{i\omega t} d\omega$$

$$G(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t).e^{i\omega t} dt$$

$$= \frac{1}{2\pi} \left[\int_{-\infty}^{0} e^{t}.e^{-i\omega t} dt + \int_{0}^{1} e^{-t}.e^{-i\omega t} dt \right]$$

$$= \frac{1}{2\pi} \left[\frac{e^{(1-i\omega)t}}{1-i\omega} \Big|_{-\infty}^{0} + \frac{e^{-(1+i\omega)t}}{-(1+i\omega)} \Big|_{0}^{\infty} \right]$$

Note:
$e^{-\infty} = 0$

$$= \frac{1}{2\pi} \left[\frac{1}{1-i\omega}(1-0) - \frac{1}{1+i\omega}(0-1) \right] = \frac{1}{2\pi} \left[\frac{1}{1-i\omega} + \frac{1}{1+i\omega} \right]$$

$$G(\omega) = \frac{1}{2\pi} \left[\frac{1+i\omega+1-i\omega}{1+\omega^2} \right] = \frac{1}{\pi(1+\omega^2)}$$

$$\therefore f(t) = \int_{-\infty}^{\infty} G(\omega) e^{i\omega t} d\omega = \int_{-\infty}^{\infty} \frac{1}{\pi(1+\omega^2)}.e^{i\omega t} d\omega$$

H.W. Problems P.200 & P. 220 "Wylie"

3. The Laplace Transform

$$F(s)=\mathcal{L}\{f(t)\}=\int_0^\infty e^{-st}f(t)\,dt$$ **"Laplace transform"**

$$f(t)=\mathcal{L}^{-1}F(s)=\frac{1}{i\,2\pi}\int_{\sigma-i\infty}^{\sigma+i\infty}F(s)e^{st}ds$$ **"Complex Inversion formula"**

Ex. Find Laplace transform for the function f(t) = 1

$$F(s)=\mathcal{L}\{f(t)\}=\mathcal{L}\{1\}=\int_0^\infty e^{-st}.1dt=\left.\frac{e^{-st}}{-s}\right|_0^\infty=\frac{-1}{s}[0-1]=\frac{1}{s}$$

Table (1) Elementary Laplace transforms

f(t)	F(s)	f(t)	F(s)
a	$\frac{a}{s}$	e^{at}	$\frac{1}{s-a}$
t^n	$\begin{cases}\frac{\Gamma(n+1)}{s^{n+1}} & n>-1 \\ \frac{n!}{s^{n+1}} & n\text{ a positive integer}\end{cases}$		
$\sin at$	$\frac{a}{s^2+a^2}$	$\cos at$	$\frac{s}{s^2+a^2}$
$\sinh at$	$\frac{a}{s^2-a^2}$	$\cosh at$	$\frac{s}{s^2-a^2}$
$f_1(t)\mp f_2(t)$	$F_1(s)\mp F_2(s)$	$\int_a^t f(t)dt$	$\frac{1}{s}F(s)+\frac{1}{s}\int_a^0 f(t)dt$
$e^{at}f(t)$	$F(s-a)$	$t^n f(t)$	$(-1)^n\frac{d^nF(s)}{ds^n}$

- **Laplace Transform of Derivative**

$$\mathcal{L}\{f'(t)\}=\int_0^{\infty} e^{-st}.f'(t)dt=e^{-st}f(t)\Big|_0^{\infty}-\int_0^{\infty} f(t)(-s)e^{-st}dt$$

$$0-f(0)+s\int_0^{\infty} f(t)e^{-st}dt=s\mathcal{L}\{f(t)\}-f(0)$$

$$\therefore \mathcal{L}\{f'(t)\}=sF(s)-f(0)$$

Generally:

$$\mathcal{L}\{f^{(n)}(t)\}=s^nF(s)-s^{n-1}f(0)-s^{n-2}f'(0)-s^{n-3}f''(0)-\cdots\cdots f^{(n-1)}(0)$$

- **Laplace Transform for integral**

$$\mathcal{L}\left\{\int_a^t f(t)dt\right\}=\int_0^{\infty}\left(\int_a^t f(t)dt\right)e^{-st}dt=\left[\int_a^t f(t)dt\left(\frac{e^{-st}}{-s}\right)\right]_0^{\infty}-\int_0^{\infty} f(t)\left(\frac{e^{-st}}{-s}\right)dt$$

$$=\cancel{\int_a^{\infty} f(t)dt\left(\frac{e^{-\infty}}{-s}\right)}^{=0}-\int_a^0 f(t)dt\left(\frac{e^0}{-s}\right)+\frac{1}{s}\int_0^{\infty} f(t)e^{-st}dt$$

$$=\frac{1}{s}\int_a^0 f(t)dt+\frac{1}{s}\int_0^{\infty} f(t)e^{-st}dt$$

$$\therefore \mathcal{L}\left\{\int_a^t f(t)dt\right\}=\frac{1}{s}F(s)-\frac{1}{s}\int_0^a f(t)dt$$

This can be extended to double, triple and higher integration.

$$\mathcal{L}\left\{\int_0^t f(t)dt\right\}=\frac{1}{s}F(s)$$

Ex. Find the Laplace transform for the functions

$$1-f(t)=\sin^2 t$$
$$2-f(t)=t^2e^{2t}$$

1. $f(t)=\sin^2 t \quad ,f'(t)=2\sin t\cos t=\sin 2t$

$$\therefore f'(t) = \sin 2t, \qquad \mathcal{L}\{f'(t)\} = \frac{2}{s^2+4}$$

$$\mathcal{L}\{f'(t)\} = sF(s) - f(0) \qquad \because f(0) = \sin 0 = 0$$

$$\therefore sF(s) = \frac{2}{s^2+4} \quad \rightarrow \quad \therefore F(s) = \mathcal{L}\{f(t)\} = \frac{2}{s(s^2+4)}$$

$$Or \qquad \sin^2 t = \frac{1}{2}(1-\cos 2t) = f(t)$$

$$\mathcal{L}\{f(t)\} = \frac{1}{2s} - \frac{1}{2}\frac{s}{s^2+4} = \frac{2}{s(s^2+4)}$$

2. $f(t) = t^2 e^{2t}$

$$\because \mathcal{L}(t^2) = \frac{2!}{s^3} \quad \rightarrow \quad \therefore \mathcal{L}\{f(t)\} = \frac{2!}{(s-2)^3}$$

$$Or \ \because \mathcal{L}(e^{2t}) = \frac{1}{s-2} \quad \rightarrow \quad \mathcal{L}\{t^2 e^{2t}\} = (-1)^2 \frac{d^2}{ds^2}\left(\frac{1}{s-2}\right) = \frac{2}{(s-2)^3}$$

Ex. Find the Laplace transform for $\int_0^t \cos 4t \, dt$

$$\because \mathcal{L}\{\cos 4t\} = \frac{s}{s^2+16}$$

$$\therefore \mathcal{L}\left\{\int_0^t \cos 4t \, dt\right\} = \frac{1}{s} \cdot \frac{s}{s^2+16} = \frac{1}{s^2+16}$$

$$Or \ \int_0^t \cos 4t \, dt = \frac{\sin 4t}{4} \quad \rightarrow \quad \mathcal{L}\left\{\frac{\sin 4t}{4}\right\} = \frac{1}{4} \cdot \frac{4}{s^2+16} = \frac{1}{s^2+16}$$

Ex. Find the inverse Laplace transform for $F(s) = \frac{1}{s(s^2-4)}$

$$\mathcal{L}^{-1}\left\{\frac{1}{s(s^2-4)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s} \cdot \frac{1}{2} \cdot \frac{2}{s^2-4}\right\}$$

$$= \int_0^t \frac{1}{2}\sinh 2t \, dt = \frac{1}{2}\left[\frac{\cosh 2t}{2}\right]_0^t = \frac{1}{4}(\cosh 2t - 1)$$

Note: if case of ($1/s^2$) using double integral; become $\int_0^t \left(\int_0^t \frac{1}{2} \sinh 2t \, dt \right)$

H.W Problems P. 241 " Wylie"

H.W (1) Find Laplace transform for the following function;

$1-f(t)=t\sin\omega t$

$2-f(t)=t\cosh at$

$3-f(t)=t^2\cos 2t$

$4-f(t)=e^{2t}\sin 4t$

$5-f(t)=\sin 2t-t^2e^{-2t}$

$6-\int_2^t \sin t\,dt$

(2) Find Inverse Laplace transform for;

$1-F(s)=\dfrac{1}{s(s+1)}$

$2-F(s)=\dfrac{3}{s^2+4}$

$3-F(s)=\dfrac{3}{s(s^2+16)}$

$4-F(s)=\dfrac{2}{s^2(s+4)}$

$5-F(s)=\dfrac{1}{s}\cdot\left(\dfrac{s-1}{s+1}\right)$