University of Basrah - College of Engineering

Department of Mechanical Engineering

M. Sc. In Mechanical Engineering

UBAS/MEPC/ETH5

Lect.: 2 HPW, 1 Term.

Semester: **Second/Third** Subject: **Fluid Dynamics**

Introduction to tensors and generalized coordinates, the conservation equations in fluid mechanics, the N. S. equations in vector and tensor forms, adapting the equations to different coordinate systems.

Exact solutions of N. S. equations in the limiting cases of Re $\rightarrow$ 0 and Re $\rightarrow \infty$, methods of solution of more general cases, the parabolic, elliptic and hyperbolic flows, incompressible and compressible flows, steady and unsteady flows, the concept of stream function, the vorticity equations.

Turbulence, methods of incorporating turbulence and correlation of fluctuating quantities, turbulence models.

Recommended Books:

1. **An Introduction to Fluid Dynamics**, Batchelor, G.K., Cambridge University Press, Cambridge.
2. **Boundary-Layer Theory,** Schlichting, Herrmann, Gersten, Klaus.
3. Fluid Mechanics, Landau and Lifshits, Pergamon.
4. Physical Fluid Dynamics, Trittong, Van Nostrand.
5. Measurement of Turbulence, Bradshaw, Pergamon.
6. **Vectors, Tensors and the Basic Equations of Fluid Mechanics** R. Aris, Dover publications

Introduction to Vectors and Tensors

The physical quantities encountered in fluid mechanics can be classified into three classes:

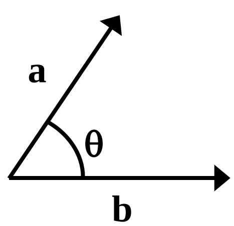
(a) ***Scalar*** - a quantity having magnitude but no direction, such as pressure, density, viscosity, temperature, length, mass, volume and time;

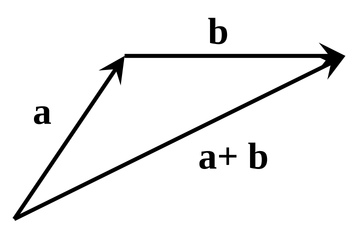
(b) ***Vector*** - (1st rank tensor) a quantity having magnitude and deriction, such as velocity, acceleration, displacement, linear momentum and force, and

(c) ***Tensor***- (2nd rank tensor) a quantity having magnitude and **two** directions (e.g. momentum flux, stress, rate of strain and vorticity).

Vector Addition

Given two arbitrary vectors **a** and **b**, by **a+b** we mean the vector formed by connecting the tail of **a** to the head of **b** when **b** is moved such that its tail coincides with head of **a** . A brief review of vector addition and multiplication can be found in Calculus, Thomas.





1. Vector Multiplication

Given two arbitrary vectors **a** and **b**, there are three types of vector products are defined:

	Notation	**Result**	**Definition**
Dot product	**a .b**	**Scalar**	$ab\cos\theta$
Cross product	**a × b**	**Vector**	$ab\lvert\sin\theta\rvert\mathbf{n}$
Dyadic product	**ab**	**Tensor**	-

where **n** is a unit vector which is normal to both **a** and **b**.

The sense of **n** is determined from the "right-hand rule". In the above definition, we denote the magnitude (or length) of vector **a** by the scalar *a*. Using the following notation:

scalar = lightface *Italic* such us *a*

vector = boldface Roman such as **a**

tensor = boldface Greek such as $\boldsymbol{\tau}$

<u>*Definition of Dyadic product*</u>

The word "dyad" comes from Greek: "dy" means two while "ad" means adjacent. Thus the name dyad refers to the way in which this product is denoted: the two vectors are written adjacent to one another with no space or other operator in between.

There is no geometrical picture that I can draw which will explain what a dyadic product. It's best to think of the dyadic product as a purely mathematical abstraction having some very useful properties:

Dyadic Product ab - that mathematical entity which satisfies they following properties (where **a, b, v,** and **w** are any four vectors)

1. **ab.v=a(b.v)** [which has the direction of **a**; note that **ba.v=b(a.v)** which has the dircetion of **b**]. Thus **ab** ≠ **ba.**
2. **v.ab=(v.a)b** [Thus **v.ab ≠ ab.v**]
3. **ab × v=a(b×v)** which is another dyad
4. **v×ab=(v×a)b**
5. **ab:vw=(a.w)(b.v)** which is sometimes known as the inner-outer product or the double-out product.
6. **a(v+w)=av+aw** (distribution for addition)
7. **(v+w)a=va+wa**
8. (s+t)**ab=**s**ab+**t**ab** (distribution for scalar multiplication)
9. s**ab = (**s**a)b=a(**s**b)**

2. Decomposition into Scalar Components (System of Coordinates)

A coordinate system in the three-dimensional space is defined by choosing a set of three ***linearly independent*** vectors, $B=\{\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}\}$, if none can be expressed as a linear combination of the other two (e.g. **i, j, and k)**. The set B is a ***basis*** of the three-dimensional space, i.e., each vector **v** of this space is uniquely written as a *linear combination* of this ***basis***:

$$\mathbf{v} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + v_3\mathbf{e}_3$$

where the v_i are called the ***scalar components*** of **v**. In most cases, the vectors $\mathbf{e_1}$, $\mathbf{e_2}$ **and** $\mathbf{e_3}$ are ***unit*** vectors. In the three coordinate systems, i.e., Cartesian, cylindrical and spherical coordinates, the three vectors are, in addition, orthogonal. Hence, in all these systems, the basis $B=\{\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}\}$ is orthonormal:

$$\mathbf{e_i}.\mathbf{e_j} = \delta_{\mathbf{ij}} \begin{cases} 1 & if \quad \mathbf{i}=\mathbf{j} \\ 0 & if \quad \mathbf{i}\neq\mathbf{j} \end{cases}$$

where $\delta_{\mathbf{ij}}$ is called the ***Kronecker delta.***

For the cross products of $\mathbf{e_1}$, $\mathbf{e_2}$ **and** $\mathbf{e_3}$, one gets:

$$\mathbf{e_i}\times\mathbf{e_j} = \sum_{\mathbf{k}=1}^{3}\varepsilon_{\mathbf{ijk}}\mathbf{e_k}$$

where $\varepsilon_{\mathbf{ijk}}$ is the ***permutation symbo*** *or* **(Levi-Civita)**, defined as:

$$\varepsilon_{\mathbf{ijk}} \equiv \begin{cases} 1 & \text{, if ijk} = 123, 231, \text{or } 312 \text{ (i.e, an even permutation of } 123) \\ -1 & \text{, if ijk} = 321, 132, \text{or } 213 \text{ (i.e, an odd permutation of } 123) \\ 0 & \text{, if any two indices are equal} \end{cases}$$

The Cartesian (or rectangular) system of coordinates (x,y,z), with

$$-\infty < x < \infty, \quad -\infty < y < \infty \quad \text{and} \quad -\infty < z < \infty$$

Its basis is often denoted by {**i**, **j**, **k**} or $\{\mathbf{e_x}, \mathbf{e_y}, \mathbf{e_z}\}$. The decomposition of a vector **v** into (v_x, v_y, v_z) is depicted in figure (1).

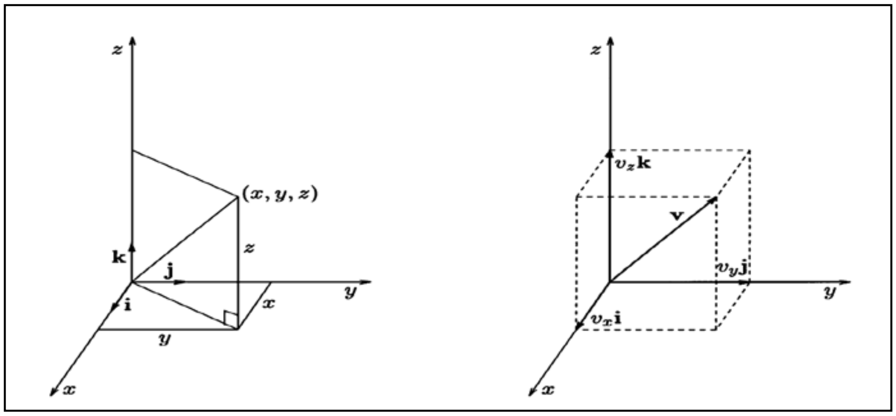


Figure (1) Cartesian coordinate (x,y,z)

The cylindrical and spherical polar coordinates are the two most important orthogonal ***curvilinear*** coordinate systems. The cylindrical polar coordinates (r, θ, z),with $r \geq 0$, $0 \leq \theta < 2\pi$ and $-\infty < z < \infty$ are shown in fig (2)

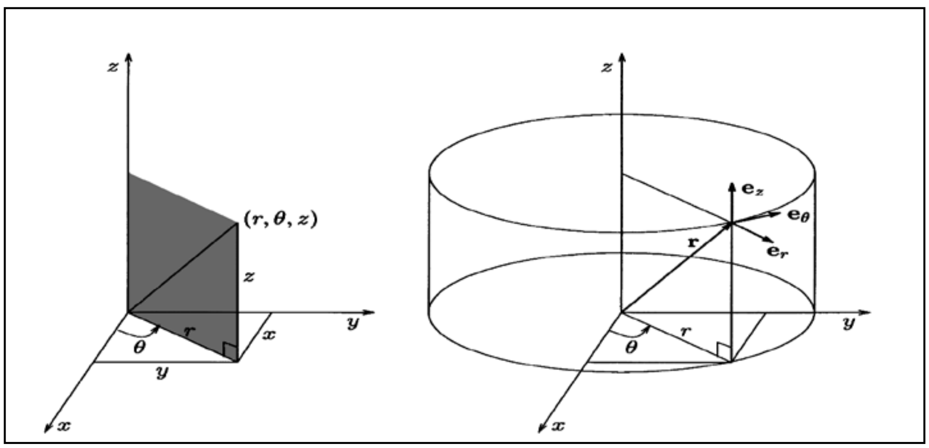


Figure (2) Cylindrical polar coordinate (r,θ,z)

By invoking simple trigonometric relations, any vector, including those of the bases, can be transformed from one system to another. Table (1) lists the formulas for making coordinate conversions from cylindrical to Cartesian coordinates and vice versa.

Table (1) Relations between Cartesian and cylindrical polar coordinates.

$(r,\theta,z) \longrightarrow (x,y,z)$	$(x,y,z) \longrightarrow (r,\theta,z)$
Coordinates $x = r\cos\theta$ $y = r\sin\theta$ $z = z$	$r = \sqrt{x^2+y^2}$ $\theta = \begin{cases} \arctan\frac{y}{x}, & x>0,\ y\geq 0 \\ \pi + \arctan\frac{y}{x}, & x<0 \\ 2\pi + \arctan\frac{y}{x}, & x>0,\ y<0 \end{cases}$ $z = z$
Unit vectors $\mathbf{i} = \cos\theta\,\mathbf{e}_r - \sin\theta\,\mathbf{e}_\theta$ $\mathbf{j} = \sin\theta\,\mathbf{e}_r + \cos\theta\,\mathbf{e}_\theta$ $\mathbf{k} = \mathbf{e}_z$	$\mathbf{e}_r = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}$ $\mathbf{e}_\theta = -\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j}$ $\mathbf{e}_z = \mathbf{k}$

The spherical polar coordinates (r,θ, ϕ) with $r \geq 0$, $0 \leq \theta < 2\pi$ and $0 \leq \phi < 2\pi$ together with Cartesian coordinate with the same origin, are in figure (2).

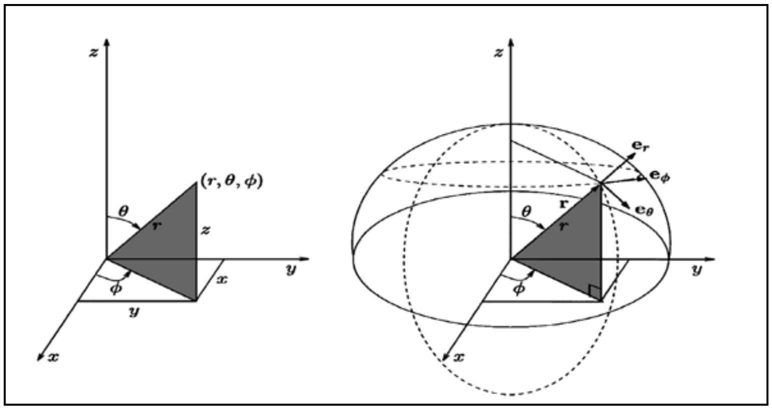


Figure (2) Spherical polar coordinate (r,θ,ϕ)

The transformation of a vector from spherical to Cartesian coordinates (sharing the same origin) and vice-versa obeys the relations of Table (2).

Table (2) Relations between Cartesian and spherical polar coordinates

$(r,\theta,\phi) \longrightarrow (x,y,z)$	$(x,y,z) \longrightarrow (r,\theta,\phi)$
Coordinates	
$x = r\sin\theta\cos\phi$	$r = \sqrt{x^2+y^2+z^2}$
$y = r\sin\theta\sin\phi$	$\theta = \begin{cases} \arctan\frac{\sqrt{x^2+y^2}}{z}, & z>0 \\ \frac{\pi}{2}, & z=0 \\ \pi + \arctan\frac{\sqrt{x^2+y^2}}{z}, & z<0 \end{cases}$
$z = r\cos\theta$	$\phi = \begin{cases} \arctan\frac{y}{x}, & x>0,\ y\geq 0 \\ \pi + \arctan\frac{y}{x}, & x<0 \\ 2\pi + \arctan\frac{y}{x}, & x>0,\ y<0 \end{cases}$
Unit vectors	
$\mathbf{i} = \sin\theta\cos\phi\,\mathbf{e}_r + \cos\theta\cos\phi\,\mathbf{e}_\theta - \sin\phi\,\mathbf{e}_\phi$	$\mathbf{e}_r = \sin\theta\cos\phi\,\mathbf{i} + \sin\theta\sin\phi\,\mathbf{j} + \cos\theta\,\mathbf{k}$
$\mathbf{j} = \sin\theta\sin\phi\,\mathbf{e}_r + \cos\theta\sin\phi\,\mathbf{e}_\theta + \cos\phi\,\mathbf{e}_\phi$	$\mathbf{e}_\theta = \cos\theta\cos\phi\,\mathbf{i} + \cos\theta\sin\phi\,\mathbf{j} - \sin\theta\,\mathbf{k}$
$\mathbf{k} = \cos\theta\,\mathbf{e}_r - \sin\theta\,\mathbf{e}_\theta$	$\mathbf{e}_\phi = -\sin\phi\,\mathbf{i} + \cos\phi\,\mathbf{j}$

Example. Show that the basis $B=\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$ of the cylindrical system is orthonormal.

Sol. since $\mathbf{i}.\mathbf{i} = \mathbf{j}.\mathbf{j} = \mathbf{k}.\mathbf{k} = 1$ and $\mathbf{i}.\mathbf{j} = \mathbf{j}.\mathbf{k} = \mathbf{k}.\mathbf{i} = 0$, we obtain;

$$\mathbf{e_r}.\mathbf{e_r} = (\cos\theta\mathbf{i} + \sin\theta\mathbf{j}).(\cos\theta\mathbf{i} + \sin\theta\mathbf{j}) = \cos^2\theta + \sin^2\theta = 1$$

$$\mathbf{e}_\theta.\mathbf{e}_\theta =$$

$$\mathbf{e_z}.\mathbf{e_z} = \mathbf{k}.\mathbf{k} = 1$$

$$\mathbf{e_r}.\mathbf{e}_\theta = (\cos\theta\mathbf{i} + \sin\theta\mathbf{j}).(-\sin\theta\mathbf{i} + \cos\theta\mathbf{j}) = 0$$

$$\mathbf{e_r}.\mathbf{e_z} =$$

$$\mathbf{e}_\theta.\mathbf{e_z} =$$

H.W. The position vector **r** defines the position of a point in space, with respect to coordinate system. In Cartesian coordinate, $\mathbf{r} = x\ \mathbf{i} + y\ \mathbf{j} + z\ \mathbf{k}$, show that in cylindrical coordinates, the position vector is given by $\mathbf{r} = r\ \mathbf{e_r} + z\ \mathbf{e_z}$, and in spherical coordinate, $\mathbf{r} = r\ \mathbf{e_r}$.

In the following subsections, we will make use of the vector differential operator *nabla* (or *del*), ∇. In Cartesian coordinates, ∇ is defined by

$$\nabla \equiv \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}$$

The ***gradient*** of a scalar field $f(x, y, z)$ is a vector field defined by

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

The ***divergence*** of a vector field $\mathbf{v}(x, y, z)$ is a scalar field defined by

$$\nabla.\mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

More details about ∇ and its forms in curvilinear coordinates are given in

3. Tensors

In the previous sections, two kinds of products that can be formed with any two unit basis vectors were defined, i.e. the *dot product*, $\mathbf{e}_i \cdot \mathbf{e}_j$, and the *cross product*, $\mathbf{e}_i \times \mathbf{e}_j$. A third kind of product is the *dyadic product*, $\mathbf{e}_i\mathbf{e}_j$, also referred to as a *unit dyad*. The unit dyad **e***i***e***j* represents *an ordered pair of coordinate directions*. The nine possible unit dyads, {**e**1**e**1, **e**1**e**2, **e**1**e**3, **e**2**e**1, **e**2**e**2, **e**2**e**3, **e**3**e**1, **e**3**e**2, **e**3**e**3}

A second-order tensor, $\boldsymbol{\tau}$, can thus be written as a linear combination of the unit dyads:

$$\tau = \sum_{\mathbf{i}=1}^{3}\sum_{\mathbf{j}=1}^{3}\tau_{\mathbf{ij}}\mathbf{e_i}\mathbf{e_j}$$

where the scalars $\boldsymbol{\tau}_{ij}$ are referred to as the *components* of the tensor $\boldsymbol{\tau}$. Similarly, a third-order tensor can be defined as the linear combination of all possible *unit triads* $\mathbf{e}_i\mathbf{e}_j\mathbf{e}_k$, etc. Scalars can be viewed as zero-order tensors, and vectors as first-order tensors.

A tensor, $\boldsymbol{\tau}$, can be represented by means of a square matrix as

$$\boldsymbol{\tau} = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) \begin{bmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{bmatrix} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}$$

and often simply by the matrix of its components,

$$\boldsymbol{\tau} = \begin{bmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{bmatrix}$$

Note that the equality sign "=" is loosely used, since $\boldsymbol{\tau}$ is a tensor and *not* a matrix. For a complete description of a tensor by means of matrix, the basis {**e**1, **e**2, **e**3} should be provided.

The *unit* (or *identity*) *tensor*, **I**, is defined by

$$\mathbf{I} = \sum_{\mathbf{i}=1}^{3} \sum_{\mathbf{j}=1}^{3} \delta_{\mathbf{ij}} \mathbf{e}_{\mathbf{i}} \mathbf{e}_{\mathbf{j}} = \mathbf{e}_1 \mathbf{e}_1 + \mathbf{e}_2 \mathbf{e}_2 + \mathbf{e}_3 \mathbf{e}_3$$

Each diagonal component of the matrix form of **I** is unity and the non-diagonal components are zero:

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The sum of two tensors, $\boldsymbol{\sigma}$ and $\boldsymbol{\tau}$, is the tensor whose components are the sums of the corresponding components of the two tensors:

$$\boldsymbol{\sigma} + \boldsymbol{\tau} = \sum_{\mathbf{i}=1}^{3} \sum_{\mathbf{j}=1}^{3} \sigma_{\mathbf{ij}} \mathbf{e}_{\mathbf{i}} \mathbf{e}_{\mathbf{j}} + \sum_{\mathbf{i}=1}^{3} \sum_{\mathbf{j}=1}^{3} \tau_{\mathbf{ij}} \mathbf{e}_{\mathbf{i}} \mathbf{e}_{\mathbf{j}} = \sum_{\mathbf{i}=1}^{3} \sum_{\mathbf{j}=1}^{3} (\sigma_{\mathbf{ij}} + \tau_{\mathbf{ij}}) \mathbf{e}_{\mathbf{i}} \mathbf{e}_{\mathbf{j}}$$

The product of a tensor, $\boldsymbol{\tau}$, and a scalar, m, is the tensor whose components are equal to the components of $\boldsymbol{\tau}$ multiplied by m:

$$m\boldsymbol{\tau} = m \left(\sum_{\mathbf{i}=1}^{3} \sum_{\mathbf{j}=1}^{3} \tau_{\mathbf{ij}} \mathbf{e}_{\mathbf{i}} \mathbf{e}_{\mathbf{j}} \right) = \sum_{\mathbf{i}=1}^{3} \sum_{\mathbf{j}=1}^{3} (m \tau_{\mathbf{ij}}) \mathbf{e}_{\mathbf{i}} \mathbf{e}_{\mathbf{j}}$$

The *transpose*, $\boldsymbol{\tau}^T$, of a tensor $\boldsymbol{\tau}$ is defined by

$$\boldsymbol{\tau}^{\mathbf{T}} = \sum_{\mathbf{i}=1}^{3}\sum_{\mathbf{j}=1}^{3}\tau_{\mathbf{ji}}\mathbf{e}_{\mathbf{i}}\mathbf{e}_{\mathbf{j}}$$

The matrix form of $\boldsymbol{\tau}^T$ is obtained by interchanging the rows and columns of the matrix form of $\boldsymbol{\tau}$:

$$\boldsymbol{\tau}^{\mathbf{T}} = \begin{bmatrix} \tau_{11} & \tau_{21} & \tau_{31} \\ \tau_{12} & \tau_{22} & \tau_{32} \\ \tau_{13} & \tau_{23} & \tau_{33} \end{bmatrix}$$

If $\boldsymbol{\tau}^T = \boldsymbol{\tau}$, i.e., if $\boldsymbol{\tau}$ is equal to its transpose, the tensor $\boldsymbol{\tau}$ is said to be *symmetric*. If $\boldsymbol{\tau}^T = -\boldsymbol{\tau}$, the tensor $\boldsymbol{\tau}$ is said to be *antisymmetric* (or *skew symmetric*).

The dyadic product of two vectors **a** and **b** can easily be constructed as follows:

$$\mathbf{ab} = \left(\sum_{\mathbf{i}=1}^{3} a_{\mathbf{i}}\mathbf{e}_{\mathbf{i}}\right)\left(\sum_{\mathbf{j}=1}^{3} b_{\mathbf{j}}\mathbf{e}_{\mathbf{j}}\right) = \sum_{\mathbf{i}=1}^{3}\sum_{\mathbf{j}=1}^{3} a_{\mathbf{i}} b_{\mathbf{j}}\mathbf{e}_{\mathbf{i}}\mathbf{e}_{\mathbf{j}}$$

Obviously, **ab** is a tensor, referred to as *dyad* or *dyadic tensor*. Its matrix form is

$$\mathbf{ab} = \begin{bmatrix} a_1 b_1 & a_1 b_2 & a_1 b_3 \\ a_2 b_1 & a_2 b_2 & a_2 b_3 \\ a_3 b_1 & a_3 b_2 & a_3 b_3 \end{bmatrix}$$

Note that $\mathbf{ab} \neq \mathbf{ba}$ unless **ab** is symmetric. Given that $(\mathbf{ab})^T = \mathbf{ba}$, the dyadic product of a vector by itself, **aa**, is symmetric.

The unit dyads $\mathbf{e}_i\mathbf{e}_j$ are dyadic tensors, the matrix form of which has only one unitary nonzero entry at the (i, j) position. For example,

$$\mathbf{e}_2\mathbf{e}_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

The most important operations involving unit dyads are the following:

(i) The *single-dot product* (or *tensor product*) of two unit dyads is a tensor defined by

$$\left(\mathbf{e_i e_j}\right)\cdot\left(\mathbf{e_k e_l}\right)\equiv \mathbf{e_i}\left(\mathbf{e_j}\cdot\mathbf{e_k}\right)\mathbf{e_l}=\delta_{\mathbf{jk}}\mathbf{e_i e_l}$$

This operation is not commutative

(ii) The *double-dot product* (or *scalar product* or *inner product*) of two unit dyads is a scalar defined by;

$$\left(\mathbf{e_i e_j}\right):\left(\mathbf{e_k e_l}\right)\equiv\left(\mathbf{e_i}\cdot\mathbf{e_l}\right)\left(\mathbf{e_j}\cdot\mathbf{e_k}\right)=\delta_{\mathbf{il}}\delta_{\mathbf{jk}}$$

It is easily seen that this operation is commutative.

(iii) The dot product of a unit dyad and a unit vector is a vector defined by

$$\left(\mathbf{e_i e_j}\right)\cdot\mathbf{e_k}\equiv\mathbf{e_i}\left(\mathbf{e_j}\cdot\mathbf{e_k}\right)=\delta_{\mathbf{ik}}\mathbf{e_i}$$

or

$$\mathbf{e_i}\cdot\left(\mathbf{e_j e_k}\right)\equiv\left(\mathbf{e_i}\cdot\mathbf{e_j}\right)\mathbf{e_k}=\delta_{\mathbf{ij}}\mathbf{e_k}$$

Obviously, this operation is *not* commutative.

Operations involving tensors are easily performed by expanding the tensors into components with respect to a given basis and using the elementary unit dyad operations. The most important operations involving tensors are summarized below.

1. ***The single-dot product of two tensors (Tensor product)***

If $\boldsymbol{\sigma}$ and $\boldsymbol{\tau}$ are tensors, then

$$\boldsymbol{\sigma}\cdot\boldsymbol{\tau}=\left(\sum_{\mathbf{i}=1}^{3}\sum_{\mathbf{j}=1}^{3}\sigma_{\mathbf{ij}}\mathbf{e_i e_j}\right)\cdot\left(\sum_{k=1}^{3}\sum_{\iota=1}^{3}\tau_{\mathbf{k}\iota}\mathbf{e_k e}_\iota\right)=\sum_{\mathbf{i}=1}^{3}\sum_{\mathbf{j}=1}^{3}\sum_{k=1}^{3}\sum_{\iota=1}^{3}\sigma_{\mathbf{ij}}\tau_{\mathbf{k}\iota}\left(\mathbf{e_i e_j}\right)\cdot\left(\mathbf{e_k e}_\iota\right)$$

$$=\sum_{\mathbf{i}=1}^{3}\sum_{\mathbf{j}=1}^{3}\sum_{k=1}^{3}\sum_{\iota=1}^{3}\sigma_{\mathbf{ij}}\tau_{\mathbf{k}\iota}\delta_{\mathbf{jk}}\mathbf{e_i e}_\iota=\sum_{\mathbf{i}=1}^{3}\sum_{\mathbf{j}=1}^{3}\sum_{\iota=1}^{3}\sigma_{\mathbf{ij}}\tau_{\mathbf{j}\iota}\mathbf{e_i e}_\iota$$

$$\boldsymbol{\sigma}\cdot\boldsymbol{\tau}=\sum_{\mathbf{i}=1}^{3}\sum_{\iota=1}^{3}\left(\sum_{\mathbf{j}=1}^{3}\sigma_{\mathbf{ij}}\tau_{\mathbf{j}\iota}\right)\mathbf{e_i e}_\iota$$

The operation is not commutative. It is easily verified that $\boldsymbol{\sigma}\cdot\mathbf{I}=\mathbf{I}\cdot\boldsymbol{\sigma}=\boldsymbol{\sigma}$

2. ***The double-dot product of two tensors (Scalar product)***

$$\boldsymbol{\sigma} : \boldsymbol{\tau} = \sum_{\mathbf{i}=1}^{3}\sum_{\mathbf{j}=1}^{3} \sigma_{\mathbf{ij}}\tau_{\mathbf{ij}}\mathbf{e}_{\mathbf{i}}\mathbf{e}_{\mathbf{j}}$$

3. ***The dot product of a tensor and a vector (vector product)***

This is a very useful operation in fluid mechanics. If **a** is a vector, we have:

$$\boldsymbol{\sigma} \cdot \mathbf{a} = \left(\sum_{\mathbf{i}=1}^{3}\sum_{\mathbf{j}=1}^{3} \sigma_{\mathbf{ij}}\mathbf{e}_{\mathbf{i}}\mathbf{e}_{\mathbf{j}} \right) \cdot \left(\sum_{k=1}^{3} a_{\mathbf{k}}\mathbf{e}_{\mathbf{k}} \right) = \sum_{\mathbf{i}=1}^{3}\sum_{\mathbf{j}=1}^{3}\sum_{\mathbf{k}=1}^{3} \sigma_{\mathbf{ij}} a_{\mathbf{k}} \left(\mathbf{e}_{\mathbf{i}}\mathbf{e}_{\mathbf{j}}\right) \cdot \mathbf{e}_{\mathbf{k}}$$

$$= \sum_{\mathbf{i}=1}^{3}\sum_{\mathbf{j}=1}^{3}\sum_{\mathbf{k}=1}^{3} \sigma_{\mathbf{ij}} a_{\mathbf{k}} \delta_{\mathbf{ik}}\mathbf{e}_{\mathbf{k}} = \sum_{\mathbf{i}=1}^{3}\sum_{\mathbf{j}=1}^{3} \sigma_{\mathbf{ij}} a_{\mathbf{j}} \delta_{\mathbf{ij}}\mathbf{e}_{\mathbf{i}}$$

$$\boldsymbol{\sigma} \cdot \mathbf{a} = \sum_{\mathbf{i}=1}^{3} \left(\sum_{\mathbf{j}=1}^{3} \sigma_{\mathbf{ij}} a_{\mathbf{j}} \right) \mathbf{e}_{\mathbf{i}}$$

Similarly, we find that

$$\mathbf{a} \cdot \boldsymbol{\sigma} = \sum_{\mathbf{i}=1}^{3} \left(\sum_{\mathbf{j}=1}^{3} \sigma_{\mathbf{ji}} a_{\mathbf{j}} \right) \mathbf{e}_{\mathbf{i}}$$

The vectors $\boldsymbol{\sigma} \cdot \mathbf{a}$ and $\mathbf{a} \cdot \boldsymbol{\sigma}$ are not, in general, equal.

4. Index Notation and Summation Convention

So far, we have used three different ways for representing tensors and vectors:

(a) the compact *symbolic notation*, e.g., **u** for a vector and $\boldsymbol{\tau}$ for a tensor;

(b) the so-called *Gibbs' notation*, e.g.,

$$\sum_{\mathbf{i}=1}^{3} u_{\mathbf{i}}\mathbf{e}_{\mathbf{i}} \quad \text{and} \quad \sum_{\mathbf{i}=1}^{3}\sum_{\mathbf{j}=1}^{3} \tau_{\mathbf{ij}}\mathbf{e}_{\mathbf{i}}\mathbf{e}_{\mathbf{j}}$$

for **u** and $\boldsymbol{\tau}$, respectively; and

(c) the *matrix notation*, e.g.,

$$\boldsymbol{\tau} = \begin{bmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{bmatrix} \qquad \text{for } \boldsymbol{\tau}.$$

Very frequently, in the literature, use is made of the *index notation* and the so-called *Einstein's summation convention*, in order to simplify expressions involving vector and tensor operations by omitting the summation symbols.

In index notation, a vector **v** is represented as

$$v_{\mathbf{i}} \equiv \sum_{\mathbf{i}=1}^{3} v_{\mathbf{i}} \mathbf{e}_{\mathbf{i}} = \mathbf{v}$$

A tensor $\boldsymbol{\tau}$ is represented as

$$\tau_{\mathbf{ij}} \equiv \sum_{\mathbf{i}=1}^{3} \sum_{\mathbf{j}=1}^{3} \tau_{\mathbf{ij}} \mathbf{e}_{\mathbf{i}} \mathbf{e}_{\mathbf{j}} = \boldsymbol{\tau}$$

The nabla operator, for example, is represented as

$$\frac{\partial}{\partial x_i} \equiv \sum_{\mathbf{i}=1}^{3} \frac{\partial}{\partial x_i} \mathbf{e}_{\mathbf{i}} = \frac{\partial}{\partial x} \mathbf{i} + \frac{\partial}{\partial y} \mathbf{j} + \frac{\partial}{\partial z} \mathbf{k} = \nabla$$

where x_i is the general Cartesian coordinate taking on the values of x, y and z. The unit tensor **I** is represented by Kronecker's delta:

$$\delta_{\mathbf{ij}} \equiv \sum_{\mathbf{i}=1}^{3} \sum_{\mathbf{j}=1}^{3} \delta_{\mathbf{ij}} \mathbf{e}_{\mathbf{i}} \mathbf{e}_{\mathbf{j}} = \mathbf{I}$$

It is evident that an explicit statement must be made when the tensor τ_{ij} is to be distinguished from its (i, j) element.

With *Einstein's summation convention*, if an index appears twice in an expression, then summation is implied with respect to the *repeated* index, over the range of that index. The number of the *free indices*, i.e., the indices that appear only once, is the number of directions associated with an expression; it thus determines whether an expression is a scalar, a vector or a tensor. In the following expressions, there are no free indices, and thus these are scalars:

$$u_{\mathbf{i}} v_{\mathbf{i}} \equiv \sum_{\mathbf{i}=1}^{3} u_{\mathbf{i}} v_{\mathbf{i}} = \mathbf{u} \cdot \mathbf{v}$$

$$\tau_{\mathbf{ii}} \equiv \sum_{\mathbf{i}=1}^{3} \sum_{\mathbf{j}=1}^{3} \tau_{\mathbf{ii}} = tr\, \boldsymbol{\tau} \qquad \text{trace for } \boldsymbol{\tau}$$

$$\frac{\partial u_{\mathbf{i}}}{\partial x_i} \equiv \sum_{\mathbf{i}=1}^{3} \frac{\partial u_{\mathbf{i}}}{\partial x_i} = \frac{\partial u_x}{\partial x} + \frac{\partial u_{\mathbf{y}}}{\partial y} + \frac{\partial u_{\mathbf{z}}}{\partial z} = \nabla \cdot \mathbf{u}$$

$$\frac{\partial^2 f}{\partial x_i \partial x_i} \mathbf{or} \frac{\partial^2 f}{\partial x_{\mathbf{i}}^2} \equiv \sum_{\mathbf{i}=1}^{3} \frac{\partial^2 f}{\partial x_{\mathbf{i}}^2} = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \nabla^2 f$$

where ∇^2 is the ***Laplacian operator*** to be discussed in more detail in Section (5). In the following expression, there are two sets of double indices, and summation must be performed over both sets:

$$\sigma_{\mathbf{ij}} \tau_{\mathbf{ji}} \equiv \sum_{\mathbf{i}=1}^{3} \sum_{\mathbf{j}=1}^{3} \sigma_{\mathbf{ij}} \tau_{\mathbf{ii}} = \boldsymbol{\sigma} : \boldsymbol{\tau}$$

The following expressions, with one free index, are vectors:

$$\varepsilon_{\mathbf{ijk}} u_{\mathbf{i}} v_{\mathbf{j}} \equiv \sum_{\mathbf{k}=1}^{3} \left(\sum_{\mathbf{i}=1}^{3} \sum_{\mathbf{j}=1}^{3} \varepsilon_{\mathbf{ijk}} u_{\mathbf{i}} v_{\mathbf{j}} \right) \mathbf{e_k} = \mathbf{u} \times \mathbf{v}$$

$$\frac{\partial f}{\partial x_i} \equiv \sum_{\mathbf{i}=1}^{3} \frac{\partial f}{\partial x_{\mathbf{i}}} \mathbf{e_i} = \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j} + \frac{\partial f}{\partial z} \mathbf{k} = \nabla f$$

$$\tau_{\mathbf{ji}} v_{\mathbf{j}} \equiv \sum_{\mathbf{i}=1}^{3} \left(\sum_{\mathbf{j}=1}^{3} \tau_{\mathbf{ii}} v_{\mathbf{j}} \right) \mathbf{e_i} = \boldsymbol{\tau} \cdot \mathbf{v}$$

Finally, the following quantities, having two free indices, are tensors:

$$u_{\mathbf{i}} v_{\mathbf{j}} \equiv \sum_{\mathbf{i}=1}^{3} \sum_{\mathbf{j}=1}^{3} u_{\mathbf{i}} v_{\mathbf{j}} \mathbf{e_i} \mathbf{e_j} = \mathbf{uv}$$

$$\sigma_{\mathbf{ik}} \tau_{\mathbf{kj}} \equiv \sum_{\mathbf{i}=1}^{3} \sum_{\mathbf{j}=1}^{3} \left(\sum_{\mathbf{k}=1}^{3} \sigma_{\mathbf{ik}} \tau_{\mathbf{kj}} \right) \mathbf{e_i} \mathbf{e_j} = \boldsymbol{\sigma} \cdot \boldsymbol{\tau}$$

$$\frac{\partial u_{\mathbf{j}}}{\partial x_i} \equiv \sum_{\mathbf{i}=1}^{3} \sum_{\mathbf{j}=1}^{3} \frac{\partial u_{\mathbf{j}}}{\partial x_i} \mathbf{e_i} \mathbf{e_j} = \nabla \mathbf{u}$$

Note that $\nabla \mathbf{u}$ in the last equation is a dyadic tensor.

<u>N.B.</u> Some authors use even simpler expressions for the nabla operator. For example, $\nabla \cdot \mathbf{u}$ is also represented as $\partial_i u_i$ or $u_{i,i}$,with a comma to indicate the derivative, and the dyadic ∇u is represented as $\partial_i u_j$ or $u_{i,j}$.

5. Differential Operators

The nabla operator ∇, already encountered in previous sections, is a differential operator. In a Cartesian system of coordinates (x1, x2, x3), defined by the orthonormal basis ($\mathbf{e_1}$, $\mathbf{e_2}$, $\mathbf{e_3}$),

$$\nabla \equiv \frac{\partial}{\partial x_1}\mathbf{e}_1 + \frac{\partial}{\partial x_2}\mathbf{e}_2 + \frac{\partial}{\partial x_3}\mathbf{e}_3 = \sum_{\mathbf{i}=1}^{3} \frac{\partial}{\partial x_{\mathbf{i}}}\mathbf{e_i}$$

or, in index notation,

$$\nabla \equiv \frac{\partial}{\partial x_{\mathbf{i}}}$$

The nabla operator is a vector operator which acts on scalar, vector, or tensor fields. The result of its action is another field the order of which depends on the type of the operation. In the following, we will first define the various operations of ∇ in Cartesian coordinates, and then discuss their forms in curvilinear coordinates.

- The ***gradient*** of a differentiable scalar field f, denoted by ∇f or *gradf*, is a vector field:

$$\nabla f \equiv \left(\sum_{\mathbf{i}=1}^{3} \frac{\partial}{\partial x_{\mathbf{i}}}\mathbf{e_i} \right) f = \sum_{\mathbf{i}=1}^{3} \frac{\partial f}{\partial x_{\mathbf{i}}}\mathbf{e_i} = \frac{\partial f}{\partial x_1}\mathbf{e}_1 + \frac{\partial f}{\partial x_2}\mathbf{e}_2 + \frac{\partial f}{\partial x_3}\mathbf{e}_3$$

- The ***gradient*** of a differentiable vector field $\mathbf{u}$ is a dyadic tensor field:

$$\nabla \mathbf{u} \equiv \left(\sum_{\mathbf{i}=1}^{3} \frac{\partial}{\partial x_{\mathbf{i}}}\mathbf{e_i} \right)\left(\sum_{\mathbf{j}=1}^{3} u_{\mathbf{j}}\mathbf{e_j} \right) = \sum_{\mathbf{i}=1}^{3}\sum_{\mathbf{i}=1}^{3} \frac{\partial u_{\mathbf{j}}}{\partial x_{\mathbf{i}}}\mathbf{e_i}\mathbf{e_j}$$

if $\mathbf{u}$ is the velocity, then $\nabla\mathbf{u}$ is called the *velocity gradient tensor*.

- The ***divergence*** of a differentiable vector field $\mathbf{u}$, denoted by $\nabla \cdot \mathbf{u}$ or *div* $\mathbf{u}$, is a scalar field

$$\nabla \cdot \mathbf{u} \equiv \left(\sum_{\mathbf{i}=1}^{3} \frac{\partial}{\partial x_{\mathbf{i}}}\mathbf{e_i} \right) \cdot \left(\sum_{\mathbf{j}=1}^{3} u_{\mathbf{j}}\mathbf{e_j} \right) = \sum_{\mathbf{i}=1}^{3} \frac{\partial u_{\mathbf{i}}}{\partial x_{\mathbf{i}}}\delta_{\mathbf{ij}} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}$$

$\nabla \cdot \mathbf{u}$ measures changes in magnitude, or flux through a point. If **u** is the velocity, then $\nabla \cdot \mathbf{u}$ measures the rate of volume expansion per unit volume; hence, it is zero for incompressible fluids.

- The ***curl*** or *rotation* of a differentiable vector field **u**, denoted by $\nabla\times\mathbf{u}$ or *curl* **u** or *rot* **u**, is a vector field:

$$\nabla \times \mathbf{u} \equiv \left(\sum_{\mathbf{i}=1}^{3} \frac{\partial}{\partial x_{\mathbf{i}}} \mathbf{e}_{\mathbf{i}} \right) \times \left(\sum_{\mathbf{j}=1}^{3} u_{\mathbf{j}} \mathbf{e}_{\mathbf{j}} \right) = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ \dfrac{\partial}{\partial x_1} & \dfrac{\partial}{\partial x_2} & \dfrac{\partial}{\partial x_3} \\ u_1 & u_2 & u_3 \end{vmatrix}$$

or

$$\nabla \times \mathbf{u} \equiv \left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3} \right) \mathbf{e}_1 + \left(\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1} \right) \mathbf{e}_2 + \left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} \right) \mathbf{e}_3$$

The field $\nabla\times\mathbf{u}$ is often called the ***vorticity*** (or *chirality*) of **u**.

- The ***divergence*** of a differentiable tensor field $\boldsymbol{\tau}$ is a vector field:

$$\nabla \cdot \boldsymbol{\tau} \equiv \left(\sum_{\mathbf{k}=1}^{3} \frac{\partial}{\partial x_{\mathbf{k}}} \mathbf{e}_{\mathbf{k}} \right) \cdot \left(\sum_{\mathbf{i}=1}^{3} \sum_{\mathbf{j}=1}^{3} \tau_{\mathbf{ij}} \mathbf{e}_{\mathbf{i}} \mathbf{e}_{\mathbf{j}} \right) = \sum_{\mathbf{i}=1}^{3} \sum_{\mathbf{j}=1}^{3} \frac{\partial \tau_{\mathbf{ij}}}{\partial x_{\mathbf{i}}} \mathbf{e}_{\mathbf{j}}$$

Example. Consider the position vector in Cartesian coordinates,

$$\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}\,.$$

For its divergence and curl, we obtain:

$$\nabla \cdot \mathbf{r} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} \quad \Rightarrow \quad \nabla \cdot \mathbf{r} = 3$$

and

$$\nabla \times \mathbf{r} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x & y & z \end{vmatrix} \quad \Rightarrow \quad \nabla \times \mathbf{r} = 0$$

Other useful operators involving the nabla operator are the ***Laplace operator*** ∇^2 and the operator $\mathbf{u} \cdot \nabla$, where $\mathbf{u}$ is a vector field. The ***Laplacian*** of a scalar f with continuous second partial derivatives is defined as the divergence of the gradient:

$$\nabla^2 f \equiv \nabla \cdot \nabla f = \frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2} + \frac{\partial^2 f}{\partial x_3^2}$$

i.e.,
$$\nabla^2 \equiv \nabla \cdot \nabla = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$$

For the operator $\mathbf{u} \cdot \nabla$, we obtain:

$$\mathbf{u} \cdot \nabla \equiv \left(u_1 \mathbf{e}_1 + u_2 \mathbf{e}_2 + u_3 \mathbf{e}_3\right) \cdot \left(\frac{\partial}{\partial x_1}\mathbf{e}_1 + \frac{\partial}{\partial x_2}\mathbf{e}_2 + \frac{\partial}{\partial x_3}\mathbf{e}_3\right)$$

$$\therefore \mathbf{u} \cdot \nabla = u_1 \frac{\partial}{\partial x_1} + u_2 \frac{\partial}{\partial x_2} + u_3 \frac{\partial}{\partial x_3}$$

The above expressions are valid only for Cartesian coordinate systems. In curvilinear coordinate systems, the basis vectors are not constant and the forms of ∇ are quite different. Notice that gradient always raises the order by one (the gradient of a scalar is a vector, the gradient of a vector is a tensor and so on), while divergence reduces the order of a quantity by one.

For any scalar function f with continuous second partial derivatives, the curl of the gradient is zero,

$$\nabla \times \left(\nabla f\right) = 0 \qquad (1)$$

For any vector function $\mathbf{u}$ with continuous second partial derivatives, the divergence of the curl is zero,

$$\nabla \cdot \left(\nabla \times \mathbf{u}\right) = 0 \qquad (2)$$

Equations (1) and (2) are valid independently of the coordinate system.

In fluid mechanics, the ***vorticity*** $\boldsymbol{\omega}$ of the velocity vector $\mathbf{u}$ is defined as the curl of $\mathbf{u}$, $\omega \equiv \nabla \times \mathbf{u}$

Example.

(a) Express the nabla operator

$$\nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k} \quad \text{in cylindrical polar coordinates.}$$

(b) Determine ∇c and $\nabla \cdot \mathbf{u}$, where c is a scalar and $\mathbf{u}$ is a vector.

(c) Derive the operator $\mathbf{u} \cdot \nabla$ and the dyadic product $\nabla\mathbf{u}$ in cylindrical polar coordinates.

Sol.

(a) From Table (1), we have:

$$\mathbf{i} = \cos\theta\, \mathbf{e}_r - \sin\theta\, \mathbf{e}_\theta \quad , \quad \mathbf{j} = \sin\theta\, \mathbf{e}_r + \cos\theta\, \mathbf{e}_\theta \quad \textit{and} \quad \mathbf{k} = \mathbf{e}_z$$

Therefore, we just need to convert the derivatives with respect to x, y and z into derivatives with respect to r, θ and z. Using the chain rule, we get:

$$\frac{\partial}{\partial x} = \frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial x}\frac{\partial}{\partial \theta} = \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}$$

$$\frac{\partial}{\partial y} = \qquad\qquad = \sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta}$$

$$\frac{\partial}{\partial z} = \frac{\partial}{\partial z}$$

Substituting now into nabla operator, gives

$$\nabla = \left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\right)\left(\cos\theta\mathbf{e_r} - \sin\theta\mathbf{e}_\theta\right) + \left(\sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta}\right)\left(\sin\theta\mathbf{e_r} + \cos\theta\mathbf{e}_\theta\right) + \frac{\partial}{\partial z}\mathbf{e_z}$$

After some simplifications and using the trigonometric identity $\sin^2\theta+\sin^2\theta=1$, we get

$$\nabla = \frac{\partial}{\partial r}\mathbf{e_r} + \frac{1}{r}\frac{\partial}{\partial \theta}\mathbf{e}_\theta + \frac{\partial}{\partial z}\mathbf{e_z}$$

(b) The gradient of the scalar c is given by;

$$\nabla c = \frac{\partial c}{\partial r}\mathbf{e_r} + \frac{1}{r}\frac{\partial c}{\partial \theta}\mathbf{e}_\theta + \frac{\partial c}{\partial z}\mathbf{e_z}$$

For the divergence of the vector **u**, we have;

$$\nabla\cdot\mathbf{u}=\left(\frac{\partial}{\partial r}\mathbf{e_r}+\frac{1}{r}\frac{\partial}{\partial\theta}\mathbf{e}_\theta+\frac{\partial}{\partial z}\mathbf{e_z}\right)\cdot\left(u_r\mathbf{e_r}+u_\theta\mathbf{e}_\theta+u_\mathbf{z}\mathbf{e_z}\right)$$

Noting that the only nonzero spatial derivatives of the unit vectors are

$$\frac{\partial\mathbf{e_r}}{\partial\theta}=\mathbf{e}_\theta \qquad \textit{and} \qquad \frac{\partial\mathbf{e}_\theta}{\partial\theta}=-\mathbf{e_r} \qquad \text{we obtain;}$$

$$\nabla\cdot\mathbf{u}=\frac{\partial u_r}{\partial r}+\frac{1}{r}\left(\frac{\partial\mathbf{e_r}}{\partial\theta}u_\mathbf{r}+\frac{\partial u_\theta}{\partial\theta}\mathbf{e}_\theta+\frac{\partial\mathbf{e}_\theta}{\partial\theta}u_\theta\right)\cdot\mathbf{e}_\theta+\frac{\partial u_z}{\partial z}$$

$$=\frac{\partial u_r}{\partial r}+\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}+\frac{1}{r}\left(u_\mathbf{r}\mathbf{e}_\theta-u_\theta\mathbf{e_r}\right)\cdot\mathbf{e}_\theta+\frac{\partial u_z}{\partial z}$$

$$=\frac{\partial u_r}{\partial r}+\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}+\frac{u_\mathbf{r}}{r}+\frac{\partial u_z}{\partial z}$$

$$\nabla\cdot\mathbf{u}=\frac{1}{r}\frac{\partial}{\partial r}\left(ru_r\right)+\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}+\frac{\partial u_z}{\partial z}$$

(c) $\mathbf{u}\cdot\nabla=\left(u_r\mathbf{e_r}+u_\theta\mathbf{e}_\theta+u_\mathbf{z}\mathbf{e_z}\right)\cdot\left(\frac{\partial}{\partial r}\mathbf{e_r}+\frac{1}{r}\frac{\partial}{\partial\theta}\mathbf{e}_\theta+\frac{\partial}{\partial z}\mathbf{e_z}\right)$

$$\mathbf{u}\cdot\nabla=u_r\frac{\partial}{\partial r}+\frac{u_\theta}{r}\frac{\partial}{\partial\theta}+u_z\frac{\partial}{\partial z}$$

Finally, for the dyadic product $\nabla\mathbf{u}$, we have;

$$\nabla\mathbf{u}=\left(\frac{\partial}{\partial r}\mathbf{e_r}+\frac{1}{r}\frac{\partial}{\partial\theta}\mathbf{e}_\theta+\frac{\partial}{\partial z}\mathbf{e_z}\right)\left(u_r\mathbf{e_r}+u_\theta\mathbf{e}_\theta+u_\mathbf{z}\mathbf{e_z}\right)$$

$$=\frac{\partial u_r}{\partial r}\mathbf{e_r}\mathbf{e_r}+\frac{\partial u_\theta}{\partial r}\mathbf{e_r}\mathbf{e}_\theta+\frac{\partial u_\mathbf{z}}{\partial r}\mathbf{e_r}\mathbf{e_z}$$

$$+\frac{1}{r}\frac{\partial u_r}{\partial\theta}\mathbf{e}_\theta\mathbf{e_r}+\frac{1}{r}u_\mathbf{r}\frac{\partial\mathbf{e_r}}{\partial\theta}\mathbf{e}_\theta+\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}\mathbf{e}_\theta\mathbf{e}_\theta+\frac{1}{r}u_\theta\frac{\partial\mathbf{e}_\theta}{\partial\theta}\mathbf{e}_\theta+\frac{1}{r}\frac{\partial u_\mathbf{z}}{\partial\theta}\mathbf{e}_\theta\mathbf{e_z}$$

$$+\frac{\partial u_r}{\partial z}\mathbf{e_z}\mathbf{e_r}+\frac{\partial u_\theta}{\partial z}\mathbf{e_z}\mathbf{e}_\theta+\frac{\partial u_\mathbf{z}}{\partial z}\mathbf{e_z}\mathbf{e_z}$$

$$
\begin{aligned}
\nabla \mathbf{u} = {} & \frac{\partial u_r}{\partial r}\mathbf{e_r}\mathbf{e_r} + \frac{\partial u_\theta}{\partial r}\mathbf{e_r}\mathbf{e}_\theta + \frac{\partial u_\mathbf{z}}{\partial r}\mathbf{e_r}\mathbf{e_z} \\
& + \frac{1}{r}\left(\frac{\partial u_r}{\partial \theta} - u_\theta\right)\mathbf{e}_\theta\mathbf{e_r} + \frac{1}{r}\left(\frac{\partial u_\theta}{\partial \theta} + u_\mathbf{r}\right)\mathbf{e}_\theta\mathbf{e}_\theta + \frac{1}{r}\frac{\partial u_\mathbf{z}}{\partial \theta}\mathbf{e}_\theta\mathbf{e_z} \\
& + \frac{\partial u_r}{\partial z}\mathbf{e_z}\mathbf{e_r} + \frac{\partial u_\theta}{\partial z}\mathbf{e_z}\mathbf{e}_\theta + \frac{\partial u_\mathbf{z}}{\partial z}\mathbf{e_z}\mathbf{e_z}
\end{aligned}
$$

The Conservation Equations In Fluid Mechanics

The basic equation considered here are the three laws of conservation for physical systems:

1. Conservation of mass (continuity)
2. Conservation of momentum (Newton's second law)
3. Conservation of energy (first law of thermodynamics)

The three unknowns that must be obtained simultaneously from these three basic equations are the velocity **u** (three components), the thermodynamic pressure *p*, and the absolute temperature *T*. However, the final forms of the conservation equations also contain four other thermodynamics variables: the density ρ, the enthalpy h, and have two transport properties μ and *k*.

1.2 Conservation of mass

The first step in the derivation of the mass conservation equation is to write down a mass balance for the fluid element:

Rate of increase of mass in fluid element	**=**	**Net rate of flow of mass into fluid element**

The rate of increase of mass in the fluid element is;

$$\frac{\partial}{\partial t}\left(\rho \Delta x\, \Delta y\, \Delta z\right) = \frac{\partial \rho}{\partial t}\left(\Delta x\, \Delta y\, \Delta z\right)$$

Next we need to account for the mass flow rate across a face of the element, which is given by the product of density, area and the velocity component normal to the face. From Figure (1).

Net rate of flow of mass into the element = mass in flow - mass out flow

Velocities
u in x direction
v in y direction
w in z direction

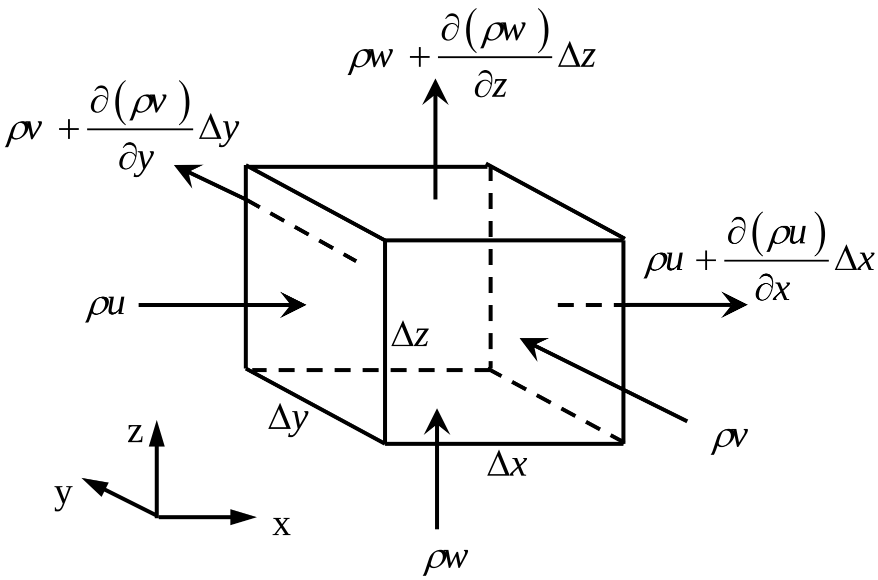


Fig. (1) Mass flows in and out of fluid element

Mass in flow - Mass out flow = Rate of increase of mass

The mass balance for the fluid element is given by;

In x - dir. $$(\rho u)\Delta y\,\Delta z - \left(\rho u + \frac{\partial(\rho u)}{\partial x}\Delta x\right)\Delta y\,\Delta z$$

In y - dir. $$+(\rho v)\Delta x\,\Delta z - \left(\rho v + \frac{\partial(\rho v)}{\partial y}\Delta y\right)\Delta x\,\Delta z$$

In z - dir. $$+(\rho w)\Delta x\,\Delta y - \left(\rho w + \frac{\partial(\rho w)}{\partial z}\Delta z\right)\Delta x\,\Delta y = \frac{\partial \rho}{\partial t}(\Delta x\,\Delta y\,\Delta z)$$

Rearrangement the above equ. we have;

$$-\left(\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z}\right)\Delta x\,\Delta y\,\Delta z = \frac{\partial \rho}{\partial t}(\Delta x\,\Delta y\,\Delta z)$$

Finally, divided by the element volume, we get general form of continuity in Cartesian coordinate form as;

$$\underbrace{\frac{\partial \rho}{\partial t}}_{\text{the rate of change in time}} + \underbrace{\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z}}_{\text{Convective term}} = 0 \qquad (1)$$

- Or In vector notation

$$\frac{\partial \rho}{\partial t}+\mathbf{div}(\rho \mathbf{u})=0 \qquad \mathbf{or} \qquad \frac{\partial \rho}{\partial t}+(\nabla.\rho \mathbf{u})=0$$

where $\mathbf{u}=u\,i+v\,j+w\,k$ is velocity vector in Cartesian coordinate.

- Or in tensor notation $\rho_{,\mathbf{t}}+(\rho u_{\mathbf{i}})_{,\mathbf{i}}=0$

Equation (1) is the ***unsteady, three-dimensional mass conservation or continuity equation*** for ***compressible fluid***.

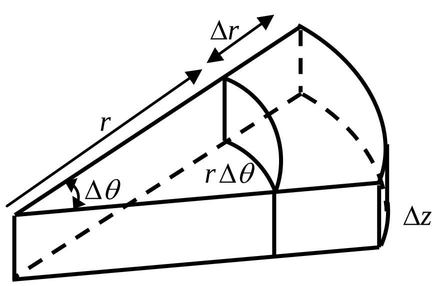
For an ***incompressible fluid*** and **steady** (i.e. a liquid) ρ = constant;

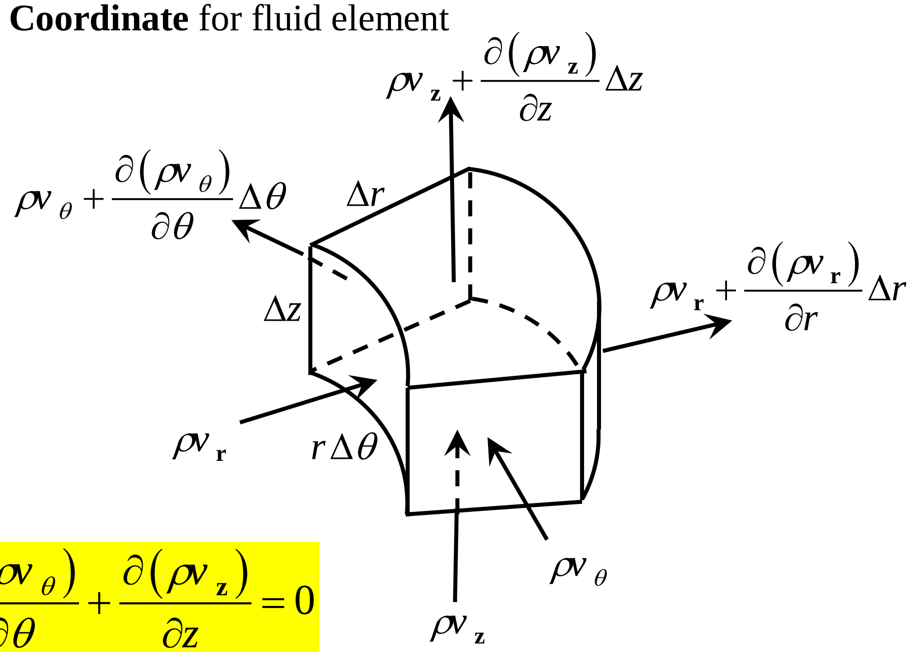
$$\text{div}\,\mathbf{u}=0 \qquad \mathbf{or} \qquad \nabla.\mathbf{u}=0$$

Or in Cartesian coordinate system; $\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0$

Or in tensor $u_{\mathrm{i,i}}=0$

H.W Continuity in cylindrical Coordinate for fluid element



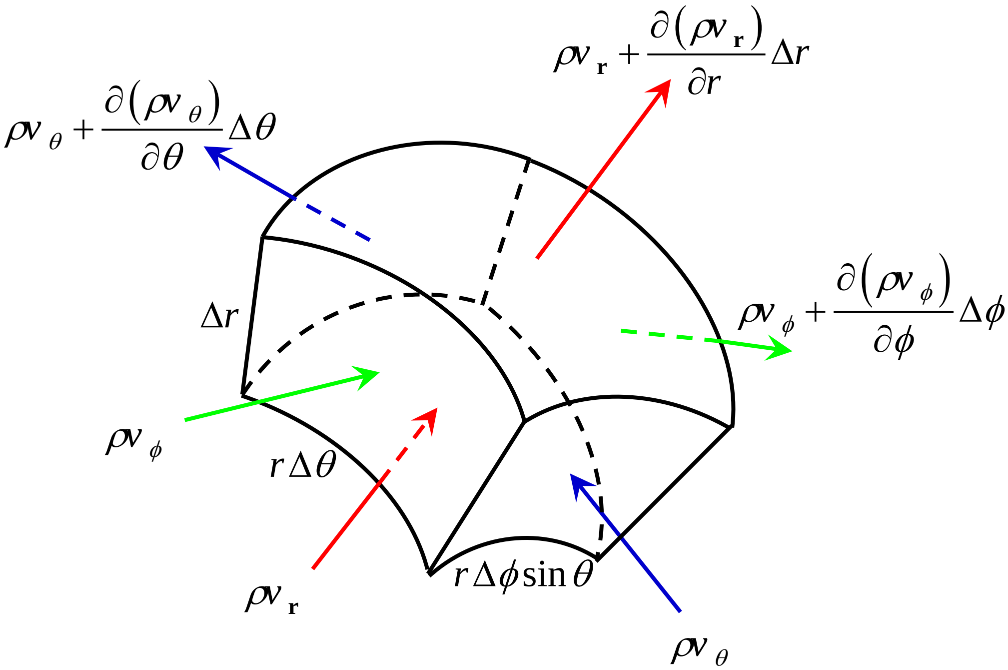


Shows that:

$$\frac{\partial \rho}{\partial t}+\frac{\partial(\rho v_{\mathbf{r}})}{\partial r}+\rho\frac{v_{\mathbf{r}}}{r}+\frac{1}{r}\frac{\partial(\rho v_{\theta})}{\partial \theta}+\frac{\partial(\rho v_{\mathbf{z}})}{\partial z}=0$$

<u>**Or**</u> using velocity vector $\mathbf{v}=v_{\mathbf{r}}\mathbf{e}_{\mathbf{r}}+v_{\theta}\mathbf{e}_{\theta}+v_{\mathbf{z}}\mathbf{e}_{\mathbf{z}}$ and nabal operator in cylindrical coordinate as; $\nabla=\frac{\partial}{\partial r}\mathbf{e}_{\mathbf{r}}+\frac{1}{r}\frac{\partial}{\partial \theta}\mathbf{e}_{\theta}+\frac{\partial}{\partial z}\mathbf{e}_{\mathbf{z}}$ and sub. in $\frac{\partial \rho}{\partial t}+(\nabla.\rho \mathbf{u})=0$.

<u>H.W</u> Continuity in spherical Coordinate for fluid element



Shows that:

$$\frac{\partial \rho}{\partial t}+\frac{1}{r^2}\frac{\partial(\rho v_r r^2)}{\partial r}+\frac{1}{r\sin\theta}\frac{\partial(\rho v_\theta \sin\theta)}{\partial \theta}+\frac{1}{r\sin\theta}\frac{\partial(\rho v_\phi)}{\partial \phi}=0$$

2.2 Conservation of Momentum

Momentum equation are derived based on **Newton's second Law** of motion;

$$\sum \mathbf{F_i} = \mathrm{m}\,\mathbf{a_i}$$

where $\sum \mathbf{F_i}$ = summation of forces in i- direction,

$\mathbf{a}_i$ = acceleration in i- direction,

m = mass of the fluid particle.

Forces: there are two types of forces on fluid particle:

Surface forces: There are acting on the surface of the element. surface forces are normal or tangential. (e.g. pressure forces, and viscous forces).

Body forces: They are acting through the material of the element. (e.g. gravity force, centrifugal force, Coriolis force, and electromagnetic force).

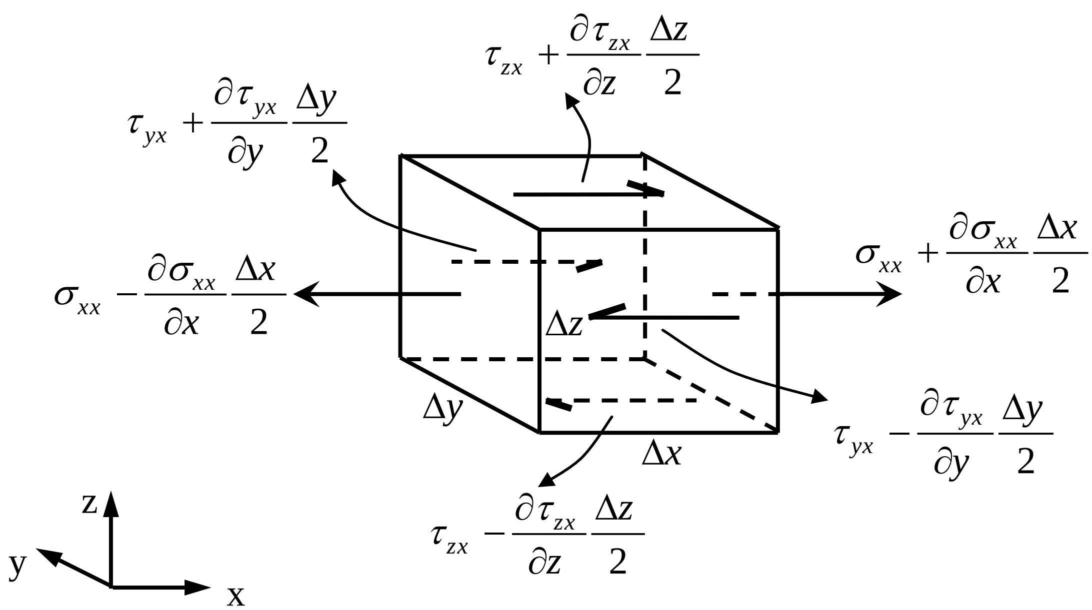
Herein, the following notation will be followed;

σ_{ij} & τ_{ij} : i -represents the direction of the normal line to the surface on which the stress is acting.
j - represents the stress direction.

τ = tangential stress
σ = normal stress.

In order to apply Newton second law, it is required to obtain the resultant of forces in x-, y -, and z - direction;

- To find net $\sum F_x$ forces;



The resultant of **surface forces** in x-direction;

$$F_x = \left(\sigma_{xx} + \frac{\partial \sigma_{xx}}{\partial x}\frac{\Delta x}{2}\right)\Delta y\,\Delta z - \left(\sigma_{xx} - \frac{\partial \sigma_{xx}}{\partial x}\frac{\Delta x}{2}\right)\Delta y\,\Delta z$$
$$+\left(\tau_{yx} + \frac{\partial \tau_{yx}}{\partial y}\frac{\Delta y}{2}\right)\Delta x\,\Delta z - \left(\tau_{yx} - \frac{\partial \tau_{yx}}{\partial y}\frac{\Delta y}{2}\right)\Delta x\,\Delta z$$
$$+\left(\tau_{zx} + \frac{\partial \tau_{zx}}{\partial z}\frac{\Delta z}{2}\right)\Delta x\,\Delta y - \left(\tau_{zx} - \frac{\partial \tau_{zx}}{\partial z}\frac{\Delta z}{2}\right)\Delta x\,\Delta y$$

$$\therefore F_x = \left(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right)\Delta x\,\Delta y\,\Delta z$$

The resultant of surface forces in x-direction per unit volume;

$$f_x = \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}$$

Similarly;

$$f_y = \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z}$$

$$f_z = \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z}$$

Also, it is proved that; for symmetrical stress;

$$\tau_{xy} = \tau_{yx} \quad , \quad \tau_{yz} = \tau_{zy} \quad , \text{ and } \quad \tau_{zx} = \tau_{xz}$$

Body forces

The mass of the fluid particle = $\rho \, \Delta x \, \Delta y \, \Delta z$

let a body force per unit volume in x - direction = X,

a body force per unit volume in y - direction = Y,

a body force per unit volume in z - direction = Z.

- If gravity is the only body force $X = \rho g_x$

$$\therefore \sum F_x \text{ (per unit volume) } = X + \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}$$

$$\therefore \sum F_y \text{ (per unit volume) } = Y + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z}$$

$$\therefore \sum F_z \text{ (per unit volume) } = Z + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z}$$

Acceleration;

$$a_x = \frac{Du}{Dt} \qquad \text{since } u = f\,(x, y, z, t)$$

$$\Delta u = \frac{\partial u}{\partial t}\Delta t + \frac{\partial u}{\partial x}\Delta x + \frac{\partial u}{\partial y}\Delta y + \frac{\partial u}{\partial z}\Delta z$$

$$\frac{\Delta u}{\Delta t} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\frac{\Delta x}{\Delta t} + \frac{\partial u}{\partial y}\frac{\Delta y}{\Delta t} + \frac{\partial u}{\partial z}\frac{\Delta z}{\Delta t}$$

$$\lim_{\Delta x, \Delta y, \Delta z \to 0} \frac{\Delta u}{\Delta t} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}$$

$$\therefore a_x = \frac{Du}{Dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}$$

Similarly;

$$a_y = \frac{Dv}{Dt} = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}$$

$$a_z = \frac{Dw}{Dt} = \underbrace{\frac{\partial w}{\partial t}}_{\text{Local Acc.}} + \underbrace{u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}}_{\text{Convective term}}$$

Where $\left(\frac{D}{Dt}\right)$ is called "**Substantive derivative** or **Material derivative.**

Apply Newton second law in x - direction;

$\sum F_x = m\,a_x$ where the mass is; $m = \rho\,\Delta x\,\Delta y\,\Delta z$

The above equation can be re-written per unit volume as;

$$\sum F_{x(\text{per unit volume})} = \rho\,a_x$$

$$\rho\frac{Du}{Dt} = X + \left(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right) \quad \text{in x-direction} \qquad \ldots(1a)$$

Similarly;

$$\rho\frac{Dv}{Dt} = Y + \left(\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z}\right) \quad \text{in y-direction} \qquad \ldots(1b)$$

$$\rho\frac{Dw}{Dt} = Z + \left(\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z}\right) \quad \text{in z-direction} \qquad \ldots(1c)$$

if the fluid is "frictionless" (μ = 0) all shearing stresses vanish $\left(\tau_{xy} = \tau_{yz} = \tau_{xz} = 0\right)$; only the normal stresses remain in this case, and they are, moreover, equal.

$$\sigma_{ii} \neq 0 \quad \Rightarrow \quad \sigma_{xx} = \sigma_{yy} = \sigma_{zz} = -\mathrm{p}$$

where p is "pressure" means the normal force per unit area acted on the fluid particle.

As the fluid is static, the pressure of the fluid is called **hydrostatic pressure**. Since the fluid is motionless, the fluid is in equilibrium, therefore the;

(Hydrostatic pressure = thermodynamic pressure)

As the fluid is in motion, the 3 principal normal stresses are not necessary equal, and the fluid is not in equilibrium. Therefore, the hydrodynamic pressure is defined by

$$\textbf{Hydrostatic pressure} \equiv -\frac{1}{3}\left(\sigma_{xx} + \sigma_{yy} + \sigma_{zz}\right) = -\frac{1}{3}\mathrm{tr}\,\boldsymbol{\tau}$$

and which is not equal to the thermodynamic pressure either. Later we will prove that

$$\textbf{Hydrostatic pressure) = thermodynamic pressure + } \frac{1}{3}\lambda'$$

Rate of linear deformation

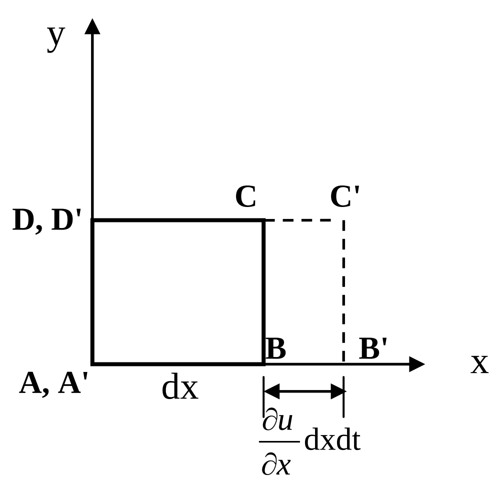
The rate of linear deformation of a fluid element has none components in three dimensions, six of which are independent in isotropic fluids.

the rate of elongation in the x- direction;

$$\varepsilon_{\mathrm{xx}} = \frac{\textit{rate change of length}}{\textit{origin length}} = \frac{\frac{\partial u}{\partial x}\mathrm{dxdt}}{\mathrm{dxdt}} = \frac{\partial u}{\partial x}$$

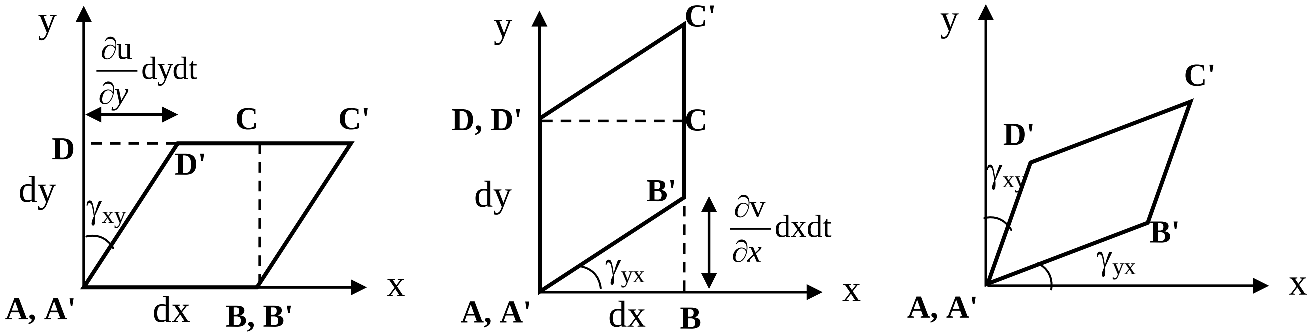
Similarly;

$$\varepsilon_{\mathrm{yy}} = \frac{\partial v}{\partial y} \quad \text{and} \quad \varepsilon_{\mathrm{zz}} = \frac{\partial w}{\partial z}$$



Volumetric deformation = $\varepsilon_{\mathrm{v}} = \varepsilon_{\mathrm{xx}} + \varepsilon_{\mathrm{yy}} + \varepsilon_{\mathrm{zz}} = \mathrm{div}(\mathbf{v}) = \nabla.\mathbf{v}$

velocity strain → the rate of change in angle



$$2\varepsilon_{xy} = \frac{\frac{\partial u}{\partial y}dydt}{dydt} + \frac{\frac{\partial v}{\partial x}dxdt}{dxdt} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

$$\therefore \varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \qquad \text{...(2a)}$$

Similarly;

$$\varepsilon_{yz} = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) \qquad \text{...(2b)}$$

$$\varepsilon_{zx} = \frac{1}{2}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) \qquad \text{...(2c)}$$

where; $\varepsilon_{xy} = \varepsilon_{yx}$, $\varepsilon_{xz} = \varepsilon_{zx}$, and $\varepsilon_{zy} = \varepsilon_{yz}$

Stress - strain relation

Solids when an elastic solid is subjected to normal stresses $\sigma_{xx}, \sigma_{yy}, and\ \sigma_{zz}$, the corresponding strain;

$$\varepsilon_{xx} = \frac{1}{E}\left(\sigma_{xx} - \upsilon\left(\sigma_{yy} + \sigma_{zz}\right)\right) \qquad \text{...(3a)}$$

$$\varepsilon_{yy} = \frac{1}{E}\left(\sigma_{yy} - \upsilon\left(\sigma_{xx} + \sigma_{zz}\right)\right) \qquad \text{...(3b)}$$

$$\varepsilon_{xx} = \frac{1}{E}\left(\sigma_{zz} - \upsilon\left(\sigma_{xx} + \sigma_{yy}\right)\right) \qquad \text{...(3c)}$$

$$\therefore \varepsilon_{v} = \frac{1}{E}(1-2\upsilon)\left(\sigma_{xx} + \sigma_{yy} + \sigma_{zz}\right)$$

$$\therefore \sigma_{xx} + \sigma_{yy} + \sigma_{zz} = \frac{E\varepsilon_v}{(1-2\upsilon)} = 3\bar{\sigma} \qquad ...(4)$$

where, $\bar{\sigma}$ = the average of normal stresses $= \dfrac{\sigma_{xx} + \sigma_{yy} + \sigma_{zz}}{3}$

The modulus of shear is; $G = \dfrac{E}{2(1+\upsilon)}$

From eq. (4) we have ; $\bar{\sigma} = \dfrac{2G}{3}\dfrac{(1+\upsilon)}{(1-2\upsilon)}\varepsilon_v \qquad ...(5)$

where the stress in terms strain, from elastic solid;

$$\because \sigma_{xx} = \frac{E}{(1+\upsilon)(1-2\upsilon)}\left[(1-\upsilon)\varepsilon_{xx} + \upsilon\left(\varepsilon_{yy} + \varepsilon_{zz}\right)\right]$$

$$\therefore \sigma_{xx} = 2G\varepsilon_{xx} + 2G\frac{\upsilon}{1-2\upsilon}\left(\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}\right) \qquad ...(6)$$

Subtract eq. (5) from eq. (6), gives;

$$\sigma_{xx} - \bar{\sigma} = 2G\varepsilon_{xx} + 2G\frac{\upsilon}{1-2\upsilon}\varepsilon_v - \frac{2G}{3}\frac{(1+\upsilon)}{(1-2\upsilon)}\varepsilon_v$$

From the above eq., we have;

$$\sigma_{xx} - \bar{\sigma} = 2G\varepsilon_{xx} - \frac{2}{3}G\,\varepsilon_v \qquad ...(7a)$$

Similarly;

$$\sigma_{yy} - \bar{\sigma} = 2G\varepsilon_{yy} - \frac{2}{3}G\,\varepsilon_v \qquad ...(7b)$$

$$\sigma_{zz} - \bar{\sigma} = 2G\varepsilon_{zz} - \frac{2}{3}G\,\varepsilon_v \qquad ...(7c)$$

Shear stresses;

$$\tau_{xy} = 2G\varepsilon_{xy} = G\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \qquad ...(8a)$$

$$\tau_{yz} = G\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) \qquad ...(8b)$$

$$\tau_{xy} = G\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) \qquad ...(8c)$$

Fluid;

Stress - rate of strain relation;

For fluid G is replaced by μ and strain is replaced by rate of strain. Also take $\bar{\sigma} = -p$, i.e., the average normal stress represents the normal pressure. Therefore Eqs. (7) and (8) can be written as;

$$\sigma_{xx} = -p + 2\mu\frac{\partial u}{\partial x} - \frac{2}{3}\mu\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) \quad \Rightarrow \quad \sigma_{xx} = -p + 2\mu\frac{\partial u}{\partial x} - \frac{2}{3}\mu \,\mathrm{div}\,\mathbf{V}$$

$$\therefore \sigma_{xx} = -p + \mu\left(2\frac{\partial u}{\partial x} - \frac{2}{3}\mu(\nabla.\mathbf{V})\right) \qquad \text{...(9a)}$$

Similarly;

$$\sigma_{yy} = -p + \mu\left(2\frac{\partial v}{\partial y} - \frac{2}{3}\mu(\nabla.\mathbf{V})\right) \qquad \text{..(9b)}$$

$$\sigma_{zz} = -p + \mu\left(2\frac{\partial w}{\partial z} - \frac{2}{3}\mu(\nabla.\mathbf{V})\right) \qquad \text{...(9c)}$$

and;

$$\tau_{xy} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \qquad \text{...(10a)}$$

$$\tau_{yz} = \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) \qquad \text{...(10b)}$$

$$\tau_{xy} = \mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) \qquad \text{...(10c)}$$

Sub. of Eqs. (9) and (10) into Eq. (1) yields;

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = X - \frac{\partial p}{\partial x} + \frac{\partial}{\partial x}\left[\mu\left(2\frac{\partial u}{\partial x} - \frac{2}{3}\mathrm{div}\,\mathbf{V}\right)\right] + \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right] + \frac{\partial}{\partial z}\left[\mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)\right] \qquad (11a)\ \textit{x-dir.}$$

$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) = Y - \frac{\partial p}{\partial y} + \frac{\partial}{\partial y}\left[\mu\left(2\frac{\partial v}{\partial y} - \frac{2}{3}\mathrm{div}\,\mathbf{V}\right)\right] + \frac{\partial}{\partial x}\left[\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right] + \frac{\partial}{\partial z}\left[\mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)\right] \qquad (11b)\ \textit{y-dir.}$$

$$\rho\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right)=Z-\frac{\partial p}{\partial z}+\frac{\partial}{\partial z}\left[\mu\left(2\frac{\partial w}{\partial z}-\frac{2}{3}\text{div}\,\mathbf{V}\right)\right]$$
$$+\frac{\partial}{\partial x}\left[\mu\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right)\right]+\frac{\partial}{\partial y}\left[\mu\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)\right] \quad (11c)\ \textit{z-dir.}$$

The above equation are momentum equations, in x-, y -, and z- directions or; they are called "**Navier-Stokes equations**" (NSE) or equations of motion.

where the body forces $X=\rho g_x$, $Y=\rho g_y$, and $Z=\rho g_z$ (if gravity effect only)

- There are 3 Spatial (z, y, z) + 1 temporal (t) variables = 4 independent variables
- There are 3 velocity components (u, v, w) + pressure + density _ viscosity = 6 dependent variables.
- Navier Stokes equations (3) + Continuity equation (1) + Equation of state (1)

2.3 Conservation of Energy

For incompressible viscous flow , the energy equation is;

$$\rho c_P\left(\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}+w\frac{\partial T}{\partial z}\right)=\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)+\frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)+\mu\Phi$$

For constant properties (k = constant), yields;

$$\rho c_P\left(\frac{\partial T}{\partial t}+\underbrace{u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}+w\frac{\partial T}{\partial z}}_{\text{Convective heat}}\right)=k\underbrace{\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}\right)}_{\text{Condition heat}}+\underbrace{\mu\Phi}_{\text{Viscous heat}}$$

Where;

$$\Phi=2\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2+\left(\frac{\partial w}{\partial z}\right)^2\right]+\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right)^2+\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)^2$$

Φ = the viscous - dissipation heat (using this term for high Re)

c_p = specific heat, and k = thermal conductivity.

Characteristics of NSEs.

- Non-linear equations
- Partial differential equations
- Used for; - compressible and uncompressible flows
  - viscous flow
  - three dimensional flow system
  - time dependent → unsteady flow system.

2.4 The NSEs in Vector and Tensor Forms

Can be rewriting the NSEs in general form in **vector forms**:

$$\rho \frac{DV}{Dt} = \rho g - \nabla p + (\nabla . \boldsymbol{\tau})$$

where $\boldsymbol{\tau} = -\mu\left(\nabla \mathbf{V} + (\nabla \mathbf{V})^T\right) + \frac{2}{3}\mu(\nabla . \mathbf{V})$

and the term $\left(\nabla \mathbf{V} + (\nabla \mathbf{V})^T\right)$ is **rate of strain**

Tensor form:

$$\rho\left(u_{i,t} + u_j u_{i,j}\right) = \rho g_i + \sigma_{ij,j}$$

where $\sigma_{ij} = -\delta_{ij} P + \mu\left(u_{i,j} + u_{j,i}\right) - \frac{2}{3}\mu \delta_{ij} u_{kk}$

$$\rho\left(u_{i,t} + u_j u_{i,j}\right) = \rho g_i + -\delta_{ij} P_{,j} + \left[\mu\left(u_{i,j} + u_{j,i}\right) - \frac{2}{3}\mu \delta_{ij} u_{kk}\right]_{,j}$$

Energy Equation:

Vector form:

$$\rho c_P \frac{DT}{Dt} = \nabla . k \nabla T + \mu[\boldsymbol{\tau} : \nabla \mathbf{V}]$$

for constant properaties (k = cont.)

$$\rho c_P \frac{DT}{Dt} = k \nabla^2 T + \mu[\boldsymbol{\tau} : \nabla \mathbf{V}]$$

2.4.1 Simplifications for NSEs

(i) For incompressible flow (ρ = c) and negligible body forces

continuity eq. in vector form as ; $\frac{\partial \rho}{\partial t} + \rho(\nabla.\mathbf{V}) = 0$

(ii) For incompressible and steady flow with negligible body forces;

- Continuity eq. in **vector** form $\nabla.\mathbf{V} = 0$ or tensor form as $u_{i,i} = 0$
- NSE in vector form as;

$$\rho(\mathbf{V}.\nabla u) = -\nabla p + (\nabla.\boldsymbol{\tau})$$

where $\boldsymbol{\tau} = -\mu\left(\nabla\mathbf{V} + (\nabla\mathbf{V})^T\right)$

- In Cartesian coordinate for x-dir. as;

$$\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = \frac{\partial p}{\partial x} + \frac{\partial}{\partial x}\left[\mu\left(2\frac{\partial u}{\partial x}\right)\right] + \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right] + \frac{\partial}{\partial z}\left[\mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)\right]$$

Similarly for equation of motion in y and z

- NSE in Tensor form

$$\rho(u_j u_{i,j}) = -\delta_{ij}P_{,j} + \left[\mu(u_{i,j} + u_{j,i})\right]_{,j}$$

(iii) For incompressible with Newtonian fluid (μ = constant) and unsteady flow.

$$\rho\frac{DV}{Dt} = -\nabla p + \mu(\nabla^2\mathbf{V})$$

Steady flow $\rho(\mathbf{V}.\nabla u) = -\nabla p + \mu(\nabla^2\mathbf{V})$

- **Tensor** form:

$$\rho(u_j u_{i,j}) = -\delta_{ij}P_{,j} + \mu u_{i,jj}$$

In Cartesian coordinate in x-dir

$$\rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)=-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)$$

Similarly for y & z directions

(iv) For frictionless ($\tau = 0$) for unsteady with body forces the term $(\nabla.\boldsymbol{\tau})=0$

$$\rho\frac{DV}{Dt}=\rho g-\nabla p$$

This equation is the famous Euler equation, first derived in 1755. It has been widely used for describing flow systems in which viscous effects are relatively unimportant. (ideal flow).

(v) For two dimension (2D) flow in x-y plane; w=0 and there are no variations with respect to z;

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=\rho g_x-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right) \quad \text{x-dir.}$$

$$\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right)=\rho g_y-\frac{\partial p}{\partial y}+\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right) \quad \text{y-dir.}$$

where $\nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$ Laplace operator in 2D

2.5 Types of Coordinate Systems

There are two types of coordinate systems using for fluid flow;

1- Cartesian coordinate system (x, y, z),

2- Cylindrical coordinate system (r, θ, z).

The relation between the above two system is mentioned in previous section (table 1).

Also, Cylindrical coordinate system is divided into;

I - Polar coordinate system (r, θ) (There is no variation with respect to z)

II- Axisymmetrical coordinate system (r, z) (There is no variation with respect to θ)

2.6 The NSEs in Cylindrical coordinate

If r, q, and z are the 3-D cylindrical coordinate and v_r, v_θ, and v_z denote the velocity components in the respective directions,

- For Newtonian (μ = cont.), incompressible viscous flow, steady, Navier stokes equations are;

$$\rho\left(v_r\frac{\partial v_r}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta}+v_z\frac{\partial v_r}{\partial z}-\frac{v_\theta^2}{r}\right)=\rho g_r-\frac{\partial p}{\partial r}+\mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_r)\right)+\frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2}+\frac{\partial^2 v_r}{\partial z^2}-\frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}\right]\qquad r-\mathrm{dir}$$

$$\rho\left(v_r\frac{\partial v_\theta}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta}+v_z\frac{\partial v_\theta}{\partial z}+\frac{v_r v_\theta}{r}\right)=\rho g_\theta-\frac{1}{r}\frac{\partial p}{\partial \theta}+\mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_\theta)\right)+\frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2}+\frac{\partial^2 v_\theta}{\partial z^2}+\frac{2}{r^2}\frac{\partial v_r}{\partial \theta}\right]\qquad \theta-\mathrm{dir}$$

$$\rho\left(v_r\frac{\partial v_z}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta}+v_z\frac{\partial v_z}{\partial z}\right)=\rho g_z-\frac{\partial p}{\partial z}+\mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2}+\frac{\partial^2 v_z}{\partial z^2}\right]\qquad z-\mathrm{dir}$$

- **Continuity eq.**

$$\frac{\partial v_r}{\partial r}+\frac{v_r}{r}+\frac{1}{r}\frac{\partial v_\theta}{\partial \theta}+\frac{\partial v_z}{\partial z}=0$$

- **The stress components in cylindrical coordinate as;**

$$\tau_{rr}=2\mu\frac{\partial v_r}{\partial r}\quad,\ \tau_{\theta\theta}=2\mu\left(\frac{1}{r}\frac{\partial v_\theta}{\partial \theta}+\frac{v_r}{r}\right)\quad \text{and}\quad \tau_{zz}=2\mu\frac{\partial v_z}{\partial z}$$

$$\tau_{r\theta}=\mu\left[r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right)+\frac{1}{r}\frac{\partial v_r}{\partial \theta}\right]$$

$$\tau_{\theta z}=\mu\left[\frac{\partial v_\theta}{\partial z}+\frac{1}{r}\frac{\partial v_z}{\partial \theta}\right]$$

$$\tau_{rz}=\mu\left[\frac{\partial v_r}{\partial z}+\frac{\partial v_z}{\partial r}\right]$$

Axisymmetric Problems

Here, $v_\theta = 0$ and there is no variation with respect to $\theta = 0$. Equation of motion are;

$$\rho\left(v_r\frac{\partial v_r}{\partial r}+v_z\frac{\partial v_r}{\partial z}\right)=\rho g_r-\frac{\partial p}{\partial r}+\mu\left[\frac{\partial^2 v_r}{\partial r^2}+\frac{1}{r}\frac{\partial v_r}{\partial r}-\frac{v_r}{r^2}+\frac{\partial^2 v_r}{\partial z^2}\right] \qquad r-\text{dir}$$

$$\rho\left(v_r\frac{\partial v_z}{\partial r}+v_z\frac{\partial v_z}{\partial z}\right)=\rho g_z-\frac{\partial p}{\partial z}+\mu\left[\frac{\partial^2 v_z}{\partial r^2}+\frac{1}{r}\frac{\partial v_z}{\partial r}+\frac{\partial^2 v_z}{\partial z^2}\right] \qquad z-\text{dir}$$

- **Continuity eq.**

$$\frac{\partial v_\mathbf{r}}{\partial r}+\frac{v_\mathbf{r}}{r}+\frac{\partial v_\mathbf{z}}{\partial z}=0$$

- **The stress components in cylindrical coordinate as;**

$$\tau_{\mathbf{rr}}=2\mu\frac{\partial v_\mathbf{r}}{\partial r} \qquad , \tau_{\theta\theta}=2\mu\left(\frac{v_r}{r}\right) \qquad \text{and} \qquad \tau_{zz}=2\mu\frac{\partial v_z}{\partial z}$$

$$\tau_{r\theta}=0$$

$$\tau_{\theta z}=0$$

$$\tau_{rz}=\mu\left[\frac{\partial v_r}{\partial z}+\frac{\partial v_z}{\partial r}\right]$$

Boundary conditions for viscous fluid flow

The equations of motions will require mathematically tenable and physically realistic boundary conditions. The fluid flow, there are five types of boundary considered:

1. A solid surface (which may be porous)
2. A free liquid surface
3. A liquid - vapour interface
4. An inlet or exit section

(i) Conditions at a Solid surface

Wall boundary conditions depend upon whether the fluid is a liquid or gas. For macroflows, system dimensions are large compared to molecular spacing, so that both liquid and gas particles contacting the wall must essentially be in equilibrium with the solid. For the solid surface;

$V_{fluid} = V_{sol}$ (***no-slip*** condition)

$T_{fluid} = T_{sol}$ (***no-temperature-jump*** condition)

However, certain liquid/solid combinations are known to ***slip*** under small-scale ***microflow*** conditions. One way to characterize slip in liquid is the slip length L_{slip} relating slip velocity to the local velocity gradient, a model first suggested by Navier himself:

$$u_{wall} = L_{slip} \left(\frac{\partial u}{\partial n} \right)_{wall}$$

The slip length depends upon the liquid, the geometry, and the shear rate.

In many cases, of course, the coordinate system is such that the wall is stationary, so that the velocity conditions is simply $V_{fluid} = 0$.

Conditions at a Permeable Wall

In the event that the wall is porous and can permit fluid to pass through. The proper conditions are complicated by the type of porosity of the wall, but in general, we assume;

$$V_{tangential} = 0 \quad (\textit{\textbf{no-slip}} \text{ condition})$$

$$V_{normal} \neq 0 \quad (\textit{\textbf{Flow through wall}})$$

➢ The temperature condition is also complicated by a porous wall.

❶ For wall ***suction***, where the fluid leaves the main flow and passes into the wall, we assume;

$$T_{Fluid} = T_w \quad \text{(suction)}$$

❷ For ***injection*** through a porous wall into the main stream (sometime called transpiration), the injected fluid may be, say a coolant at a temperature. A good approximation using;

$$k\left.\frac{dT}{dy}\right|_w \approx \rho_w V_n Cp\left(T_w - T_{coolant}\right) \quad \text{(Injection)}$$

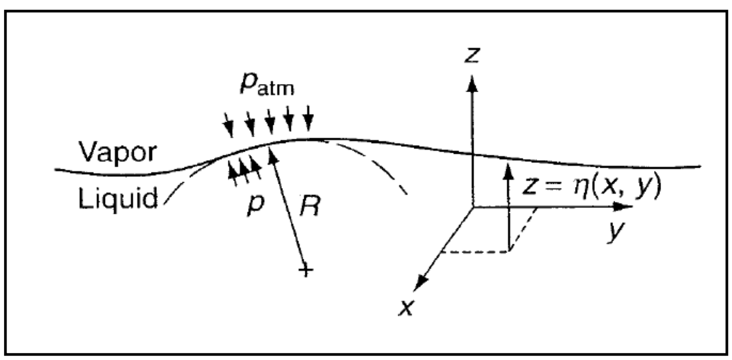
where $\rho_w V_n$ is the mass flow of coolant per unit area through the wall.

(ii) Conditions at a Free Liquid Surface

There are many flow problems where the liquid fluids ends, not at a solid wall, but at an open or free surface exposed to an atmosphere of either gas or vapor. Two cases

1. The ideal or classic free surface that exerts only a known pressure on the liquid boundary.
2. Complicated case where the atmosphere exerts not only pressure but also shear, heat flux, and mass flux at the surface.

ideal free surface, $z = \eta(x, y)$



In ideal free surface, the two required conditions:

(1) The fluid particles at the surface must remain attached (Kinematic condition).

(2) The liquid and the atmospheric pressure must balance except for surface-tension effects.

$$w(x,y,\eta)=\frac{D\eta}{Dt}=\frac{\partial\eta}{\partial t}+u\frac{\partial\eta}{\partial x}+v\frac{\partial\eta}{\partial y}$$

The pressure equilibrium; $p(x,y,\eta)=p_a-\tau\left(\frac{1}{R_x}+\frac{1}{R_y}\right)$

where R_x and R_y are the radii of curvature of the surface and

τ : is the coefficient of surface tension of the interface. (N/m)

For 2D surface $\eta=\eta(x)$ only the above eq. becomes;

$$p(x,\eta)=p_a-\frac{\tau\, d^2\eta/dx^2}{\left[1+(d\eta/dx)^2\right]^{3/2}}$$

We see from this relation that,

at $p<p_a$ (concave upward, positive curvature) smile interface

at $p>p_a$ (concave downward) frowning interface

Note:

In large-scale problems, such as open-channel or river flow, the free surface deforms only slightly and surface tension effects are negligible;

$$w\approx\frac{\partial\eta}{\partial t}\qquad \text{and} \qquad p\approx p_a$$

(iii) Conditions at a Liquid-vapor of Liquid-Liquid interface

The term *free surface* means that the gas lying over the liquid has no effect except to impose pressure on the interface. Heat transfer and shear effects are negligible.

In a true liquid-vapor or liquid-liquid interface, the upper fluid is strongly coupled and exerts kinematic, stress, and energy constraints on the lower fluid. The motions of the two fluids are solved simultaneously and must match in certain ways at the interface.

at true fluid-fluid interface : $V_1 = V_2$, $\tau_1 = \tau_2$, $T_1 = T_2$

the normal velocities match; $V_{n1} = V_{n2}$

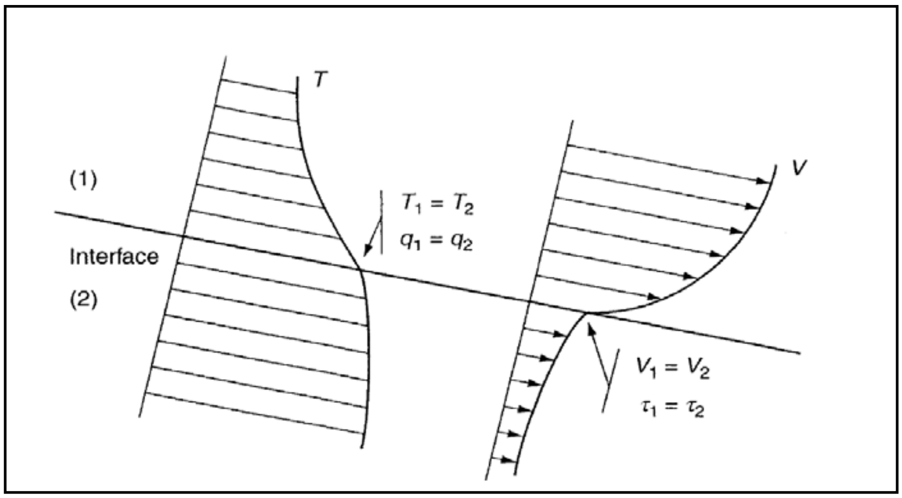
the tangential velocities also match; $V_{t1} = V_{t2}$

Although velocities and temperature are continuous across the interface, their slopes generally do not match because of differing transport coefficients;

$$\tau_1 = \mu_1 \frac{\partial V_{t1}}{\partial n} = \tau_2 = \mu_2 \frac{\partial V_{t2}}{\partial n}$$

$$q_1 = -k_1 \frac{\partial T_1}{\partial n} = q_2 = -k_2 \frac{\partial T_2}{\partial n}$$

where n is the coordinate normal to the interface. The slopes are not equal if $\mu_1 \neq \mu_2$ or $k_1 \neq k_2$



Since k and μ for a vapor are usually much smaller than for a liquid, we can often approximate liquid conditions at the interface as;

$$\left.\frac{\partial V_t}{\partial n}\right|_{liq} \approx 0 \qquad \left.\frac{\partial T}{\partial n}\right|_{liq} \approx 0$$

Finally, if there is evaporation, condensation, or diffusion at the interface, the mass flows must also balance, $\dot{m}_1 = \dot{m}_2$

(iv) Inlet and Exit boundary conditions

At the inlet, we would specify the distributions of V, T, and P. Often the inlet pressure is assumed uniform as a simplification. At the exit, we specify V and T. No exit condition is required upon P, which is then found from the solution.

(v) conditions at a Symmetry plane

The symmetry plane boundary condition imposes constraints that 'mirror' the flow on either side of it. The symmetry boundary condition can therefore be summarized as follows:

❶ zero normal velocity at a symmetry plane;

$$V_n = 0$$

❷ zero normal gradients of all scalar variables at a symmetry plane;

$$\frac{\partial \phi}{\partial n} = 0$$

Exact Solutions of Navier-Stokes Equations

The NSEs for a general viscous flow are nonlinear partial differential equations, for most applications, it is impossible to obtain solution for the complete NSEs, even for constant property flows. In most flow, problems simplifying assumptions are made so that approximate solution can be obtained in order to generate design information.

As we might expect, almost all the known particular solutions are for incompressible Newtonian flow with constant transport properties, for which the basic equations of continuity, NSEs, and energy reduce to;

Continuity: $$\nabla.\mathbf{V} = 0 \tag{3.1}$$

Momentum : $$\rho \frac{DV}{Dt} = \rho g_i - \nabla p + \mu\left(\nabla^2 \mathbf{V}\right) \tag{3.2}$$

Where Φ = the viscous - dissipation heat (using this term for high Re)

Basically, there are two types of exact solution of Eq. (3.2)

I - Linear solution, where the convective acceleration $\left(\mathbf{V}.\nabla\right)$ vanishes.

ii- Non-linear solution, where $\left(\mathbf{V}.\nabla\right)$ does not vanish.

It is also possible to classify solution by the type or geometry of flow involved:

3.1 Couette (wall-driven) steady flow.

(i) Steady flow between a fixed and moving plate

In figure (1), two infinite plate are 2H apart, and the upper plate moves at speed U_1 relative to the lower. The pressure p is assumed constant. These boundary conditions are independent of x or z (infinite plate).

Take the NSE for one dimensional steady flow for incompressible fluid in x-dir. only,

$$\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = \rho g_x - \frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

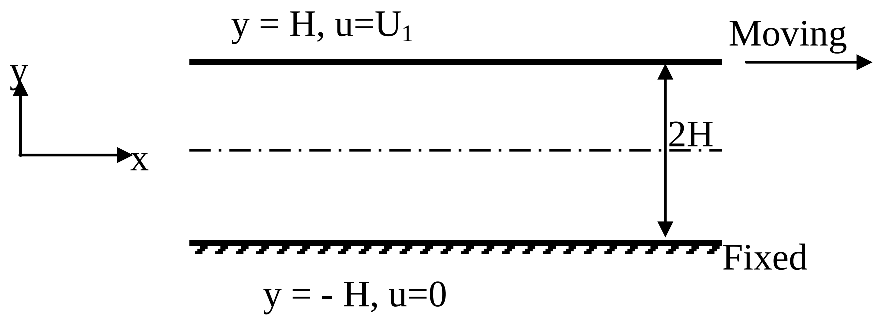
Can be simplification,

v = w = 0, and $\frac{\partial u}{\partial z} = 0$, No body force, $\rho g_x=0$, and because p = c yield to $\frac{\partial p}{\partial x}=0$

The continuity , and momentum can be reduce as;

Continuity eq. $\frac{\partial u}{\partial x}=0$ (1)

Momentum eq. $0=\mu\frac{\partial^2 u}{\partial y^2}$ (2)



Using eq. (2)

$$0=\mu\frac{\partial^2 u}{\partial y^2}\xrightarrow{\int}\frac{\partial^2 u}{\partial y^2}=c_1\xrightarrow{\int}u=c_1 y+c_2 \quad (4)$$

B.C. (1) at y = -H , u = 0,

$$u=c_1 y+c_2 \rightarrow 0=-c_1 H+c_2 \rightarrow \therefore c_2=c_1 H$$

B. C. (2) at y = H, u = U_1

$$U_1=c_1 H+c_1 H \rightarrow \therefore c_1=\frac{U_1}{2H} \quad \text{and} \quad c_2=\frac{U_1}{H} \quad \text{Subs. in eq. (4)}$$

The velocity distribution as;

$$\therefore u=\frac{U_1}{2}\left(\frac{y}{H}+1\right)$$

To find the shear stress

$$\tau_{xx}=2\mu\frac{\partial u}{\partial x} \qquad \rightarrow\therefore \tau_{xx}=0$$

$$\tau_{xy} = \tau_{yx} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \qquad \rightarrow \ \therefore \tau_{xy} = \tau_{yx} = \mu\frac{U}{2H} = \text{const.}$$

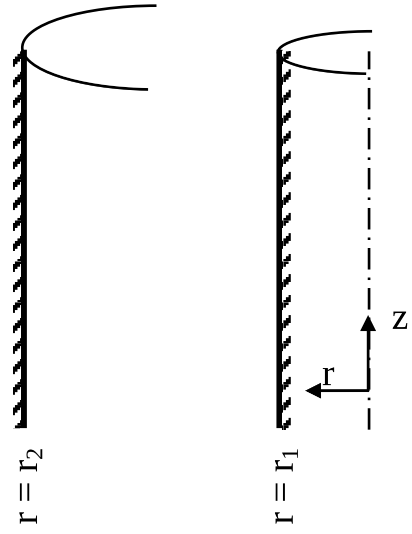
$$\tau_{xz} = \tau_{zx} = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \qquad \rightarrow \therefore \tau_{xz} = \tau_{zx} = 0$$

The shear stress is constant throughout the fluid flow.

H.W. find the velocity distribution and shear stress for the about Ex. with p = variable with x-dir. $\frac{\partial p}{\partial x} \neq 0$

(ii) Axially moving concentric cylinders

Consider two long concentric cylinder with a viscous fluid between them, as in figure below, Let with the inner ($r = r_1$) cylinder move axially at $u = U_1$ or the outer ($r = r_2$) cylinder move at $u = U_2$, as shown. the pressure gradient and gravity are assumed to be negligible. The no-slip condition will set the fluid into steady motion u(r), and u_θ and u_r will be zero.



Continuity eq.

$$\frac{\partial v_{\mathbf{r}}}{\partial r} + \frac{v_{\mathbf{r}}}{r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_{\mathbf{z}}}{\partial z} = 0$$

$$Reduce\ to \quad \therefore \frac{\partial v_{\mathbf{z}}}{\partial z} = 0$$

Take momentum eq. in z - dir. only

$$\rho\left(v_r\frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z\frac{\partial v_z}{\partial z}\right) = \rho g_z - \frac{\partial p}{\partial z} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2}\right] \qquad z - \text{dir}$$

reduce to

$$0 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) \qquad \textbf{(1)}$$

Using eq. (1)

$$\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)=0 \quad \xrightarrow{\int} \quad r\frac{\partial v_z}{\partial r}=c_1 \quad \xrightarrow{\int} \quad v_z=c_1\ln r+c_2 \qquad (2)$$

(1) If the inner cylinder moves, and outer cylinder fixed (no-slip condition)

Apply the B.C. (1) at r = r_2 , v_z = 0

$$v_z=c_1\ln r+c_2 \quad \Rightarrow \quad 0=c_1\ln r_2+c_2 \quad \rightarrow \therefore c_2=-c_1\ln r_2$$

Apply the B.C. (2) at r = r_1 , v_z = U_1

$$v_z=c_1\ln r-c_1\ln r_2 \quad \Rightarrow \quad U_1=c_1\ln r_1-c_1\ln r_2=c_1\ln\frac{r_1}{r_2}$$

$$\therefore c_1=\frac{U_1}{\ln\frac{r_1}{r_2}} \quad \text{and} \quad c_2=-c_1\ln r_1 \quad \text{subs. in eq. (2)}$$

$$v_z=\frac{U_1}{\ln\left(r_1/r_2\right)}\ln r-\frac{U_1}{\ln\left(r_1/r_2\right)}\ln r_1=\frac{U_1}{\ln\left(r_1/r_2\right)}\left(\ln r-\ln r_1\right)$$

The velocity distribution is $\therefore v_z=U_1\frac{\ln\left(r_1/r\right)}{\ln\left(r_2/r_1\right)}$

- **The stress components in cylindrical coordinate as;**

$$\tau_{zz}=2\mu\frac{\partial v_z}{\partial z} \qquad \rightarrow \therefore \tau_{zz}=0$$

$$\tau_{\theta z}=\mu\left[\frac{\partial v_\theta}{\partial z}+\frac{1}{r}\frac{\partial v_z}{\partial \theta}\right] \qquad \rightarrow \therefore \tau_{\theta z}=0$$

$$\tau_{rz}=\mu\left[\frac{\partial v_r}{\partial z}+\frac{\partial v_z}{\partial r}\right] \qquad \rightarrow \therefore \tau_{rz}=\frac{-\mu U_1}{r\ln\left(r_2/r_1\right)}$$

<u>**H.W**</u> Find velocity distribution at inner cylinder is fixed and outer cylinder is moved.

(iii) Flow between rotating concentric cylinders

Consider the steady flow maintained between two concentric cylinders by steady angular velocity of one or both cylinders. Let the inner cylinder have radius r_1, angular velocity ω_1, and the outer cylinder has r_2, ω_2, respectively. The geometry is such that the only nonzero velocity component is v_θ and the variable v_θ and p must be functions only of radius r. The equations of motion in polar coordinates

Continuity eq. $\dfrac{\partial v_r}{\partial r} + \dfrac{v_r}{r} + \dfrac{1}{r}\dfrac{\partial v_\theta}{\partial \theta} + \dfrac{\partial v_z}{\partial z} = 0 \xrightarrow{\text{reduce to}} \therefore \dfrac{\partial v_\theta}{\partial \theta} = 0$

Momentum eq.

$$\rho\left(v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} + v_z\frac{\partial v_r}{\partial z} - \frac{v_\theta^2}{r}\right) = \rho g_r - \frac{\partial p}{\partial r} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r v_r)\right) + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2} + \frac{\partial^2 v_r}{\partial z^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}\right] \qquad r-\text{dir}$$

Reduce to $\rho\dfrac{v_\theta^2}{r} = \dfrac{\partial p}{\partial r}$ $\qquad r-\text{dir}.$

$$\rho\left(v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + v_z\frac{\partial v_\theta}{\partial z} + \frac{v_r v_\theta}{r}\right) = \rho g_\theta - \frac{1}{r}\frac{\partial p}{\partial \theta} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r v_\theta)\right) + \frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{\partial^2 v_\theta}{\partial z^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta}\right] \qquad \theta-\text{dir}$$

Reduce to $0 = \dfrac{\partial}{\partial r}\left(\dfrac{1}{r}\dfrac{\partial}{\partial r}(r v_\theta)\right) \rightarrow \dfrac{\partial^2 v_\theta}{\partial r^2} + \dfrac{\partial}{\partial r}\left(\dfrac{v_\theta}{r}\right) = 0 \qquad \theta-\text{dir}$

With boundary conditions

At $r = r_1$: $\quad v_\theta = r_1\omega_1$, $p = p_1$
At $r = r_2$: $\quad v_\theta = r_2\omega_2$

Show that the solution to the q -dir. of momentum equation has the form

$$v_\theta = c_1 r + \frac{c_2}{r}$$

After find c_1 and c_2 from the boundary conditions. The velocity distribution as;

$$v_\theta = \omega_1 r_1 \frac{r_2/r - r/r_2}{r_2/r_1 - r_1/r_2} + \omega_2 r_2 \frac{r/r_1 - r_1/r}{r_2/r_1 - r_1/r_2}$$

Some special cases are of interest. In the limit as the inner cylinder vanishes ($r_1 = w_1 = 0$) show that the velocity distribution as;

$$v_\theta = \omega_2 r$$

3.2 Poiseuille (pressure-driven) steady duct flows.

Whereas Couette flows are driven by moving walls, Poiseuille flows are generated by pressure gradients, with applications primarily to ducts. They are named after J.L. Poiseuille (1984), as a French physician who experimented with low speed flow in tubes.

(i) Steady flow between two fixed parallel plates (Fully-developed plane Poiseuille flow)

When a liquid is forced between two stationary infinite plates are 2H apart, under constant pressure gradient $\delta p/\delta x$ and zero gravity. For steady state.

The continuity , and momentum can be reduce as;

Continuity eq. $$\frac{\partial u}{\partial x} = 0 \qquad (1)$$

Momentum eq. $$0 = -\frac{\partial p}{\partial x} + \mu \frac{\partial^2 u}{\partial y^2} \qquad (2)$$

Using eq. (2)

$$\int \mu \frac{\partial^2 u}{\partial y^2} dy = \int \frac{\partial p}{\partial x} dy \;\rightarrow\; \frac{\partial u}{\partial y} = \frac{1}{\mu}\left(\frac{\partial p}{\partial x} y + c_1\right) \;\Rightarrow\; u = \frac{1}{\mu}\left(\frac{\partial p}{\partial x}\frac{y^2}{2} + c_1 y + c_2\right)$$

B.C. (1) at y = - H , u = 0,

(2) at y = H , u =0

Show that parabolic velocity profile after apply the boundary conditions as;

$$u = -\frac{1}{2\mu}\frac{\partial p}{\partial x}\left(H^2 - y^2\right)$$

If the pressure gradient is negative, then the flow is in the positive direction.

Maximum flow velocity is at the center; (y = 0)

$$u_{max} = -\frac{1}{2\mu}\frac{\partial p}{\partial x} H^2$$

To find the velocity average

The volumetric flow rate per unit width is

$$Q = \int_{-H}^{H} u\,dy = 2\int_{0}^{H} -\frac{1}{2\mu}\frac{\partial p}{\partial x}\left(H^2 - y^2\right)dy$$

Show that $Q = -\frac{2}{3\mu}\frac{\partial p}{\partial x}H^3$ (4)

From the above equ.(4) indicates that the volumetric flow rate Q is proportional to the pressure gradient, $\partial p/\partial x$, and inversely proportional to the viscosity μ. Note also that, since $\partial p/\partial x$ is negative, Q is positive. The average velocity u_{ave} in the channel is

$$u_{ave} = \frac{Q}{A} = \frac{Q}{2H} \quad \Rightarrow \quad \therefore u_{ave} = -\frac{1}{3\mu}\frac{\partial p}{\partial x}H^2$$

$$\therefore \frac{u_{ave}}{u_{max}} = \frac{2}{3}$$

To find the shear stress distribution is given by

$$\tau_{xy} = \tau_{yx} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \quad \rightarrow \therefore \tau_w = \tau_{xy} = \tau_{yx} = \frac{\partial p}{\partial x}y$$

i.e τ_{xy} varies linearly with y

$\tau_{xy} = 0$ at the centerline (y = 0)

τ_{xy} maximum absolute at the wall (y = H or y = - H)

$$\tau_{max} = \tau_{xy}\Big|_{y=H} = \frac{\partial p}{\partial x}H$$

<u>H.W</u> Consider steady flow between two parallel inclined plates, with constant pressure gradient and gravity. The distance between the two plates is 2*H*. The angle formed by the two plates and the horizontal direction is *θ*. Find velocity distribution.

Ans. $u = \frac{1}{2\mu}\left(-\frac{\partial p}{\partial x} + \rho g\sin\theta\right)\left(H^2 - y^2\right)$

(ii) For Circular pipe (Fully-developed axisymmetric Poiseuille flow, or Hagen-Poiseuille flow)

The circular pipe is perhaps our most celebrated viscous flow, first studies by Hagen (1839) and Poiseuille (1984), is the pressure-driven flow in infinitely long cylindrical tubes. The geometry of the flow is shown in Fig. (3). Assuming that gravity is zero, and with the assumptions; $v_r = v_\theta = 0, \quad \frac{\partial v_z}{\partial \theta} = 0, \quad \frac{\partial p}{\partial z} = \text{const.}$

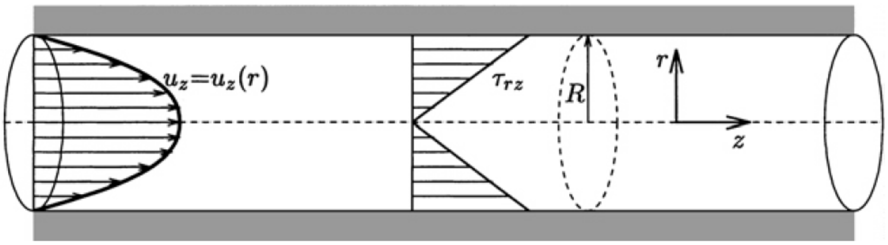


Fig. (3) Axisymmetric Poiseuille flow

Continuity eq. $\frac{\partial v_r}{\partial r} + \frac{v_r}{r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z} = 0$ *Reduce to* $\therefore \frac{\partial v_z}{\partial z} = 0$

Take momentum eq. in z - dir. only

$$\rho\left(v_r \frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z \frac{\partial v_z}{\partial z}\right) = \rho g_z - \frac{\partial p}{\partial z} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2}\right] \qquad z-\text{dir}$$

reduce to

$$0 = -\frac{\partial p}{\partial z} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)\right] \qquad \textbf{(1)}$$

Show that the general solution by using eq. (1)

$$v_z = \frac{1}{4\mu}\frac{\partial p}{\partial z} r^2 + c_1 \ln r + c_2$$

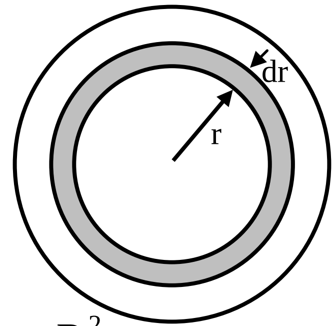
Apply the B.C. (1) at r = 0 , $\frac{\partial v_z}{\partial r} = 0$

Apply the B.C. (2) at r = R , $\mathbf{v_z = 0}$

Show that parabolic velocity profile after apply the boundary conditions as;

$$v_z = -\frac{1}{4\mu}\frac{\partial p}{\partial z}\left(R^2 - r^2\right)$$

$v_{z,max}$ at $r = 0$ $\quad v_{z,max} = -\frac{1}{4\mu}\frac{\partial p}{\partial x}R^2$



Show that $\frac{v_{z,\,ave}}{v_{z,\,max}} = \frac{1}{2}$

Hint: The volumetric flow rate is $Q = \int_0^R v_z\, 2\pi\, r\, dr$ and $A = \pi R^2$

To find the shear stress distribution is given by

$$\tau_w = \tau_{rz} = \mu\left[\frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r}\right] \qquad \rightarrow \qquad \therefore \tau_w = \frac{1}{2}\frac{\partial p}{\partial z}r$$

$\tau_w = 0$ at the centerline (r = 0) and τ_w maximum at the wall (r = R)

<u>H.W.</u> Consider fully-developed pressure-driven flow of a Newtonian liquid in a sufficiently long annulus of radii R and κR, where κ <1 (Fig. 4). For zero gravity.

Boundary conditions:

$v_z = 0 \quad$ at $\quad r = \kappa R$

$v_z = 0 \quad$ at $\quad r = R$

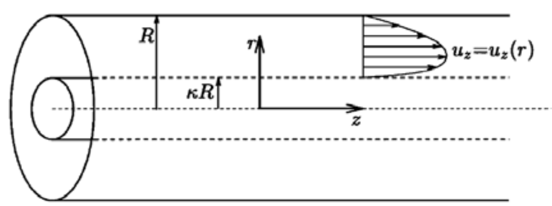


Fig. (4) Fully-developed flow in an annulus

show that

$$v_z = -\frac{1}{4\mu}\frac{\partial p}{\partial z}R^2\left[1 - \left(\frac{r}{R}\right)^2 + \frac{1-\kappa^2}{\ln(1/\kappa)}\ln\frac{r}{R}\right]$$

and the shear stress is

$$\tau_{rz} = \frac{1}{4}\frac{\partial p}{\partial z}R\left[2\left(\frac{r}{R}\right) - \frac{1-\kappa^2}{\ln(1/\kappa)}\left(\frac{R}{r}\right)\right]$$

Find the maximum velocity (Hint: occurs at the point where $\tau_{rz} = 0$ or $dv_z/dr = 0$)

3.3 Thin film flow (gravity-driven).

Consider a thin film of an incompressible Newtonian liquid flowing down an inclined plane (Fig. 5). The ambient air is assumed to be stationary, and, therefore, the flow is driven by gravity alone. Assuming that the surface tension of the liquid is negligible, and that the film is of uniform thickness δ, calculate the velocity and the volumetric flow rate per unit width. For steady state.

The continuity , and momentum can be reduce as;

- Continuity eq. $\frac{\partial u}{\partial x} = 0$ (1)
- Momentum eq. $0 = \rho g_x + \mu \frac{\partial^2 u}{\partial y^2}$

$$\mu \frac{\partial^2 u}{\partial y^2} = -\rho g \sin\theta \quad (2)$$

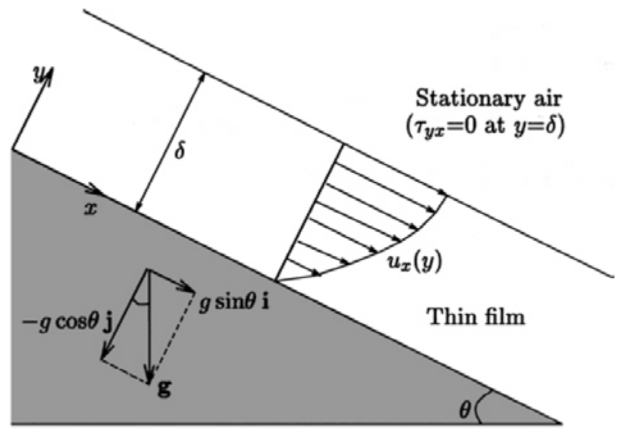


Fig. (5) Film flow down an inclined plane.

B.C. no slip along the solid boundary (1) at y = 0 , u = 0,

no shearing at the free surface (2) $\tau_{xy} = \mu \frac{\partial u}{\partial y} = 0$ at $y = \delta$

The general solution of the Eq. (2) as;

$$u = -\frac{\rho g \sin\theta}{\mu} \frac{y^2}{2} + c_1 y + c_2$$

Show that velocity profile after apply the boundary conditions as;

$$u = \frac{\rho g \sin\theta}{\mu} \left(\delta y - \frac{y^2}{2} \right)$$

The maximum velocity occurs at the free surface,

$$u_{max} = \frac{\rho g \sin\theta}{2\mu}\delta^2$$

volumetric flow rate per unit width is;

$$\frac{Q}{W} = \int_0^\delta u dy =$$

and the average velocity, u_{ave}, over a cross section of the film is given by;

$$u_{ave} = \frac{Q}{W\delta} = \frac{\rho g \sin\theta}{3\mu}\delta^2$$

Note that if the film is horizontal, then $\sin\theta = 0$ and u is zero, i.e., no flow occurs.

If the film is vertical, then $\sin\theta = 1$, and

$$u = \frac{\rho g}{\mu}\left(\delta y - \frac{y^2}{2}\right)$$

3.4 **Transient One-Dimensional Flows**

In the above examples, we studied three classes of steady-state flows, where the dependent variable, i.e., the nonzero velocity component, was assumed to be a function of a single spatial independent variable. The governing equation for such a flow is a linear second-order ordinary differential equation which is integrated to arrive at a general solution. The general solution contains two integration constants which are determined by the boundary conditions at the endpoints of the one-dimensional domain over which the analytical solution is sought.

In the present section, we consider one-dimensional, *transient* flows. Hence, the dependent variable is now a function of two independent variables, one of which is time, *t*. The governing equations for these flows are partial differential equations. In fact, we have already encountered some of these PDEs, while simplifying the corresponding components of the Navier-Stokes equation. For the sake of convenience, these are listed below.

(a) For transient one-dimensional rectilinear flow in Cartesian coordinates with v=w=0 and u=u (y , t),

$$\rho \frac{\partial u}{\partial t} = \rho g_x - \frac{\partial p}{\partial x} + \mu \frac{\partial^2 u}{\partial y^2} \qquad (1)$$

(b) For transient axisymmetric rectilinear flow with v_r=v_θ=0 and v_z=$v_z(r, t)$,

$$\rho \frac{\partial v_z}{\partial t} = \rho g_z - \frac{\partial p}{\partial z} + \mu \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial v_z}{\partial r} \right) \right] \qquad z - \text{dir.} \qquad (2)$$

(c) For transient axisymmetric torsional flow with v_z=v_r=0 and v_θ=$v_\theta(r, t)$,

$$\rho \frac{\partial v_\theta}{\partial t} = \mu \left[\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial}{\partial r} (r v_\theta) \right) \right] \qquad \theta - \text{dir} \qquad (3)$$

The above equations are all *parabolic* PDEs. For any particular flow, they are supplemented by appropriate boundary conditions at the two endpoints of the onedimensional flow domain, and by an *initial condition* for the entire flow domain. Note that the pressure gradients in Eqs. (1) and (2) may be functions of time. These two equations are *inhomogeneous* due to the presence of the pressure gradient and gravity terms. The inhomogeneous terms can be eliminated by decomposing the dependent variable into a properly chosen steady-state component (satisfying the corresponding steady-state problem and the boundary conditions) and a transient one which satisfies the *homogeneous* problem. A similar decomposition is often used for transforming inhomogeneous boundary conditions into homogeneous ones.

Separation of variables and the *similarity solution* method are the standard methods for solving Eq. (3) and the homogeneous counterparts of Eqs. (1) and (2).

In homogeneous problems admitting separable solutions, the dependent variable $u(xi, t)$ is expressed in the form;

$$u(xi, t) = X(xi).T(t)$$

Substitution of the above expression into the governing equation leads to the equivalent problem of solving two ordinary differential equations with X and T as the dependent variables.

In similarity methods, the two independent variables, *xi* and *t*, are combined into the *similarity variable*

$$\xi = \xi(xi, t)$$

If a similarity solution does exist, then the original partial differential equation for $u(xi, t)$ is reduced to an ordinary differential equation for $u(\xi)$.

(i) **Transient plane Couette flow**

Consider a Newtonian liquid of density ρ and viscosity μ bounded by two infinite parallel plates separated by a distance *H*, as shown in Figure below. The liquid and the two plates are initially at rest. At time *t*=0+, the lower plate is suddenly brought to a steady velocity *V* in its own plane, while the upper plate is held stationary. Assuming that gravity and pressure gradient are zero

The governing equation (1) is *homogeneous* can be reduce to,

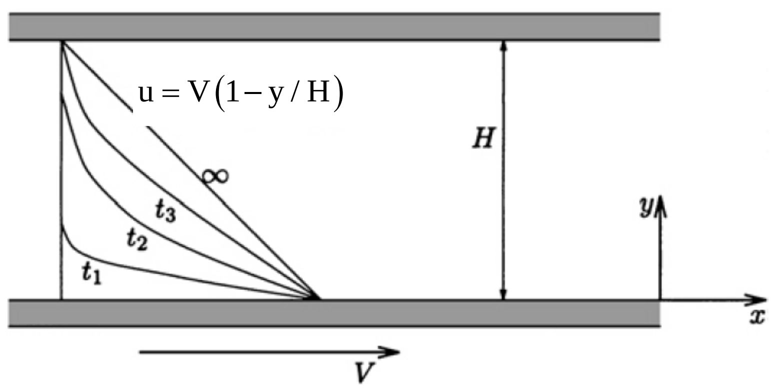
$$\frac{\partial u}{\partial t} = \upsilon \frac{\partial^2 u}{\partial y^2} \qquad (4)$$

where $\upsilon = \mu / \rho$ is the kinematic viscosity. Mathematically, Eq. (4) is called the *heat* or *diffusion* equation. The boundary and initial conditions are:

$$u = V \quad \text{at} \quad y = 0,\ t > 0$$

$$u = 0 \quad \text{at} \quad y = H,\ t \geq 0$$

$$u = 0 \quad \text{at} \quad t = 0,\ 0 \leq y \leq H$$



Note that, while the governing equation is homogeneous, the boundary conditions are inhomogeneous. Therefore, separation of variables cannot be applied directly. We first have to transform the problem so that the governing equation and the two boundary conditions are homogeneous. This can be achieved by decomposing *u*(*y*, *t*) into the steady plane Couette velocity profile, which is expected to prevail at large times, and a transient component:

$$u(y,t) = V\left(1-\frac{y}{H}\right) - u'(y,t)$$

Substituting into Eqs. (4), we obtain the following equation;

$$\frac{\partial u'}{\partial t} = \upsilon \frac{\partial^2 u'}{\partial y^2} \qquad (5)$$

with the boundary conditions;

$$u' = 0 \qquad \text{at} \quad y = 0,\ t > 0$$

$$u' = 0 \qquad \text{at} \quad y = H,\ t \geq 0$$

$$u' = V\left(1-\frac{y}{H}\right) \qquad \text{at} \quad t = 0,\ 0 \leq y \leq H$$

Therefore, separation of variables can now be used. The first step is to express u'(y , t) in the form;

Assume $u'(y,t) = Y(y).T(t)$

Substituting into Eq. (5) and separating the functions *Y* and *T*, we get;

$$\frac{1}{\upsilon}\frac{T'}{T} = \frac{Y''}{Y} = \text{constant} = -\frac{\lambda^2}{H^2}$$

The only way a function of t can be equal to a function of y is for both functions to be equal to the same constant. For convenience, we choose this constant to be $-\lambda^2/H^2$. We thus obtain two ordinary differential equations:

$$\frac{1}{\upsilon}\frac{T'}{T}=\frac{Y''}{Y}=\text{constant}=-\frac{\lambda^2}{H^2}$$

$$\frac{1}{\upsilon}\frac{T'}{T}=-\lambda^2 \qquad \Rightarrow T'+\frac{\upsilon\lambda^2}{H^2}T=0 \;\rightarrow\; (D^2+\frac{\upsilon\lambda^2}{H^2})T=0 \;\rightarrow \therefore T(t)=Ce^{-\frac{\upsilon\lambda^2}{H^2}t}$$

$$\frac{Y''}{Y}=-\frac{\lambda^2}{H^2} \qquad \Rightarrow Y''+\frac{\lambda^2}{H^2}Y=0 \;\rightarrow\; (D^2+\frac{\lambda^2}{H^2})X=0$$

$$(D-i\tfrac{\lambda}{H})\left(D+i\tfrac{\lambda}{H}\right)Y=0 \qquad \Rightarrow\; \therefore Y(y)=c_1e^{i\frac{\lambda}{H}x}+c_2e^{-i\frac{\lambda}{H}x}$$

$$\therefore Y(\text{y})=A\cos\left(\frac{\lambda y}{H}\right)+B\sin\left(\frac{\lambda y}{H}\right)$$

In this case the solution

$$u'(\text{y},t)=Y.T=\left(A\cos\left(\frac{\lambda y}{H}\right)+B\sin\left(\frac{\lambda y}{H}\right)\right)\left(Ce^{-\frac{\upsilon\lambda^2}{H^2}t}\right)$$

$$\therefore u'(\text{y},t)=\left(A^*\cos\left(\frac{\lambda y}{H}\right)+B^*\sin\left(\frac{\lambda y}{H}\right)\right)e^{-\frac{\upsilon\lambda^2}{H^2}t}$$

Apply the B.C.

$$(1)\; u'=0=\left(A^*\cos(0)+B^*\sin(0)\right)e^{-\frac{\upsilon\lambda^2}{H^2}t}=A^*e^{-\alpha\lambda^2t}$$

$$e^{-\alpha\lambda^2t}\neq 0 \qquad \Rightarrow \qquad \therefore A^*=0$$

$$\therefore u'(\text{y},t)=B^*\sin\left(\frac{\lambda y}{H}\right).e^{-\frac{\upsilon\lambda^2}{H^2}t}$$

Apply at $u'=0 \quad at \quad y=H$

$$(2) u'=0=B^*\sin\lambda.e^{-\frac{\upsilon\lambda^2}{H^2}t} \qquad \rightarrow B^*\neq 0 \quad ,e^{-\alpha\lambda^2t}\neq 0$$

$$\therefore \sin\lambda=0 \;\rightarrow\; \lambda=n\pi \quad \Rightarrow \therefore \lambda=n\pi \quad \rightarrow \lambda_n=n\pi, \qquad n=1,\;2,\cdots$$

$$\therefore u'(y,t)=\sum_{n=1}^{\infty} B_n{}^* \sin\left(\frac{\lambda_n y}{H}\right).e^{-\frac{\upsilon\lambda_n^2}{H^2}t} = \sum_{n=1}^{\infty} B_n{}^* \sin\left(\frac{n\pi y}{H}\right).e^{-\frac{n^2\pi^2}{H^2}\upsilon t}$$

Apply the I.C.

$$\therefore u(y,0)=V\left(1-\frac{y}{H}\right)=\sum_{n=1}^{\infty} B_n{}^* \sin\left(\frac{n\pi y}{H}\right).e^{0}$$

$$f(y)=\sum_{n=1}^{\infty} B_n{}^* \sin\frac{n\pi}{L} y\,.$$

Apply Fourier series $B_n^* = \frac{2}{H}\int_0^H f(y)\sin\frac{n\pi y}{H}dy$

$$B_n^* = \frac{2V}{n\pi}$$

$$\therefore u'(y,t)=\frac{2V}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{n\pi y}{H}\right).e^{-\frac{n^2\pi^2}{H^2}\upsilon t}$$

Finally, for the original dependent variable ux(y , t) we get

$$u(y,t)=V\left(1-\frac{y}{H}\right)-\frac{2V}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{n\pi y}{H}\right).e^{-\frac{n^2\pi^2}{H^2}\upsilon t}$$

Stream Function

For two dimensional and axisymmetric flows, the continuity can be used to show that the complete velocity field can be described in terms of a single, scalar field variable, which is called stream function, *ψ(x, y, t)*. In this development, we will consider only the case of constant-density flow.

❶ **2-D Case**

When nothing happens along one of the three directions in rectangular coordinate system, we have 2-D flow:

$$\mathbf{V} = u(x,y)i + v(x,y)j \qquad \text{and } w = 0, \quad \partial / \partial z = 0$$

For such a flow,

$$\mathbf{V} = \nabla \times \left[\psi(x,y)e_z\right] = \nabla\psi \times e_z + \psi \overbrace{\nabla \times e_z}^{0}$$

$$= \left(\frac{\partial \psi}{\partial x}e_x + \frac{\partial \psi}{\partial y}e_y\right) \times e_z$$

$$= \frac{\partial \psi}{\partial x}\underbrace{e_x \times e_z}_{-e_y} + \frac{\partial \psi}{\partial x}\underbrace{e_y \times e_z}_{e_x}$$

In terms of its scalar components, the velocity is:

$$u = \frac{\partial \psi}{\partial y} \qquad \text{and} \qquad v = -\frac{\partial \psi}{\partial x}$$

Next we substitute this form for **V** into continuity equation for steady state, incompressible flow;

$$\nabla.V = \nabla.\left[\nabla \times (\psi e_z)\right] = 0$$

which automatically satisfies continuity, for any choice of $\psi(x,y)$. The scalar field $\psi(x,y)$ is called the **stream function**.

- Consider a line along which ψ is a constant:

$$d\psi = 0 = udy - vdx \qquad \Rightarrow \left(\frac{dy}{dx}\right)_{\psi} = \frac{v}{u}$$

This is the Equation for the **streamline**. Thus, streamlines are lines of constant ψ.

- For irrotational flow, the problem would be to determine $\psi(x,y)$ such that $(\nabla \times \mathbf{V} = 0)$ is satisfied:

$$\nabla \times \left[\nabla \times (\psi e_z)\right] = 0$$

We can reduce this to a scalar equation. using identities for vector notation;

$$\nabla \times \left[\nabla \times (\psi e_z)\right] = \nabla\left[\nabla.(\psi e_z)\right] - \nabla^2(\psi e_z)$$

but $\nabla.(\psi e_z) = \dfrac{\partial \psi}{\partial z} = 0$

and $\nabla^2(\psi e_z) = (\nabla^2 \psi) e_z$

thus $\nabla \times \mathbf{V} = -(\nabla^2 \psi) e_z$

So for irrotational flow, the stream function must also satisfy Laplace's equation;

$$\nabla \times \mathbf{V} = 0 \qquad\qquad \nabla^2 \psi = 0$$

Unlike the scalar potential, the stream function can be used in all 2D flows, including those for which the flow is not irrotational.

❸ <u>Axisymmetric flow (Cylindrical)</u>

Another general class of flow for which a stream function exists is axisymmetric flow. In cylindrical coordinates (r, θ, z), this corresponds to:

$$\mathbf{V} = v_r(r,z)e_r + v_z(r,z)e_z \qquad \text{and } v_\theta = 0, \quad \partial/\partial\theta = 0$$

Then $\nabla.\mathbf{V} = 0$ can be satisfied be seeking **V** of the form:

$$\mathbf{V} = \nabla \times \left[f(r,z)e_\theta\right]$$

or

$$\mathbf{V} = \nabla \times \left[\frac{\psi(r,z)}{r} e_\theta\right]$$

Using the second expression to show that:

$$v_r = -\frac{1}{r}\frac{\partial \psi}{\partial z} \qquad \text{and} \qquad v_z = \frac{1}{r}\frac{\partial \psi}{\partial r}$$

where *ψ(r,z)* is called the **Stokes stream function.**

H.W. find $\nabla^2 \psi$

❸ **Axisymmetric flow (Spherical)**

In spherical coordinates (r, θ, ϕ), axisymmetric flow means;

$$\mathbf{V} = v_r(r,z)e_r + v_\theta(r,\theta)e_\theta \qquad \text{and } v_\phi = 0, \quad \partial/\partial\phi = 0$$

where ϕ is the azimuthal angle. Then $\nabla.\mathbf{V} = 0$ can be satisfied be seeking **V** of the form:

$$\mathbf{V} = \nabla \times \left[\psi'(r,\theta)e_\phi\right]$$

or

$$\mathbf{V} = \nabla \times \left[\frac{\psi(r,z)}{r\sin\theta}e_\phi\right]$$

$$= \frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta}e_r - \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r}e_\theta$$

In terms of its scalar components, the velocity is:

$$v_r = \frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta} \qquad \text{and} \qquad v_\theta = -\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r}$$

H.W. find $\nabla^2 \psi$

Streamlines, Pathlines and Streaklines

- **Streamline** - a contour in the fluid whose tangent is everywhere parallel to V at a given instant of time.
- **Path line** - trajectory swept out by a fluid element.
- **Streak line** - a contour on which lie all fluid elements which earlier past through a given point in space (e.g. dye trace)

For steady flows, these three definitions describe the same contour but, more generally, they are different.

Incompressible Fluids

By "incompressible fluid" we are usually referring to the assumption that the fluid's density is not significant function of time or of position. In other words,

$$\frac{\partial \rho}{\partial t} + \nabla.(\rho \mathbf{V}) = 0$$

can be replaced by $\nabla.\mathbf{V} = 0$

For steady flow, $\partial \rho / \partial t = 0$ already and the main further requirement is that density gradients be negligible

$$\nabla.(\rho \mathbf{V}) = \rho(\nabla.\mathbf{V}) + \mathbf{V}.\nabla \rho \approx \rho(\nabla.\mathbf{V})$$

Since flow causes the pressure to change, we might expect the fluid density to change, at least for gases. As we shall see shortly, gases as well as liquids can be treated as incompressible for some kinds of flow problems. Conversely, in other flow problems, neither gas nor liquid can be treated as incompressible. So what is the real criteria?

For an ideal fluid (i.e. no viscous dissipation to cause ∇T), density variations come about primarily because of pressure variations. For an isentropic expansion, the compressibility of the fluid turns out to be;

$$\left(\frac{\partial \rho}{\partial P}\right)_S = \frac{1}{c^2}$$

where c = speed of sound in the fluid.

Thus changes in density caused by changes in pressure can be estimated as:

$$\Delta \rho \approx \frac{1}{c^2} \Delta P \qquad (1)$$

According to Bernoulli's equation, pressure changes for steady flow are related to velocity changes;

$$\frac{P}{\rho} + \frac{v^2}{2} = \text{const.} \qquad \text{or} \qquad \Delta P = -\frac{1}{2}\rho \Delta v^2 \qquad (2)$$

Subs. eq. (2) into eq. (1) leads to $\Delta\rho \approx -\dfrac{\rho}{2}\dfrac{\Delta v^2}{c^2}$

The largest change in density corresponding to the largest change in v^2;

$$\left(\left|\frac{\Delta\rho}{\rho}\right|\right)_{max} = \frac{1}{2}\left(\frac{v_{max}}{c}\right)^2$$

If the fraction change in density is small enough, then it can be neglected:

❶ **Criteria 1:** $v_{max} << c$

for air at sea level : c = 342 m/s = 700 mph

for distilled water at 25°C: c = 1500 m/s = 3400 mph

For unsteady flows, a second criteria must be met:

- **Criteria 2 :** $\tau >> \dfrac{\ell}{c}$

where τ = time over which significant changes in v occur.

ℓ = distance over which changes in v occur.

ℓ / c = time for sound to propagate a distance ℓ.

For steady flow $\tau = \infty$ and Criteria 2 is always satisfied. And fluid can be considered incompressible if both criteria are met.

Alternative Forms of the Navier-Stokes Equations

(1) Dimensionless form of the Navier-Stokes Equations:

To write the Navier-Stokes equations in dimensionless form, the parameters of the problem should be used in order to normalize the dependent and independent variables. These parameters include the physical properties of the fluid i.e. density, *ρ*, and viscosity, *μ*, geometric variables such as some characteristic length, *L*, and other parameters which may arise from the boundary conditions, which could be some characteristic velocity, *U*.

❶ For unsteady, low viscosity flows it is customary to make the pressure dimensionless with ρU^2. This results in:

continuity $\frac{\partial \rho}{\partial t} + \rho\left(\nabla.\mathbf{V}\right) = 0$

NSE $\rho\left[\frac{\partial \mathbf{V}}{\partial t} + \mathbf{V}.\nabla\mathbf{V}\right] = \rho g - \nabla p + \mu\left(\nabla^2\mathbf{V}\right)$

Using these characteristic variables, we define the dimensionless variables as follows:

$$x_i^* = \frac{x_i}{L}, \quad V^* = \frac{V}{U}, \quad t^* = \frac{tU}{L}, \quad P^* = \frac{P}{\rho U^2}$$

This becomes

continuity $\frac{\partial \rho}{\partial t^*} + \rho\left(\nabla.\mathbf{V}^*\right) = 0$

NSE $\frac{\partial \mathbf{V}^*}{\partial t^*} + \mathbf{V}^*.\nabla\mathbf{V}^* = \frac{gL}{U^2} - \nabla p^* + \frac{\mu}{\rho LU}\left(\nabla^2\mathbf{V}^*\right)$

$$\frac{\partial \mathbf{V}^*}{\partial t^*} + \mathbf{V}^*.\nabla\mathbf{V}^* = \frac{1}{Fr} - \nabla p^* + \frac{1}{Re}\left(\nabla^2\mathbf{V}^*\right)$$

We see that continuity is devoid of parameters, while Navier-Stokes contains two:

Reynolds number: $Re = \frac{\rho LU}{\mu} = \frac{LU}{\upsilon}$

Froude number: $Fr = \frac{U^2}{g\rho}$

The Reynolds number is the most important dimensionless group in fluid mechanics. Almost all viscous-flow phenomena depend upon the Reynolds number. The Froude number is important only if there is a free surface in the flow.

Euler equation

- In the limit of $\mathrm{Re} \to \infty$ (the limiting case of very small viscosity) the stress term vanishes:

$$\frac{\partial \mathbf{V}^*}{\partial t^*} + \mathbf{V}^*.\nabla \mathbf{V}^* = -\nabla p^*$$

- In dimensional form, with $\mu = 0$, we get the Euler equations:

$$\rho\left[\frac{\partial \mathbf{V}}{\partial t} + \mathbf{V}.\nabla \mathbf{V}\right] = -\nabla p$$

- The flow is then **inviscid**

- For steady state, viscous flows it is customary to make the pressure dimensionless with $\mu U/L$. This results in:

$$\rho \mathbf{V}.\nabla \mathbf{V} = -\nabla p + \mu \nabla^2 \mathbf{V}$$

with dimensionless variables: $x_i^* = \frac{x_i}{L}, \quad V^* = \frac{V}{U}, \quad P^* = \frac{P}{\mu U / L}$

This becomes

NSE $\quad \mathrm{Re}\left(\mathbf{V}^*.\nabla \mathbf{V}^*\right) = -\nabla p^* + \nabla^2 \mathbf{V}^*$

- In the limit of $\mathrm{Re} \to 0$ the convective term vanishes:

$$-\nabla p^* + \nabla^2 \mathbf{V}^* = 0$$

(2) The Vorticity Transport Equation

In the case of two-dimensional unsteady flow in the x-y plane the velocity vector becomes: $\mathbf{V} = u(x, y, t)i + v(x, y, t)j$

and the system of equations of NSE and continuity transforms into;

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{1}{\rho}X - \frac{1}{\rho}\frac{\partial p}{\partial x} + \upsilon\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \quad (1)$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = \frac{1}{\rho}Y - \frac{1}{\rho}\frac{\partial p}{\partial y} + \upsilon\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \quad (2)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (3)$$

But the vector of vorticity, curl **V**, which reduces to the one component about the z-axis for 2-D flow:

$$\omega_z = \omega = \frac{1}{2}\text{curl}\,\mathbf{V} = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \quad (4)$$

For frictionless motions are irrotational so that curl **V** = 0

Differentiate eq. (1) w.r.t. (y) and eq. (2) w.r.t. (x) for eliminating pressure terms, we obtain:

$$\underbrace{\frac{\partial \omega}{\partial t}}_{\text{Local Acc.}} + \underbrace{u\frac{\partial \omega}{\partial x} + v\frac{\partial \omega}{\partial y}}_{\text{Convective Acc.}} = \underbrace{\upsilon\left(\frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2}\right)}_{\text{dissipatation of vorticity through friction}} \quad (5)$$

(5) **(vorticity transport)**

or , in shorthand form:

$$\underbrace{\frac{D\omega}{Dt}}_{\substack{\text{substantive}\\\text{variation of}\\\text{vorticity}}} = \upsilon\nabla^2\omega \quad \text{or vector form:} \quad \frac{\partial \omega}{\partial t} + \mathbf{V}.\nabla\omega = \upsilon\nabla^2\omega$$

In some case this equation called vorticity-velocity equation.

Note: if $\mu = 0 \;\Rightarrow\; \omega \neq 0$

(3 unknowns (ω, u, v) in 1 equation, we need continuity eq. to close the system, get 2 eqs. with 3 unknowns.)

- ***The boundary conditions on vorticity is also difficult to determine on solid walls.***
- ***This problems was solved by either devising formula for the values of ω on the walls or recently by not assigning any values to ω on the walls.***

(3) The Stream function Transport Equation (Biharmonic Formulation)

Because the difficulty to determine the vorticity on the walls, Finally, it is possible to transform these two equations (eq. (3) and (5)) with three unknowns into one equation with one unknown by introducing the stream function ψ (x,y):

$$u = \frac{\partial \psi}{\partial y} \quad , \quad v = -\frac{\partial \psi}{\partial x}$$

we see that the continuity equation is satisfied automatically. In addition the vorticity equation (eq. 4) becomes;

$$\omega = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = -\frac{1}{2}\nabla^2 \psi$$

The vorticity transport eq. (5) becomes;

$$\frac{\partial \nabla^2 \psi}{\partial t} + \frac{\partial \psi}{\partial y}\frac{\partial \nabla^2 \psi}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial \nabla^2 \psi}{\partial y} = \upsilon \nabla^4 \psi \qquad (6)$$

In this form contains only one unknown, ψ. The left-hand side of eq. (6) contains the inertia terms, whereas, the right-hand side contains the frictional terms. It is a fourth-order partial differential eq. in the stream function ψ. Its solution in general terms is again, very difficult, owing to its being non-linear. But, the boundary conditions in this case are easily specified especially on solid walls.

To scaling the above eq. (6) by using the dimensionless variables as follows:

$$x_i^* = \frac{x_i}{L}, \quad t^* = \frac{tU}{L}, \quad \Psi = \frac{\psi}{UL}$$

This becomes

$$\frac{\partial \nabla^2 \Psi}{\partial t^*} + \frac{\partial \Psi}{\partial y^*}\frac{\partial \nabla^2 \Psi}{\partial x^*} - \frac{\partial \Psi}{\partial x^*}\frac{\partial \nabla^2 \Psi}{\partial y^*} = \frac{1}{Re}\nabla^4 \Psi$$

- In the limit of $Re \to 0$ (the limiting case of very large viscosity)

In very slow motions or in motions with very large viscosity the viscous forces are considerably greater than the inertia forces because the latter are of the order of the velocity squared, whereas the former are linear with velocity. To a first approximation it is possible to neglect the inertia terms with respect to the viscous terms, so the eq. (6) we becomes; $\nabla^4 \psi = 0$ (7)

This is, now a linear equation. Flows described by equ. (7) proceed with very small velocities and are sometimes called ***creeping motions.***

Creeping motions can also be regarded as solution of the Navier-Stokes equations in the ***limiting case of very small Reynolds numbers*** (Re → 0), because the Reynolds number represents the ratio of inertia to friction forces.

$$Re = \frac{Ineria\ force}{Friction\ force}$$

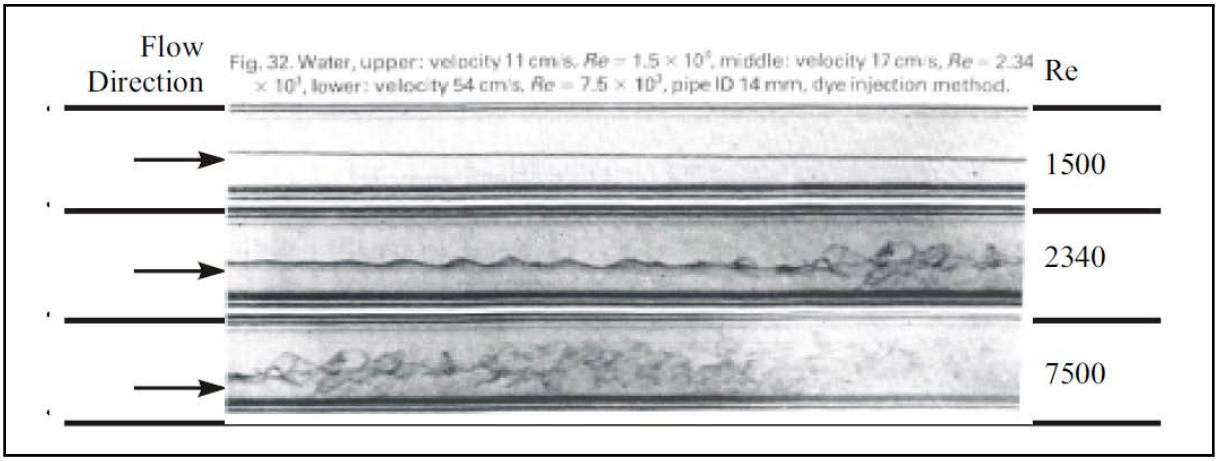
Turbulence

In all the problems we have analyzed the date, the fluid elements travel along smooth predictable trajectories. This state of affairs is called:

Laminar flow: Fluid elements travel along smooth deterministic trajectories.

These trajectories are straight parallel lines for simple pipe flows.

Consider Reynolds experiment (1882) - inject a thin stream of dye into a fully developed flow in a pipe;



- For laminar flow, dye stream appears as a straight colored thread.

- For turbulent flow, irregular radial fluctuations of dye thread.

He found in all cases, the transition occurred at a critical value of a dimensionless group:

$$\underbrace{\frac{\rho \bar{U} D}{\mu}}_{\text{Re}} = 2300 \pm 200$$

where $\bar{U}$ is the cross-sectional average velocity (= volumetric flow rate /pipe area). Today, we know this dimensionless group as the **Reynolds number**.

origin of turbulence - instability of laminar-flow solution of N-S eqs.

instability - small perturbations (caused by vibration, etc) grow rather than decay with time.

That the laminar-flow solution is metastable for Re > 2100 can be seen from Reynolds experiment performed with a pipe in which disturbance are minimized:

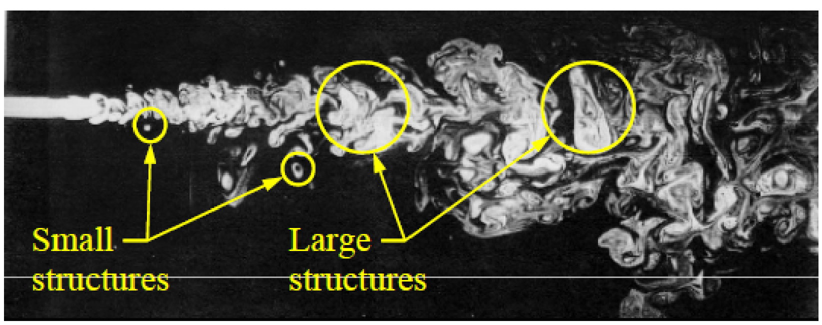
reduce vibration , fluid enters pipe smoothly , smooth pipe wall

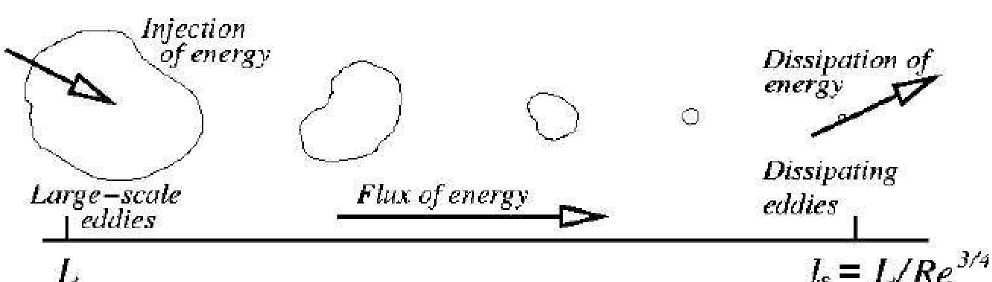
Under such conditions, laminar flow can be seen to persist up to Re = 10^4. However, just adding some vibrations (disturbance) can reduce the critical Re to 2100.

The onset of turbulence causes a number of profound changes in the nature of the flow:

- dye thread breaks up: streamlines appear contorted and random.
- sudden increase in $\Delta p/L$
- local v fluctuates wildly with time

<u>Turbulent flow structures</u>





Energy Cascade
Richardson (1922)

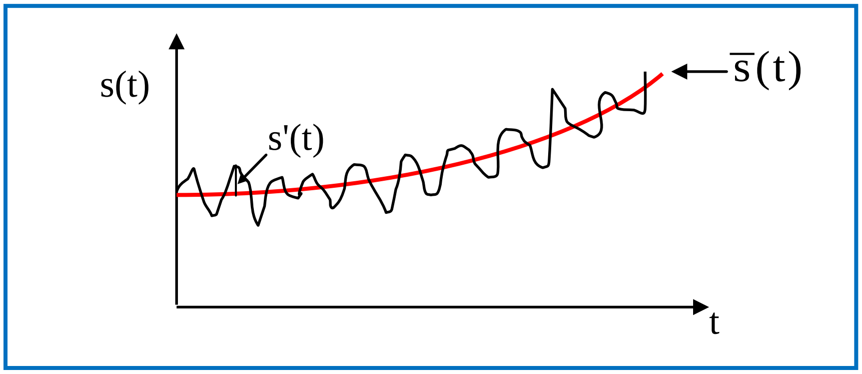
Time averaging technique (Time smoothing):

First, we need to define what we mean by a **time-averaged** quantity. Suppose we have some property like velocity or pressure which fluctuated with time:

$$s = s(t) \quad \text{where s any property}$$

we can average over some time interval;

$$\bar{s}(t) = \frac{1}{\Delta t}\int_0^{\Delta t} s(t)dt \qquad \therefore s(t) = \bar{s}(t) + s'(t)$$



Now let's define another quantity called the **fluctuation** about the mean:

$$s'(t) = s(t) - \bar{s}(t) \qquad \underbrace{\overline{s'(t)}}_{avarge\ \ of\ \ fluctuation} = \frac{1}{\Delta t}\int_0^{\Delta t} s'(t)dt = 0$$

$$u = \bar{u} + u'$$

u = instantaneous velocity component in x-dir.

$$v = \bar{v} + v' \quad \text{and} \quad w = \bar{w} + w'$$

$$\overline{u+v} = \bar{u} + \bar{v} \qquad \because \overline{u'} = \overline{v'} = 0 \quad \text{and} \quad \overline{\bar{u}u'} = 0$$

$$\overline{uv} = \overline{(\bar{u}+u')(\bar{v}+v')} = \overline{\bar{u}\,\bar{v}} + \cancel{\overline{\bar{u}v'}}^{=0} + \cancel{\overline{\bar{v}u'}}^{=0} + \overline{u'v'}$$

$$\therefore \overline{uv} = \bar{u}\,\bar{v} + \overline{u'v'}$$

$$\overline{u^2} = \bar{u}^2 + \overline{u'^2}$$

$$\left(\overline{\frac{\partial u}{\partial x}}\right) = \frac{\partial \bar{u}}{\partial x} \quad , \quad \left(\overline{\frac{\partial u}{\partial t}}\right) = \frac{\partial \bar{u}}{\partial t}$$

In general; vector quantity: $\mathbf{V} = \overline{\mathbf{V}} + \mathbf{V}'$ and scalar quantity **:** $\phi = \overline{\phi} + \phi'$

$$\overline{\text{div}\,\mathbf{V}} = \text{div}\,\overline{\mathbf{V}} \quad , \overline{\text{div}(\phi\mathbf{V})} = \text{div}\left(\overline{\phi\mathbf{V}}\right) = \text{div}\left(\overline{\phi}\,\overline{\mathbf{V}}\right) + \text{div}\left(\overline{\phi'\mathbf{V}'}\right)$$

$$\overline{\text{div}\,\text{grad}\,\phi} = \text{div}\,\text{grad}\,\overline{\phi}$$

Variance r.m.s and turbulence kinetic energy

The descriptors used to indicate the spread of the fluctuations ϕ' about the mean value $\overline{\phi}$ are the variance and root mean square (r.m.s):

$$\overline{(\phi')^2} = \frac{1}{\Delta t}\int_0^{\Delta t}(\phi')^2\,dt \quad \Rightarrow \quad \phi_{rms} = \sqrt{\overline{(\phi')^2}} = \left[\frac{1}{\Delta t}\int_0^{\Delta t}(\phi')^2\,dt\right]^{1/2}$$

The total kinetic energy per unit mass k of the turbulence at a given location can be found as follows:

$$k = \frac{1}{2}\left(\overline{u'^2} + \overline{v'^2} + \overline{w'^2}\right)$$

The turbulence intensity I_t is the average r.m.s. velocity divided by a reference mean flow velocity U_{ref} and is linked to the turbulence kinetic energy k as follows:

$$I_t = \frac{\sqrt{\frac{1}{3}\left(\overline{u'^2} + \overline{v'^2} + \overline{w'^2}\right)}}{U_{ref}} \quad \Rightarrow \quad I_t = \frac{\left(\frac{2}{3}k\right)^{1/2}}{U_{ref}}$$

<u>Reynolds-averaged Navier-Stokes equations for incompressible flow:</u>

(i) Mass Conservation Equation;

we will start with the equation of continuity for an incompressible flow;

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad (1) \qquad \text{where} \quad u = \overline{u} + u', v = \overline{v} + v', \text{and } w = \overline{w} + w'$$

$$\frac{\partial \overline{u}}{\partial x} + \frac{\partial u'}{\partial x} + \frac{\partial \overline{v}}{\partial y} + \frac{\partial v'}{\partial y} + \frac{\partial \overline{w}}{\partial z} + \frac{\partial w'}{\partial z} = 0$$

Integrating this eq. term by term over time, we have'

$$\frac{\partial \overline{u}}{\partial x} + \frac{\partial \overline{v}}{\partial y} + \frac{\partial \overline{w}}{\partial z} = 0 \quad (2) \Rightarrow \nabla.\overline{\mathbf{V}} = 0 \quad \text{or div}\,\overline{\mathbf{V}} = 0$$

(ii) momentum eq. in x- dir

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\upsilon\nabla^2 u \qquad (3)$$

Multiply continuity eq. (1) by u and add to the momentum eq. (3), we have;

$$u\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)=0$$

$$\frac{\partial u}{\partial t}+2u\frac{\partial u}{\partial x}+\left(u\frac{\partial u}{\partial y}+v\frac{\partial u}{\partial y}\right)+\left(u\frac{\partial w}{\partial z}+w\frac{\partial u}{\partial z}\right)=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\upsilon\nabla^2 u$$

$$\frac{\partial u}{\partial t}+\frac{\partial(uu)}{\partial x}+\frac{\partial(uv)}{\partial y}+\frac{\partial(uw)}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\upsilon\nabla^2 u$$

In **vector form**: $\frac{\partial u}{\partial t}+\operatorname{div}(u\mathbf{V})=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\upsilon\operatorname{div}(\operatorname{grad}(u))$

The time averaging each term overtime and applying the rule

$$\left(\overline{\frac{\partial u}{\partial t}}\right)=\frac{\partial\overline{u}}{\partial t} \quad , \overline{\operatorname{div}(u\mathbf{V})}=\operatorname{div}\left(\overline{u\mathbf{V}}\right)=\operatorname{div}\left(\overline{u}\,\overline{\mathbf{V}}\right)+\operatorname{div}\left(\overline{u'\mathbf{V}'}\right)$$

$$\overline{-\frac{1}{\rho}\frac{\partial p}{\partial x}}=-\frac{1}{\rho}\frac{\partial\overline{p}}{\partial x} \qquad \overline{\upsilon\operatorname{div}\operatorname{grad}u}=\upsilon\operatorname{div}\operatorname{grad}\overline{u}$$

Substitution of these results given the **time-average x-momentum equation**:

$$\frac{\partial\overline{u}}{\partial t}+\operatorname{div}\left(\overline{u}\,\overline{\mathbf{V}}\right)+\operatorname{div}\left(\overline{u'\mathbf{V}'}\right)=-\frac{1}{\rho}\frac{\partial\overline{p}}{\partial x}+\upsilon\operatorname{div}\left(\operatorname{grad}(\overline{u})\right)$$

Similarly in y and z dir;

$$\frac{\partial\overline{v}}{\partial t}+\operatorname{div}\left(\overline{v}\,\overline{\mathbf{V}}\right)+\operatorname{div}\left(\overline{v'\mathbf{V}'}\right)=-\frac{1}{\rho}\frac{\partial\overline{p}}{\partial y}+\upsilon\operatorname{div}\left(\operatorname{grad}(\overline{v})\right)$$

$$\frac{\partial\overline{w}}{\partial t}+\operatorname{div}\left(\overline{w}\,\overline{\mathbf{V}}\right)+\operatorname{div}\left(\overline{w'\mathbf{V}'}\right)=-\frac{1}{\rho}\frac{\partial\overline{p}}{\partial z}+\upsilon\operatorname{div}\left(\operatorname{grad}(\overline{w})\right)$$

In **Cartesian** form;

$$\frac{\partial \overline{u}}{\partial t}+\frac{\partial\left(\overline{u}^2\right)}{\partial x}+\frac{\partial\left(\overline{u}\,\overline{v}\right)}{\partial y}+\frac{\partial\left(\overline{u}\,\overline{w}\right)}{\partial z}+\frac{\partial\left(\overline{u'^2}\right)}{\partial x}+\frac{\partial\left(\overline{u'v'}\right)}{\partial y}+\frac{\partial\left(\overline{u'w'}\right)}{\partial z}=-\frac{1}{\rho}\frac{\partial \overline{p}}{\partial x}+\upsilon\nabla^2\overline{u}$$

$$\frac{\partial \overline{u}}{\partial t}+2\overline{u}\frac{\partial \overline{u}}{\partial x}+\overline{u}\frac{\partial \overline{v}}{\partial y}+\overline{v}\frac{\partial \overline{u}}{\partial y}+\overline{u}\frac{\partial \overline{w}}{\partial z}+\overline{w}\frac{\partial \overline{u}}{\partial z}+\frac{\partial\left(\overline{u'^2}\right)}{\partial x}+\frac{\partial\left(\overline{u'v'}\right)}{\partial y}+\frac{\partial\left(\overline{u'w'}\right)}{\partial z}=-\frac{1}{\rho}\frac{\partial \overline{p}}{\partial x}+\upsilon\nabla^2\overline{u}$$

Rearrangement the above eq.

$$\frac{\partial \overline{u}}{\partial t}+\overbrace{\overline{u}\frac{\partial \overline{u}}{\partial x}+\overline{u}\frac{\partial \overline{v}}{\partial y}+\overline{u}\frac{\partial \overline{w}}{\partial z}}^{\text{continuity}=0}+\overline{u}\frac{\partial \overline{u}}{\partial x}+\overline{v}\frac{\partial \overline{u}}{\partial y}+\overline{w}\frac{\partial \overline{u}}{\partial z}+$$

$$\frac{\partial\left(\overline{u'^2}\right)}{\partial x}+\frac{\partial\left(\overline{u'v'}\right)}{\partial y}+\frac{\partial\left(\overline{u'w'}\right)}{\partial z}=-\frac{1}{\rho}\frac{\partial \overline{p}}{\partial x}+\upsilon\nabla^2\overline{u}$$

$$\frac{\partial \overline{u}}{\partial t}+\overline{u}\frac{\partial \overline{u}}{\partial x}+\overline{v}\frac{\partial \overline{u}}{\partial y}+\overline{w}\frac{\partial \overline{u}}{\partial z}=-\frac{1}{\rho}\frac{\partial \overline{p}}{\partial x}+\upsilon\nabla^2\overline{u}+\frac{1}{\rho}\left[\frac{\partial(-\rho\overline{u'^2})}{\partial x}+\frac{\partial(-\rho\overline{u'v'})}{\partial y}+\frac{\partial(-\rho\overline{u'w'})}{\partial z}\right]$$

Similarly in y and z direction;

$$\frac{\partial \overline{v}}{\partial t}+\overline{u}\frac{\partial \overline{v}}{\partial x}+\overline{v}\frac{\partial \overline{v}}{\partial y}+\overline{w}\frac{\partial \overline{v}}{\partial z}=-\frac{1}{\rho}\frac{\partial \overline{p}}{\partial y}+\upsilon\nabla^2\overline{v}+\frac{1}{\rho}\left[\frac{\partial(-\rho\overline{u'v'})}{\partial x}+\frac{\partial(-\rho\overline{v'^2})}{\partial y}+\frac{\partial(-\rho\overline{v'w'})}{\partial z}\right]$$

$$\frac{\partial \overline{w}}{\partial t}+\overline{u}\frac{\partial \overline{w}}{\partial x}+\overline{v}\frac{\partial \overline{w}}{\partial y}+\overline{w}\frac{\partial \overline{w}}{\partial z}=-\frac{1}{\rho}\frac{\partial \overline{p}}{\partial x}+\upsilon\nabla^2\overline{w}+\frac{1}{\rho}\left[\frac{\partial(-\rho\overline{u'w'})}{\partial x}+\frac{\partial(-\rho\overline{v'w'})}{\partial y}+\frac{\partial(-\rho\overline{w'^2})}{\partial z}\right]$$

In **vector** form;

$$\frac{\partial \overline{u}}{\partial t}+\text{div}\left(\overline{u}\,\overline{\mathbf{V}}\right)=-\frac{1}{\rho}\frac{\partial \overline{p}}{\partial x}+\upsilon\,\text{div}\left(\text{grad}\left(\overline{u}\right)\right)+\frac{1}{\rho}\text{div}\left(-\rho\overline{u'\mathbf{V}'}\right)\qquad x-\text{dir.}$$

$$\frac{\partial \overline{v}}{\partial t}+\text{div}\left(\overline{v}\,\overline{\mathbf{V}}\right)=-\frac{1}{\rho}\frac{\partial \overline{p}}{\partial y}+\upsilon\,\text{div}\left(\text{grad}\left(\overline{v}\right)\right)+\frac{1}{\rho}\text{div}\left(-\rho\overline{v'\mathbf{V}'}\right)\qquad y-\text{dir.}$$

$$\frac{\partial \overline{w}}{\partial t}+\text{div}\left(\overline{w}\,\overline{\mathbf{V}}\right)=-\frac{1}{\rho}\frac{\partial \overline{p}}{\partial z}+\upsilon\,\text{div}\left(\text{grad}\left(\overline{w}\right)\right)+\frac{1}{\rho}\underbrace{\text{div}\left(-\rho\overline{w'\mathbf{V}'}\right)}_{\textit{Reynolds stresses}}\qquad z-\text{dir.}$$

The above equations is called **Reynolds-averaged Navier-Stokes equations (RANS)**

The **RANS** in ***tensor*** form:

$$\frac{\partial \bar{u}_i}{\partial t}+\frac{\partial}{\partial x_j}\left(\bar{u}_i\bar{u}_j\right)=-\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x_i}+\upsilon\frac{\partial^2 \bar{u}_i}{\partial x_j^2}+\frac{1}{\rho}\frac{\partial}{\partial x_j}\underbrace{(-\rho\overline{u_i' u_j'})}_{\text{Term}}$$

The extra stress terms have been written out in longhand to clarify their structure. They result from six additional stresses: three normal stresses:

$$\sigma'_{xx}=-\rho\overline{u'^2}\ ,\ \sigma'_{yy}=-\rho\overline{v'^2}\ ,\ \sigma'_{zz}=-\rho\overline{w'^2}$$

and shear stresses;

$$\tau'_{xy}=\tau'_{yx}=-\rho\overline{u'v'}\ ,\ \tau'_{xz}=\tau'_{zx}=-\rho\overline{u'w'}\ ,\ \tau'_{yz}=\tau'_{zy}=-\rho\overline{v'w'}$$

The process of time averaging has introducing new terms

Reynolds stress or turbulent stress

$$\left\{-\rho\overline{u_i' u_j'}=-\rho\begin{bmatrix}\overline{u_1'^2} & \overline{u_1' u_2'} & \overline{u_1' u_3'}\\ \overline{u_1' u_2'} & \overline{u_2'^2} & \overline{u_2' u_3'}\\ \overline{u_1' u_3'} & \overline{u_2' u_3'} & \overline{u_3'^2}\end{bmatrix}=-\rho\begin{bmatrix}\overline{u'^2} & \overline{u'v'} & \overline{u'w'}\\ \overline{u'v'} & \overline{v'^2} & \overline{v'w'}\\ \overline{u'w'} & \overline{v'w'} & \overline{w'^2}\end{bmatrix}\right.$$

In the RANS equations, there are six additional unknowns:

$$-\rho\overline{u_1'^2},\ -\rho\overline{u_2'^2},\ -\rho\overline{u_3'^2}, -\rho\overline{u_1' u_2'},\ -\rho\overline{u_1' u_3'}, \text{ and } -\rho\overline{u_2' u_3'}$$

Closure problem in turbulence: Necessity of turbulence modeling

In the RANS equations, Reynolds stress terms give additional unknowns $-\rho\overline{u_i' u_j'}$, but there are no explicit governing differential equations for the additional unknowns.

- 3 velocity components, one pressure and 6 Reynolds stress terms = 10 unknowns
- No. of equations = 4 (1 Continuity + 3 momentum)
- As No. of unknowns > No. of equations, the problem in indeterminate. One needs to close the problem to obtain a solution. This is known as ***closure problem in turbulence***.
- The turbulence modeling tries to represent the Reynolds stresses in terms of time-averaged velocity components.
- The common turbulence models are classified on the basis of the number of additional transport equations that need to be solved along with RANS equations.

Different types of turbulent model

The most common RANS turbulence models are classified on the basis of the number of additional transport equations that need to be solved along with the RANS flow equations. The commonly followed methodologies include

- Eddy viscosity models, and
- Reynolds stress transport models.

No. of extra transport equations	Name of the model
Zero	Mixing length model
One	Spalart-Allmaras model
Two	Standard k-e model RNG k-e model Realizable k-e model. k-ω model
Seven	Reynolds stress model

Eddy Viscosity Models

Of the tabulated models the mixing length and *k*–ε models are at present by far the most widely used and validated. They are based on the presumption that there exists an analogy between the action of viscous stresses and Reynolds stresses on the mean flow. Both stresses appear on the right hand side of the momentum equation, and in Newton's law of viscosity the viscous stresses are taken to be proportional to the rate of deformation of fluid elements. For an incompressible fluid this gives:

$$\tau_{ij} = \mu e_{ij} = \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

Boussinesq proposed in 1877 that Reynolds stresses might be proportional to mean rates of deformation. **Reynolds stress** parts can be broken up into two parts: **isotropic** and **anisotropic** parts:

$$-\rho\overline{u_i'u_j'} = \underbrace{\mu_t\left(\frac{\partial \overline{u}_i}{\partial x_j}+\frac{\partial \overline{u}_j}{\partial x_i}\right)}_{Anistropic\ part} - \underbrace{\rho\frac{\overline{u_1'^2}+\overline{u_2'^2}+\overline{u_3'^2}}{3}\delta_{ij}}_{Isotropic\ part}$$

(Bussinesq eddy-vescosity approx.)

$$-\rho\overline{u_i'u_j'} = \underbrace{\mu_t\left(\frac{\partial \overline{u}_i}{\partial x_j}+\frac{\partial \overline{u}_j}{\partial x_i}\right)}_{\text{Mean rates of deformation}} - \frac{2\rho}{3}\underbrace{\frac{\overline{u_1'^2}+\overline{u_2'^2}+\overline{u_3'^2}}{2}}_{Turbulent\ kinetic\ energy}\delta_{ij}$$

$$-\rho\overline{u_i'u_j'} = \mu_t\left(\frac{\partial \overline{u}_i}{\partial x_j}+\frac{\partial \overline{u}_j}{\partial x_i}\right) - \frac{2\rho}{3}k\delta_{ij}$$

Eddy viscosity (μ_t)

Kinetic energy of turbulent fluctuations (k)

Kronecker delta

$$\delta_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i\neq j \end{cases}$$

RHS of RANS equations: $\dfrac{\partial \sigma_{ij}}{\partial x_j}$

$$\sigma_{ij} = -\left[\overline{p}\delta_{ij}\right] + \left(\frac{\partial \overline{u}_i}{\partial x_j}+\frac{\partial \overline{u}_j}{\partial x_i}\right)\underbrace{-\rho\overline{u_i'u_j'}}$$

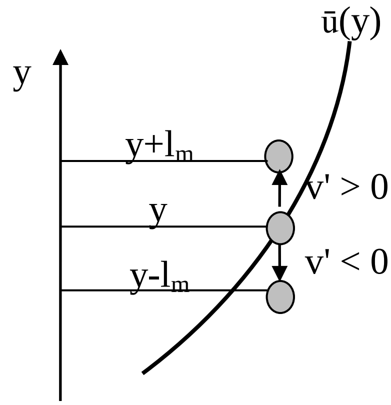
$$\mu_t\left(\frac{\partial \overline{u}_i}{\partial x_j}+\frac{\partial \overline{u}_j}{\partial x_i}\right) - \frac{2\rho}{3}k\delta_{ij}$$

$$\therefore \sigma_{ij} = -\underbrace{\overline{p}_{eff.}}_{\overline{p}+\frac{2\rho}{3}k}\delta_{ij} + \underbrace{\mu_{eff}}_{\mu+\mu_t}\left(\frac{\partial \overline{u}_i}{\partial x_j}+\frac{\partial \overline{u}_j}{\partial x_i}\right)$$

The question remains "How to model μ_t" ?

Mixing length model

Attempts to link the characteristic velocity scale of the eddies with the mean flow properties because there is a strong connect between the mean flow and the behavior of the largest eddies.



$$u' \alpha \ell \frac{\partial \bar{u}}{\partial y}$$

$$u' \alpha v'$$

$$-\rho \overline{u'v'} = \rho \ell_m^2 \left| \frac{\partial \bar{u}}{\partial y} \right| \frac{\partial \bar{u}}{\partial y}$$

$$-\rho \overline{u'v'} = \mu_t \frac{\partial \bar{u}}{\partial y}$$

$$\therefore \mu_t = \rho \ell_m^2 \left| \frac{\partial \bar{u}}{\partial y} \right|$$

$$\therefore \upsilon_t = \frac{\mu_t}{\rho} = \ell_m^2 \left| \frac{\partial \bar{u}}{\partial y} \right|$$

This is **Prandtl's mixing length model**

Algebraic expressions for mixing length in terms of the characteristic system length scale are reported for simple flows, such as fully developed pipe and channel flow, boundary layer, axisymmetric jet, wake … etc.

Advantages:

- Easy to implement.
- Cheap in terms of computing resources.
- good predictions for simple flows such as jets, mixing layers, wakes and boundary layer flow.

Disadvantages:

- Completely incapable of describing flows where the turbulent length scale varies: anything with separation or circulation.
- only calculates mean flow properties and turbulent shear stress.

Turbulent kinetic energy and dissipation

The instantaneous kinetic energy k(t) of a turbulent flow is the sum of mean kinetic energy $\overline{k}$ and turbulent kinetic energy k:

$$\overline{k} = \frac{1}{2}\left(\overline{u}_1^2 + \overline{u}_2^2 + \overline{u}_3^2\right)$$

$$k = \frac{1}{2}\overline{u_i' u_i'} = \frac{1}{2}\left(\overline{u_1'^2} + \overline{u_3'^2} + \overline{u_3'^2}\right)$$

$$k(t) = \overline{k} + k$$

The dissipation rate of k is given as:

$$\varepsilon = \upsilon \overline{\frac{\partial u_i'}{\partial x_j}\frac{\partial u_i'}{\partial x_j}}$$

We need equations for k and ε.

Turbulent kinetic energy k

Step (1) Start with the RANS derivation

From Navier-Stokes equation

$$\frac{\partial u_i}{\partial t} + \frac{\partial}{\partial x_j}\left(u_i u_j\right) = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j}\left(\upsilon \frac{\partial u_i}{\partial x_j}\right) \qquad (1)$$

Substituting $u_i = \overline{u}_i + u_i', u_j = \overline{u}_j + u_j',$ and $p = \overline{p} + p'$

$$\frac{\partial}{\partial t}\left(\overline{u}_i + u_i'\right) + \frac{\partial}{\partial x_j}\left[\left(\overline{u}_i + u_i'\right)\left(\overline{u}_j + u'\right)\right] = -\frac{1}{\rho}\frac{\partial}{\partial x_i}\left(\overline{p} + p'\right) + \frac{\partial}{\partial x_j}\left(\upsilon \frac{\partial}{\partial x_j}\left(\overline{u}_i + u_i'\right)\right)$$

Taking the average of the entire equation:

$$\frac{\partial \bar{u}_i}{\partial t} + \cancel{\frac{\partial \overline{u'_i}}{\partial t}} + \frac{\partial}{\partial x_j}\left[\bar{u}_i\,\bar{u}_j + \overline{u'_i u'_j} + \cancel{\overline{\bar{u}_i u'_j}} + \cancel{\overline{u'_i \bar{u}_j}}\right] =$$

$$-\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x_i} - \frac{1}{\rho}\cancel{\frac{\partial \overline{p'}}{\partial x_i}} + \frac{\partial}{\partial x_j}\left(\upsilon\frac{\partial \bar{u}_i}{\partial x_j}\right) + \frac{\partial}{\partial x_j}\left(\upsilon\cancel{\frac{\partial \overline{u'_i}}{\partial x_j}}\right)$$

$$\frac{\partial \bar{u}_i}{\partial t} + \frac{\partial}{\partial x_j}\left[\bar{u}_i\,\bar{u}_j + \overline{u'_i u'_j}\right] = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x_i} + \frac{\partial}{\partial x_j}\left(\upsilon\frac{\partial \bar{u}_i}{\partial x_j}\right)$$

$$\frac{\partial \bar{u}_i}{\partial t} + \frac{\partial}{\partial x_j}\left(\bar{u}_i\,\bar{u}_j\right) = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x_i} + \frac{\partial}{\partial x_j}\left(\upsilon\frac{\partial \bar{u}_i}{\partial x_j}\right) - \frac{\partial}{\partial x_j}\left(\overline{u'_i u'_j}\right) \qquad (2)$$

Step (2) Express the NS equation in terms of fluctuating components and hence obtain governing equation for turbulent kinetic energy:

Subtracting Eq. (2) from eq. (2), we have;

$$\frac{\partial u'_i}{\partial t} + \frac{\partial}{\partial x_j}\left[\bar{u}_i\,u'_j + \bar{u}_j u'_i + u'_i u'_j - \overline{u'_i u'_j}\right] = -\frac{1}{\rho}\frac{\partial p'}{\partial x_i} + \frac{\partial}{\partial x_j}\left(\upsilon\frac{\partial u'_i}{\partial x_j}\right) \qquad (3)$$

Multiplying Eq. (3) by u'_i

$$u'_i\frac{\partial u'_i}{\partial t} + u'_i\frac{\partial}{\partial x_j}\left[\bar{u}_i\,u'_j + \bar{u}_j u'_i + u'_i u'_j - \overline{u'_i u'_j}\right] = -u'_i\frac{1}{\rho}\frac{\partial p'}{\partial x_i} + u'_i\frac{\partial}{\partial x_j}\left(\upsilon\frac{\partial u'_i}{\partial x_j}\right)$$

$$\underbrace{u'_i\frac{\partial u'_i}{\partial t}}_{\text{part1}} + \underbrace{u'_i\frac{\partial}{\partial x_j}\left(\bar{u}_i\,u'_j\right)}_{2} + \underbrace{u'_i\frac{\partial}{\partial x_j}\left(\bar{u}_j u'_i\right)}_{3} + \underbrace{u'_i\frac{\partial}{\partial x_j}\left(u'_i u'_j\right)}_{4} +$$

$$\underbrace{u'_i\frac{\partial}{\partial x_j}\left(-\overline{u'_i u'_j}\right)}_{5} = -\underbrace{\frac{1}{\rho}u'_i\frac{\partial p'}{\partial x_i}}_{6} + \underbrace{u'_i\frac{\partial}{\partial x_j}\left(\upsilon\frac{\partial u'_i}{\partial x_j}\right)}_{7}$$

Part (1): $u'_i\frac{\partial u'_i}{\partial t} = \frac{1}{2}\frac{\partial}{\partial t}\left(u'_i u'_i\right) = \frac{1}{2}\frac{\partial}{\partial t}\left(u'^2_i\right) = \frac{\partial}{\partial t}\left(\frac{1}{2}u'^2_i\right)$

Part (2): $u'_i\frac{\partial}{\partial x_j}\left(\bar{u}_i\,u'_j\right) = u'_i\left[\bar{u}_i\frac{\partial u'_j}{\partial x_j} + u'_j\frac{\partial \bar{u}_i}{\partial x_j}\right] = u'_i\bar{u}_i\cancelto{0}{\frac{\partial u'_j}{\partial x_j}} + u'_i u'_j\frac{\partial \bar{u}_i}{\partial x_j} = u'_i u'_j\frac{\partial \bar{u}_i}{\partial x_j}$

Part (3): $u_i' \frac{\partial}{\partial x_j}\left(\bar{u}_j u_i'\right) = u_i'\left[\bar{u}_j \frac{\partial u_i'}{\partial x_j} + u_i' \cancelto{0}{\frac{\partial \bar{u}_j}{\partial x_j}}\right] = u_i'\bar{u}_j \frac{\partial u_i'}{\partial x_j} = \bar{u}_j \frac{\partial}{\partial x_j}\left(\frac{1}{2}u_i'^2\right)$

Part (4): $u_i' \frac{\partial}{\partial x_j}\left(u_i' u_j'\right) = u_i'\left[u_i' \frac{\partial u_j'}{\partial x_j} + u_j' \frac{\partial u_i'}{\partial x_j}\right] = u_i' u_i' \cancelto{0}{\frac{\partial u_j'}{\partial x_j}} + u_i' u_j' \frac{\partial u_i'}{\partial x_j} = u_j' \frac{\partial}{\partial x_j}\left(\frac{1}{2}u_i'^2\right)$

$$= \frac{\partial}{\partial x_j}\left(u_j' \frac{1}{2}u_i'^2\right) - \frac{1}{2}u_i'^2 \cancelto{0}{\frac{\partial u_j'}{\partial x_j}} = \frac{\partial}{\partial x_j}\left(u_j' \frac{1}{2}u_i'^2\right)$$

Part (5): $u_i' \frac{\partial}{\partial x_j}\left(-\overline{u_i' u_j'}\right)$

part (6): $\frac{1}{\rho}u_i' \frac{\partial p'}{\partial x_i} = \frac{\partial}{\partial x_i}\left(\frac{1}{\rho}u_i' p'\right) - \frac{1}{\rho}p' \cancelto{0}{\frac{\partial u_i'}{\partial x_i}} = \frac{\partial}{\partial x_i}\left(\frac{1}{\rho}u_i' p'\right)$

part (7): $u_i' \frac{\partial}{\partial x_j}\left(\upsilon \frac{\partial u_i'}{\partial x_j}\right) = \frac{\partial}{\partial x_j}\left(\upsilon u_i' \frac{\partial u_i'}{\partial x_j}\right) - \upsilon \frac{\partial u_i'}{\partial x_j}\frac{\partial u_i'}{\partial x_j}$

Substituting Part 1 to Part 7 and taking the average of the entire equation:

$$\overline{\frac{\partial}{\partial t}\left(\frac{1}{2}u_i'^2\right)} + \overline{u_i' u_j' \frac{\partial \bar{u}_i}{\partial x_j}} + \overline{\bar{u}_j \frac{\partial}{\partial x_j}\left(\frac{1}{2}u_i'^2\right)} + \overline{\frac{\partial}{\partial x_j}\left(u_j' \frac{1}{2}u_i'^2\right)} + \overline{u_i' \frac{\partial}{\partial x_j}\left(-\overline{u_i' u_j'}\right)} =$$
$$-\overline{\frac{\partial}{\partial x_i}\left(\frac{1}{\rho}u_i' p'\right)} + \overline{\frac{\partial}{\partial x_j}\left(\upsilon u_i' \frac{\partial u_i'}{\partial x_j}\right)} - \overline{\upsilon \frac{\partial u_i'}{\partial x_j}\frac{\partial u_i'}{\partial x_j}}$$

$$\Rightarrow \frac{\partial}{\partial t}\left(\frac{1}{2}\overline{u_i'^2}\right) + \overline{u_i' u'_j} \frac{\partial \bar{u}_i}{\partial x_j} + \bar{u}_j \frac{\partial}{\partial x_j}\left(\frac{1}{2}\overline{u_i'^2}\right) + \frac{\partial}{\partial x_j}\left(\frac{1}{2}\overline{u_j' u_i' u_i'}\right) + \cancelto{0}{\overline{u_i'}} \frac{\partial}{\partial x_j}\left(-\overline{u_i' u_j'}\right) =$$
$$-\frac{\partial}{\partial x_i}\left(\frac{1}{\rho}\overline{u_i' p'}\right) + \frac{\partial}{\partial x_j}\left(\upsilon \overline{u_i' \frac{\partial u_i'}{\partial x_j}}\right) - \upsilon \overline{\frac{\partial u_i'}{\partial x_j}\frac{\partial u_i'}{\partial x_j}}$$

$$\therefore \frac{\partial}{\partial t}\left(\frac{1}{2}\overline{u_i'^2}\right) + \bar{u}_j \frac{\partial}{\partial x_j}\left(\frac{1}{2}\overline{u_i'^2}\right) = -\frac{\partial}{\partial x_j}\left(\frac{1}{2}\overline{u_j' u_i' u_i'} + \frac{1}{\rho}\overline{u_j' p'} - \upsilon \overline{u_i' \frac{\partial u_i'}{\partial x_j}}\right) - \overline{u_i' u'_j} \frac{\partial \bar{u}_i}{\partial x_j} - \upsilon \overline{\frac{\partial u_i'}{\partial x_j}\frac{\partial u_i'}{\partial x_j}}$$

$$\therefore \frac{\partial k}{\partial t} + \bar{u}_j \frac{\partial k}{\partial x_j} = -\frac{\partial}{\partial x_j}\left(\frac{1}{2}\overline{u'_j u'_i u'_i} + \frac{1}{\rho}\overline{u'_j p'} - \upsilon \overline{u'_i \frac{\partial u'_i}{\partial x_j}} \right) \underbrace{- \overline{u'_i u'_j} \frac{\partial \bar{u}_i}{\partial x_j}}_{P} \underbrace{- \upsilon \overline{\frac{\partial u'_i}{\partial x_j} \frac{\partial u'_i}{\partial x_j}}}_{\varepsilon}$$

The equation for the turbulent kinetic energy k is as follows;

$$\underbrace{\frac{\partial k}{\partial t}}_{\text{rate of change of k}} + \underbrace{\bar{u}_j \frac{\partial k}{\partial x_j}}_{\text{transport of k by advection}} = -\frac{\partial}{\partial x_j}\left(\underbrace{\frac{1}{2}\overline{u'_j u'_i u'_i}}_{\text{transport of k by Reynolds stresses}} + \underbrace{\frac{1}{\rho}\overline{u'_j p'}}_{\text{transport of k by pressure}} \underbrace{- \upsilon \overline{u'_i \frac{\partial u'_i}{\partial x_j}}}_{\text{transport of k by viscous stresses}} \right) \underbrace{- \overline{u'_i u'_j} \frac{\partial \bar{u}_i}{\partial x_j}}_{\text{turbulence production}} \underbrace{- \upsilon \overline{\frac{\partial u'_i}{\partial x_j} \frac{\partial u'_i}{\partial x_j}}}_{\text{rate of dissipation of k}}$$

The first and second terms on the right-hand side represents turbulent diffusion of kinetic energy (which is actually transport of velocity fluctuations by the fluctuations themselves); it is almost modeled by use of a gradient-diffusion assumption:

$$-\left(\frac{1}{2}\overline{u'_j u'_i u'_i} + \frac{1}{\rho}\overline{u'_j p'} \right) \approx \frac{\upsilon_t}{\sigma_k} \frac{\partial k}{\partial x_j}$$

where ν_t is the ***kinematic eddy viscosity*** and σ_k is a ***turbulent Prandtl number***

The Forth term of the right-hand side represents the ***rate of production*** of turbulent kinetic energy by the mean flow. If we use the eddy-viscosity hypothesis to estimate the Reynolds stress, it can be written:

$$P = -\overline{u'_i u'_j} \frac{\partial \bar{u}_i}{\partial x_j} \approx \upsilon_t \left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i} \right) \frac{\partial \bar{u}_i}{\partial x_j}$$

The k-ε model

The model equation for the turbulent kinetic energy k is as follows:

$$\frac{Dk}{Dt}=\underbrace{\frac{\partial k}{\partial t}}_{\text{rate of increase k}}+\underbrace{\bar{u}_j\frac{\partial k}{\partial x_j}}_{\text{Convective transport}}=\underbrace{\frac{\partial}{\partial x_j}\left(\frac{\upsilon_t}{\sigma_k}\frac{\partial k}{\partial x_j}\right)}_{\text{Diffusive transport}}+\underbrace{P}_{\text{rate of production}}-\underbrace{\varepsilon}_{\text{rate of destruction}}$$

The model equation for the turbulent dissipation ε is as follows:

$$\frac{D\varepsilon}{Dt}=\underbrace{\frac{\partial \varepsilon}{\partial t}}_{\text{rate of increase }\varepsilon}+\underbrace{\bar{u}_j\frac{\partial \varepsilon}{\partial x_j}}_{\text{Convective transport}}=\underbrace{\frac{\partial}{\partial x_j}\left(\frac{\upsilon_t}{\sigma_\varepsilon}\frac{\partial \varepsilon}{\partial x_j}\right)}_{\text{Diffusive transport}}+C_{\varepsilon 1}\underbrace{\frac{P\varepsilon}{k}}_{\text{rate of production}}-C_{\varepsilon 2}\underbrace{\frac{\varepsilon^2}{k}}_{\text{rate of destruction}}$$

The standard values of all model constant as fitted with benchmark experiments are (Lauder and Sharma, Letter in heat and mass transfer, 1 (1974), 131-138):

$$C_\mu=0.09,\ \sigma_k=1.00,\ \sigma_\varepsilon=1.30,\ C_{\varepsilon 1}=1.44,\ C_{\varepsilon 2}=1.92$$

The Reynolds stresses are then calculated as follows:

$$-\rho\overline{u_i' u_j'}=\mu_t\left(\frac{\partial \bar{u}_i}{\partial x_j}+\frac{\partial \bar{u}_j}{\partial x_i}\right)-\frac{2\rho}{3}k\delta_{ij}$$

The velocity scale ϑ and length scale ℓ representative of the large-scale turbulence are define in terms of k and ε as follows:

$$\vartheta\,\alpha\,k^{1/2}\qquad \ell\,\alpha\,\frac{k^{3/2}}{\varepsilon}\qquad \upsilon_t\,\alpha\,\vartheta\ell$$

The eddy viscosity is calculated from:

$$\therefore \mu_t = \rho C_\mu \frac{k^2}{\varepsilon}$$

Advantages and disadvantages of k-e model

Advantages	Disadvantages
• Relatively simple to implement. • leads to stable calculations. • widely validated turbulence model.	♦ Poor predications for: o swirling and rotating flows. o flows with strong separation. o certain unconfined flows. o fully developed flows in non-cicular ducts. ♦ valid only for fully developed turbulent flows. ♦ more expensive than mixing length model.

More two-equation models

Many attempts have been made to develop two equation models that improve on the standard k-e model. We will discuss some here:

❖ (ReNormalization Group) RNG k-ε model

❖ k - ω model.

RNG k - ε model

Similar in form to the standard k-ε but includes:

additional term in e equation for interaction between turbulence dissipation and mean shear.

$$\frac{\partial k}{\partial t} + \bar{u}_j \frac{\partial k}{\partial x_j} = \upsilon_t \left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i} \right) \frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial}{\partial x_j} \left(\alpha_k \left(\nu + C_\mu \frac{k^2}{\varepsilon} \right) \frac{\partial k}{\partial x_j} \right) - \varepsilon$$

$$\frac{\partial \varepsilon}{\partial t} + \bar{u}_j \frac{\partial \varepsilon}{\partial x_j} = C_{1\varepsilon} \frac{\varepsilon}{k} \upsilon_t \left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i} \right) \frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial}{\partial x_j} \left(\alpha_\varepsilon \left(\nu + C_\mu \frac{k^2}{\varepsilon} \right) \frac{\partial \varepsilon}{\partial x_j} \right) - C_{1\varepsilon} \frac{\varepsilon^2}{k} - R$$

The standard values of all the model constants are;

$$C_\mu = 0.0845,\ \alpha_k = \alpha_\varepsilon = 1.39,\ \ C_{1\varepsilon} = 1.42,\ C_{2\varepsilon} = 1.68$$

Improved predications for:

- ❖ High streamline curvature and strain rate.
- ❖ Transitional flows.

Wilcox k-ω model

This model which uses the turbulence frequency $\omega = \varepsilon / k$ as the second variables. If we use this variable the length scale is $\ell = \sqrt{k} / \omega$. The eddy viscosity is given by:

$$\mu_t = \frac{\rho k}{\omega}$$

The Reynolds stresses are computed as usual in two-equation models with the Boussinesq expression:

$$-\rho \overline{u_i' u_j'} = \mu_t \left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i} \right) - \frac{2\rho}{3} k \delta_{ij}$$

The transport equation for k and ω at high Reynolds are as follows:

$$\frac{\partial k}{\partial t} + \bar{u}_j \frac{\partial k}{\partial x_j} = \frac{\partial}{\partial x_j} \left(\nu + \frac{\upsilon_t}{\sigma_k} \frac{\partial k}{\partial x_j} \right) + \upsilon_t \left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i} \right) \frac{\partial \bar{u}_i}{\partial x_j} - \beta^* k \omega$$

$$\frac{\partial \omega}{\partial t} + \bar{u}_j \frac{\partial \omega}{\partial x_j} = \frac{\partial}{\partial x_j} \left(\nu + \frac{\upsilon_t}{\sigma_\omega} \frac{\partial \omega}{\partial x_j} \right) + \gamma_1 \upsilon_t \left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i} \right) \frac{\partial \bar{u}_i}{\partial x_j} - \beta_1 k \omega$$

The standard values of all the model constants are;

$\sigma_k = 2.0, \sigma_\omega = 2.0, \;\; \beta_1 = 0.075, \;\; \beta^* = 0.09, \; \gamma_1 = 0.553$

Reynolds Stress Equation Model (RSM)

The most complex classical turbulence model, also called the second-order or second-moment closure model. Several major drawbacks of the *k*–ε model emerge when it is attempted to predict flows with complex strain fields or significant body forces. Under such conditions the individual Reynolds stresses are poorly represented by ***Boussinesq*** expression formula even if the turbulent kinetic energy is computed to reasonable accuracy. The exact Reynolds stress transport equation on the other hand can account for the directional effects of the Reynolds stress field.

The modeling strategy originates from work reported in Launder *et al*. (1975). We follow established practice in the literature and call $R_{ij} = -\tau_{ij} / \rho = \overline{u'_i u'_j}$ the Reynolds stress, although the term kinematic Reynolds stress would be more precise.

The exact equation for the transport of the Reynolds stress $R_{ij} = \overline{u'_i u'_j}$:

$$\frac{DR_{ij}}{Dt} = \overbrace{\frac{\partial R_{ij}}{\partial t}} + \overbrace{C_{ij}} = \underbrace{P_{ij}} + \underbrace{D_{ij}} - \underbrace{\varepsilon_{ij}} + \underbrace{\Pi_{ij}} + \underbrace{\Omega_{ij}}$$

- $\frac{\partial R_{ij}}{\partial t}$: **rate of change of R_{ij}**
- C_{ij}: **Transport of R_{ij} by Convective**
- P_{ij}: **Rate of production of R_{ij}**
- D_{ij}: **Transport of R_{ij} by Diffusive**
- ε_{ij}: **Rate of dissipation of R_{ij}**
- Π_{ij}: **Transport of R_{ij} due to turbulent pressure - strain interactions**
- Ω_{ij}: **Transport of R_{ij} due to rotation**

The above describes six partial differential equations: one for the transport of each of the six independent Reynolds stresses ($\overline{u_1'^2}, \overline{u_2'^2}, \overline{u_3'^2}, \overline{u'_1 u'_2}, \overline{u'_1 u'_3}$, and $\overline{u'_2 u'_3}$).

Large eddy Simulation (LES)

- Tracks the behaviour of the larger eddies
- LES involves space filtering of the unsteady Navier-Stokes equations prior to the computations, which passes the larger eddies and rejects the smaller eddies.
- The interaction effects between the larger, resolved eddies and the smaller unresolved ones, gives rise the sub-grid-scale (SGS) stresses which is described by means of an SGS model.
- The unsteady space filtered equations are solved on a grid of CVs along with the SGS model of the unresolved stresses.

<u>Advantages</u>: Can address CFD problems with complex geometry

Disadvantages: Requires substantial computing resources in terms of storage and volume.

Direct numerical simulation DNS

The instantaneous continuity and Navier–Stokes equations for an incompressible turbulent flow form a closed set of four equations with four unknowns *u*, *v*, *w* and *p*. ***Direct numerical simulation (DNS)*** of turbulent flow takes this set of equations as a starting point and develops a transient solution on a sufficiently fine spatial mesh with sufficiently small time steps to resolve even the smallest turbulent eddies and the fastest fluctuations.

Reynolds (in Lumley, 1989) and Moin and Mahesh (1998) listed the potential benefits of DNSs:

- Precise details of turbulence parameters, their transport and budgets at any point in the flow can be calculated with DNS. These are useful for the development and validation of new turbulence models.
- Instantaneous results can be generated that are not measurable with instrumentation, and instantaneous turbulence structures can be visualized and probed. For example, pressure–strain correlation terms in RSM turbulence models cannot be measured, but accurate values can be computed from DNSs.
- Advanced experimental techniques can be tested and evaluated in DNS flow fields. Reynolds (in Lumley, 1989) noted that DNS has been used to calibrate hot-wire anemometry probes in near-wall turbulence.
- Fundamental turbulence research on virtual flow fields that cannot occur in reality, e.g. by including or excluding individual aspects of flow physics.

Disadvantages:

- On the downside we note that direct solution of the flow equations is very difficult because of the wide range of length and time scales caused by the appearance of eddies in a turbulent flow.
- Highly costly in terms of computing resources.

Concluding Remarks

- Wide range of length and time scales of motion makes the prediction of the effects of turbulence so difficult.
- RANS turbulence models work well in expressing the main features of many turbulent flows by means of one length scale and one time scale.
- The standard k-ε model is widely used in industrial internal flow computation, whereas k-ω model has become established as the leading models for aerospace applications.
- Performance of the improved RANS turbulence models is not uniform. One model does not perform well for all problems.
- Although LES and DNS require substantial computing resources, but these are likely to play increasingly important role in turbulence research.

or in the other words:

- Direct Numerical Simulation (DNS)
  - Theoretically, all turbulent (and laminar / transition) flows can be simulated by numerically solving the full Navier-Stokes equations
  - Resolves the whole spectrum of scales. No modeling is required
  - But the cost is too prohibitive! Not practical for industrial flows
- **Large Eddy Simulation (LES) type models**
  - Solves the spatially averaged N-S equations
  - Large eddies are directly resolved, but eddies smaller than the mesh are modeled
  - Less expensive than DNS, but the amount of computational resources and efforts are still too large for most practical applications
- **Reynolds-Averaged Navier-Stokes (RANS) models**
  - Solve time-averaged Navier-Stokes equations
  - All turbulent length scales are modeled in RANS
    - Various different models are available
  - This is the most widely used approach for calculating industrial flows

N.B. **There is not yet a single, practical turbulence model that can reliably predict all turbulent flows with sufficient accuracy**

Analysis of Turbulent flow in Pipes

We can make the same assumptions (i.e. the same guess) about the functional form of the time-averaged velocity and pressure profile in turbulent flow that we made for laminar flow: we will assume that the time-averaged velocity profile is axisymmetric $(v_\theta = 0,\ \partial / \partial\theta = 0)$ and fully developed $(\partial / \partial z = 0)$.

$$\overline{v}_z = \overline{v}_z(r) \quad , \ \overline{v}_r = \overline{v}_\theta = 0 \quad , \overline{p} = \overline{p}(z)$$

Then the z-component for NSE.

$$\rho\overbrace{\left(\overline{v}_r\frac{\partial\overline{v}_z}{\partial r} + \frac{\overline{v}_\theta}{r}\frac{\partial\overline{v}_z}{\partial\theta} + \overline{v}_z\frac{\partial\overline{v}_z}{\partial z}\right)}^{0} = -\frac{\partial\overline{p}}{\partial z} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\overline{\tau}_{rz}^{T}\right) - \overbrace{\frac{1}{r}\frac{\partial\overline{\tau}_{\theta z}^{T}}{\partial\theta}}^{0} - \overbrace{\frac{\partial\overline{\tau}_{zz}^{T}}{\partial z}}^{0} \qquad z-\text{dir}$$

reduce to:

$$0 = -\frac{\partial\overline{p}}{\partial z} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\overline{\tau}_{rz}^{T}\right)$$

where $\overline{\tau}_{rz}^{T} = \overline{\tau}_{rz} + \overbrace{\overline{\tau}_{rz}^{t}}^{-\rho\overline{v_r'v_z'}}$

We have a function of r only equal to a function of z only. The only way these two terms can be sum to zero for all r and z is if both equal a spatial constant:

$$\frac{d\overline{p}}{dz} = \frac{1}{r}\frac{d}{dr}\left(r\overline{\tau}_{rz}^{T}\right) = -\frac{\Delta p}{L} < 0$$

This implies that pressure varies linearly with z.

Solving for the total stress $\left(\overline{\tau}_{rz}^{T}\right)$ by integrating;

$$\overline{\tau}_{rz}^{T} = -\frac{1}{2}\frac{\Delta p}{L}r + \frac{c}{r} = \overline{\tau}_{rz} + \overline{\tau}_{rz}^{t} \qquad (1)$$

The integration constant c was chosen to be zero to avoid having the stress unbounded at r = 0. Now this is the total stress: the sum of the Reynolds stress.

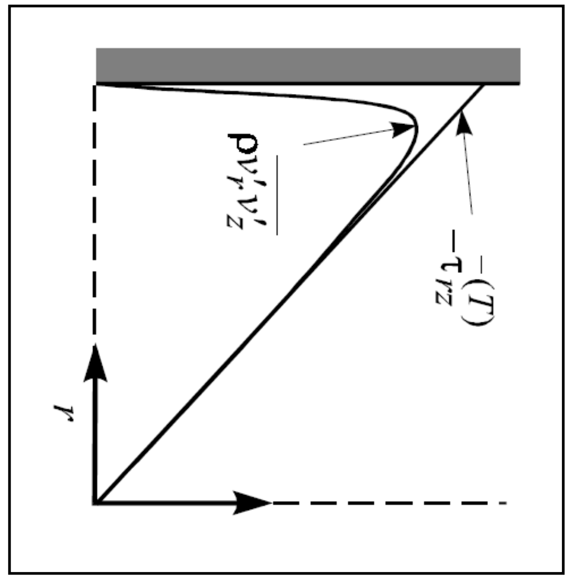
$$\overline{\tau}_{rz}^{t} = -\rho\overline{v_r'v_z'}$$

and a viscous contribution from time-smoothing

Newton's law of viscosity:

$$\bar{\tau}_{rz} = \mu \frac{d\bar{v}_z}{dr}$$

The latter can be determined by differentiating the time-averaged velocity profile. If we subtract this from the total we can determine $\bar{\tau}^t_{rz}$ - one of the component of the Reynolds stress tensor. The result is;



Notice that the Reynolds stress tends to vanish near the wall. This can be explained by noting that at the wall, "no slip" between the fluid can stationary wall requires that the instanteous velocity, as well as its time-average, must be zero:

$$v_z = \bar{v}_z = 0 \;\rightarrow\; v'_z = 0 \quad \Rightarrow \bar{\tau}^t_{rz} = -\rho\overline{v'_r v'_z} = 0$$

In terms of the relative importance of these two contributions to the total, one can define three regions;

- **turbulent core**: $\bar{\tau}^t_{rz} >> \tau$ This covers most of the cross section of the pipe.
- **laminar sublayer**; $\bar{\tau}^t_{rz} << \tau$. Very near the wall, the fluctuations must vanish (along with the Reynolds stress) but the viscous stress are largest.
- **transition zone**: $\bar{\tau}^t_{rz} \approx \tau$. Neither completely dominates the other.

When applied to the situation of fully developed pipe flows, continuity is automatically satisfied and the time-smoothed Navier-Stokes equations yields only **one equation in 2 unknowns**:

$$\bar{v}_z(r) \quad \text{and} \quad \rho\overline{v'_r v'_z}$$

Clearly another relationship is needed to complete the model. This missing relationship is the ***constitutive equation*** relating the Reynolds stress to the time-smoothed velocity profile. One might be tempted to defined a quantity like the viscosity to relate stress to the time-averaged velocity.

$$\overline{\tau}_{rz}^{t} \overset{?}{=} \mu_t \frac{d\overline{v}_z}{dr}$$

But if you define the "turbulent viscosity" this way, its value turns out to depend strongly on position.

$$\frac{\mu_t}{\mu} = \begin{cases} \approx 100 & \text{near pipe centerline} \\ 0 & \text{at pipe wall} \end{cases}$$

So unlike the usual viscosity, (μ_t) is not a material property (since it depends on position rather than just the material).

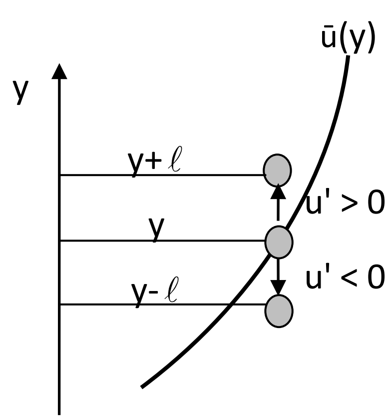
Prandtl's Mixing Length Theory

The first successful constitutive equation for turbulence was posed by Prandtl in 1952. Prandtl imagined that the fluctuations in instantaneous fluid velocity at some fixed point were caused by ***eddies*** of fluid which migrate across the flow from regions having higher or lower time-averaged velocity.

♦ **eddy**- a packet of fluid (much larger than a fluid element) which can undergo random migration across streamlines of the time-smoothed velocity field.

These eddies have a longitudinal velocity which corresponds to the time-average velocity at their previous location.

- **Mixing time**: During this time, the eddy migrates laterally a distance ℓ called the mixing length.
- **Mixing length** (ℓ): characteristic distance an eddy migrates normal to the main flow before mixing.



To estimate the magnitude of the fluctuation, we can expand the time-smoothed velocity profile in Taylor series about y.

$$\bar{u}(y+\ell)=\bar{u}(y)+\left.\frac{d\bar{u}}{dy}\right|_y \ell+\frac{1}{2}\left.\frac{d^2\bar{u}}{dy^2}\right|_y \ell^2+\cdots$$

Assuming that ℓ is sufficiently small that we can truncate this series without introducing significant error.

$$(u')_{above}=\bar{u}(y+\ell)-\bar{u}(y)\approx\ell\frac{d\bar{u}}{dy}$$

where the subscript "above" is appended to remind us that is the fluctuation resulting from an eddy migrating from above. At some later time, another eddy might migrate to our location from below, producing a negative fluctuation in velocity;

$$(u')_{below}=\bar{u}(y-\ell)-\bar{u}(y)\approx-\ell\frac{d\bar{u}}{dy}$$

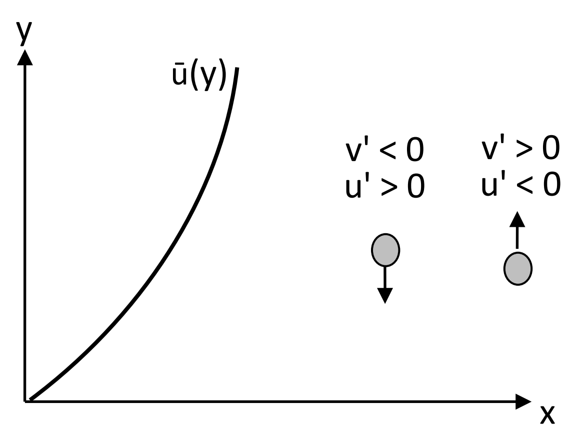
Of course the average fluctuation is zero: $\left(\overline{u'}=0\right)$, but the average of the squares is not;

$$\overline{(u')^2}\approx\frac{1}{2}\left\{(u')^2_{above}+(u')^2_{below}\right\}\approx\ell^2\left(\frac{d\bar{u}}{dy}\right)^2 \qquad (2)$$

Now let's turn our attention to v'. this is related to how fast the eddies migrate, and the sign depends on whether they are migrating upward and downward.

If the eddy migrates from above, it represents a negative y-fluctuation (it is moving in the y-direction). Such an eddy will have a greater x-velocity than the fluid receiving it, consequently generating a positive x-fluctuation:

$$v'<0\rightarrow u'>0\Rightarrow u'v'<0$$



On the other hand, if the eddy migrates from below, it represents a positive y-fluctuation but has x-velocity than the fluid receiving it, generating a negative x-fluctuation:

$$v' > 0 \rightarrow u' < 0 \Rightarrow u'v' < 0$$

Finally, if there is no vertical migration of eddies, there is no reason for the x-velocity to fluctuate:

$$v' = 0 \rightarrow u' = 0$$

These three statements suggest that the y-fluctuations are proportional to the x-fluctuations, with a negative proportionality constant;

$$v' \approx -\alpha u'$$

where α > 0. Alternatively, we can write;

$$v' > 0 \rightarrow u' < 0 \Rightarrow u'v' = -\alpha (u')^2$$

Time averaging and then subs. eq. (2);

$$\overline{u'v'} = -\alpha \overline{(u')^2} = -\alpha \ell^2 \left(\frac{d\bar{u}}{dy} \right)^2$$

Absorbing the unknown α into the (still unknown) mixing length parameter:

$$\bar{\tau}_{xy}^{t} = -\rho \overline{u'v'} = \rho \ell^2 \left(\frac{d\bar{u}}{dy} \right)^2 \qquad (3)$$

which serves as a constitutive equation for turbulent flow. Comparing this with Newton's law of viscosity;

$$\bar{\tau}_{xy} = \mu \frac{d\bar{u}}{dy}$$

we could conclude that an apparent turbulent viscosity is given by:

$$\mu_t = \rho \ell^2 \left| \frac{d\bar{u}}{dy} \right|$$

Of course, this viscosity is not a true fluid property, because it depends strongly on the velocity profile.

For this theory to be useful, we need a value for the "mixing length" ℓ. There are two properties of ℓ which we can easily deduce.

- **First** of all, ℓ was defined as the distance normal to the wall which the eddy travels before becoming mixed with local fluid. Clearly, this mixing must occur before the eddy "bumps" into the wall, so;

$$\text{property \# 1} \qquad \ell < y$$

where y is the distance from the wall.

- **Secondly**, we know from no-slip that the fluctuation all vanish at the well. Consequently, the Reynolds stress must vanish at the wall. Since the velocity gradient does not vanish, we must require that the mixing length vanish at the wall:

$$\text{property \# 2} \qquad \ell = 0 \quad \text{at} \quad y = 0$$

If it's not a constant, the next simplest functional relationship between ℓ and y which satisfies both these properties is:

$$\ell = ay \qquad (4)$$

where a is some constant and $0 < a < 1$.

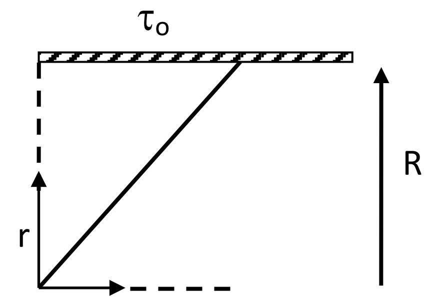
Prandtl's "Universal" Velocity Profile.

The velocity profile in turbulent flow is essentially flat, except near the wall where the velocity gradient are steep. Focusing attention on this region near the flow, Prandtl tried to deduce the form for the velocity profile in turbulent flow. Recall eq. (1) that in pipe flow, the total stress varies linearly from 0 at the center line to a maximum value at the wall:

$$\bar{\tau}_{rz}^{T} = -\frac{1}{2}\frac{\Delta p}{L} r = \tau_o \frac{r}{R} \; < \; 0 \qquad (5)$$

where we have defined

$$\tau_o \equiv -(1/2)(R/L)\Delta p \quad > 0$$



which represents the stress on the wall. In the "turbulent core", the Reynolds stress dominates the "laminar" stress; then substituting eq. (3) through eq. (5):

$$\overline{\tau}_{xy}^{t} \approx \overline{\tau}_{rz}^{T}$$

$$\rho \underbrace{\ell^2}_{a^2y^2} \left(\frac{d\overline{u}}{dy}\right)^2 \approx \tau_o \left(1 - \frac{y}{R}\right) \qquad (6)$$

Dividing through by ρ and subs. eq. (4)

$$a^2y^2 \left(\frac{d\overline{u}}{dy}\right)^2 \approx \underbrace{\frac{\tau_o}{\rho}}_{v*^2} \left(1 - \frac{y}{R}\right) \qquad (7)$$

the ratio t_o/r has units of velocity-squared, which serves as a convenient choice for a characteristic turbulent velocity:

$$v* \equiv \sqrt{\frac{\tau_o}{\rho}}$$

is called the ***friction velocity*** or ***shear velocity***. The dimensionless turbulent velocity will be denoted as;

$$u^+ = \frac{\overline{u}}{v*}$$

Taking the square-root of eq. (7);

$$a\,y \frac{d\overline{u}}{dy} = v* \sqrt{\left(1 - \frac{y}{R}\right)}$$

The general solution to this 1st order ODE is;

$$u^+\left(y^+\right) = C + \frac{2}{a} a \sqrt{1 - \frac{y^+}{R^+}} - \frac{2}{a} \tanh^{-1} \sqrt{1 - \frac{y^+}{R^+}} \qquad (8)$$

where C is the integration constant, and where we have introduced dimensionless variables:

$$u^+ = \frac{\overline{u}}{v*} \quad , \quad y^+ = \frac{v*}{\upsilon} y \quad ,\text{and} \quad R^+ = \frac{v*}{\upsilon} R$$

Near the wall (i.e. for y << R or $y^+ << R^+$), we can simplify eq. (8):

$$\sqrt{1 - \frac{y^+}{R^+}} = 1 - \frac{y^+}{2R^+} + O\left(y^{+2}\right) \qquad (9)$$

$$\tanh^{-1} \sqrt{1 - \frac{y^+}{R^+}} = \frac{1}{2} \ln \frac{4R^+}{y^+} + O\left(y^+\right)$$

Dropping the higher-order terms:

$$u^+\left(y^+\right) = \underbrace{\frac{2-\ln\left(4R^+\right)}{a} + C}_{c} + \frac{1}{a}\ln y^+ \qquad (10)$$

where c is collection of constants.

This result can be derived more easily by starting over with a simplified eq. (7) which applies when y << R.

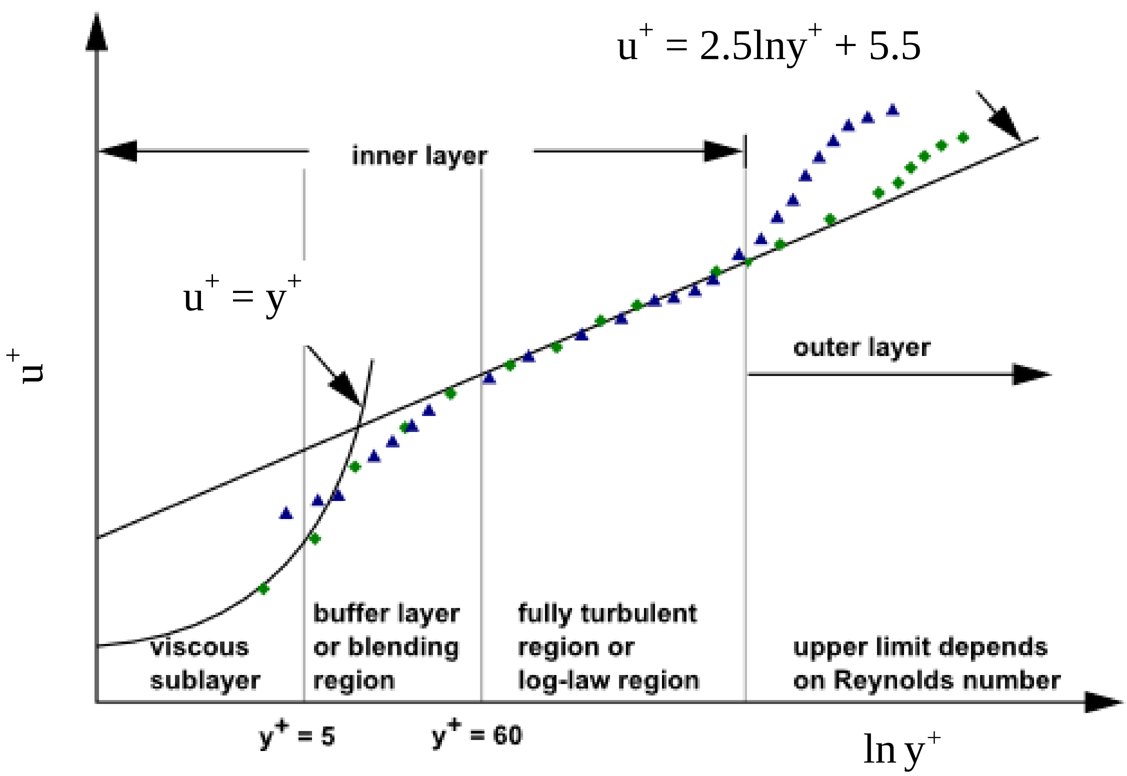
$$ay\frac{d\bar{u}}{dy} = v^*$$

or

$$\underbrace{\frac{d\bar{u}}{v^*}}_{du^+} = \frac{1}{a}\frac{dy}{y} = \frac{1}{a}\frac{dy^+}{y^+} \qquad (11)$$

which integrates to:

$$u^+ = \frac{1}{a}\ln y^+ + c \qquad (12)$$



Time-average turbulent velocity profile near wall

From the above figure on semi-log coordinate, as suggested by eq. (12), experimental velocity profiles do indeed show a linear region which extends over a couple of decades of y^+ values. Moreover, the slope and intercept of this straight line don't seem to depend on the Reynolds number. Indeed, the slope and intercept also don't seem to depends on the shape of the conduit. Rectangular conduits yields the same velocity profile on these coordinates. This is called ***Prandtl's Universal Velocity Profile***:

$y^+ > 26$: $$u^+ = 2.5 \ln y^+ + 5.5 \qquad (13)$$

which applies for $y^+ > 26$ (the **turbulent core**). This coefficient of $\ln y^+$ corresponds to a = 0.4. so eq. (4) becomes;

$$\ell = 0.4\,y$$

Of course eq. (13) also does not apply near the center of the pipe, since $y^+ \approx R^+$ there, whereas eq. (13) was derived by assuming that $y^+ << R^+$.

Laminar Sublayer

In the laminar sublayer, Reynolds stress can be totally neglected, leaving just viscous stress. This close to the wall, the total stress is practically a constant equal to the wall shear stress τ_o.

$y << R$:
$$\bar{\tau}_{xy} \approx \bar{\tau}_{rz}^{T}$$
$$\mu \frac{d\bar{u}}{dy} = \tau_o$$

Then we can integrate the above ODE for $\bar{u}$, and B.C. $\bar{u}$ at y = 0.

$$\bar{u} = \frac{\tau_o}{\mu} y \qquad (14)$$

We can make the result dimensionless;

$$\underbrace{\frac{\bar{u}}{v*}}_{u^+} = \frac{1}{v*}\frac{\tau_o/\rho}{\mu/\rho} y = \frac{1}{v*}\frac{v*^2}{\upsilon} y = \underbrace{\frac{v*}{\upsilon} y}_{y^+}$$

$$\text{or} \quad u^+ = y^+ \qquad (15)$$

which applied for $0 < y^+ < 5$ (the **laminar sublayer).**

Prandtl's Universal Law of Friction

Let's try to figure deduce the analog of Poisueille's Formula for turbulent flow. Poisueille's Formula is the relationship between volumetric flowrate through the pipe and pressure drop. Volumetric flowrate Q is calculated by integrating the axial component of fluid velocity of the cross section of the pipe.

$$\left(\bar{v}_z\right)_{ave.} = \frac{Q}{\pi R^2} = \frac{2}{R^2}\int_0^R r\,\bar{v}_z(r)\,dr \qquad (16) \qquad \text{can be used } \left(\bar{v}_z\right)_{ave.} = \left\langle \bar{v}_z \right\rangle$$

Now we are going to use eq. (13) for the velocity profile, although we assumed in eq. (11) that y << R. (where y = R - r).

The plot at below shows the velocity profiles (with different wall roughness on the walls) compared with predications based on eq. (13). The ordinate is:

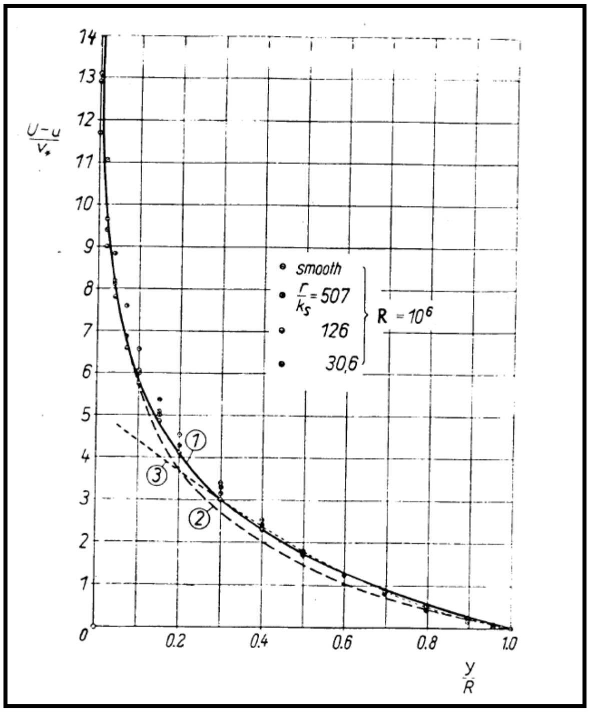
$$\frac{\bar{v}_z(R)-\bar{v}_z(r)}{v^*} = v^+(R^+) - v^+(y^+)$$

$$= \left[2.5\ln R^+ + 5.5\right] - \left[2.5\ln y^+ + 5.5\right] = 2.5\left(\ln R^+ - \ln y^+\right) = 2.5\ln\frac{R^+}{y^+}$$

$$\therefore \frac{\bar{v}_z(R)-\bar{v}_z(r)}{v^*} = 2.5\ln\frac{R}{y}$$

This equation (represented by fig. below) is compared with experimental data in the following figure.

Note that eq. (13) predicts an infinite velocity difference at y = 0, whereas the actual velocity must be finite. Of course, eq. (13) does not apply right up to the wall because very near wall the Reynolds stresses are not dominant.



Substituting eq. (13) in eq.(16) and integrating: (H.W.)

$$\langle \bar{v}_z \rangle = v^* \left[2.5 \ln\left(\frac{v^* R}{\upsilon} \right) + 1.75 \right] \qquad (17)$$

Now the friction velocity can be related to the ***friction factor***, whose usual definition can be expressed in terms of the variables in this analysis:

$$f \equiv \frac{\tau_o}{\frac{1}{2}\rho \langle \bar{v}_z \rangle^2} = 2\left(\frac{v^*}{\langle \bar{v}_z \rangle} \right)^2$$

Thus;

$$\frac{\langle \bar{v}_z \rangle}{v^*} = \sqrt{\frac{2}{f}}$$

Likewise, the usual definition of Reynolds number yields;

$$Re \equiv \frac{2\langle \bar{v}_z \rangle R}{\upsilon}$$

Thus,

$$\frac{v^* R}{\upsilon} = \underbrace{\frac{\langle \bar{v}_z \rangle R}{\upsilon}}_{Re/2} \times \underbrace{\frac{v^*}{\langle \bar{v}_z \rangle}}_{\sqrt{f/2}} = \frac{Re\sqrt{f}}{2\sqrt{2}}$$

Subs. in eq. (17), can be written as;

$$\frac{1}{\sqrt{f}} = 1.77 \ln\left(Re\sqrt{f} \right) - 0.60$$

or

$$\frac{1}{\sqrt{f}} = 4.07 \log_{10}\left(Re\sqrt{f} \right) - 0.60$$

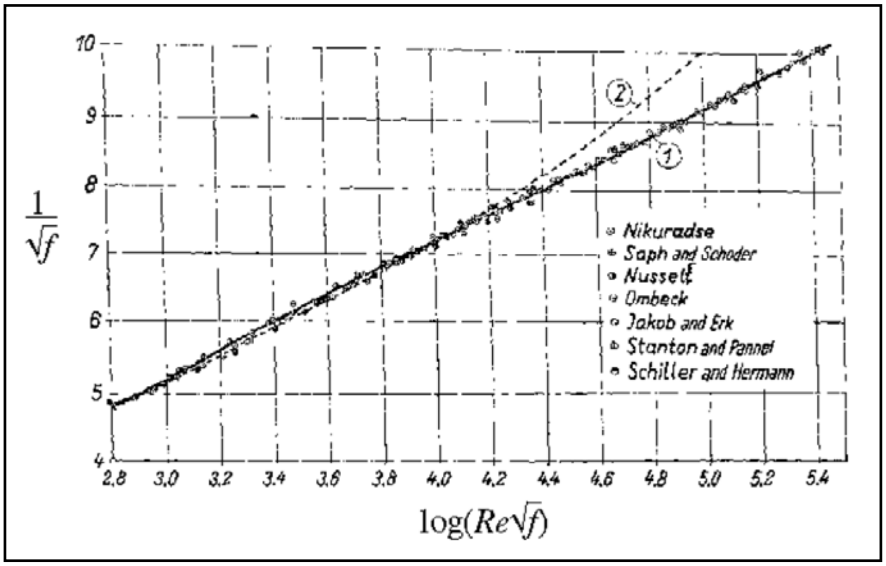


Fig. above shows that experimental data plotted as $1/\sqrt{f}$ versus $1/\sqrt{f}$ does indeed produce a linear relationship. The solid line in figure has slightly different values for the coefficients:

$$\frac{1}{\sqrt{f}} = 4.07 \log_{10}\left(Re\sqrt{f}\right) - 0.60 \qquad \text{for} \qquad 2100 < Re < 5 \times 10^6$$

which is called ***Prandtl's (universal) law of friction***.