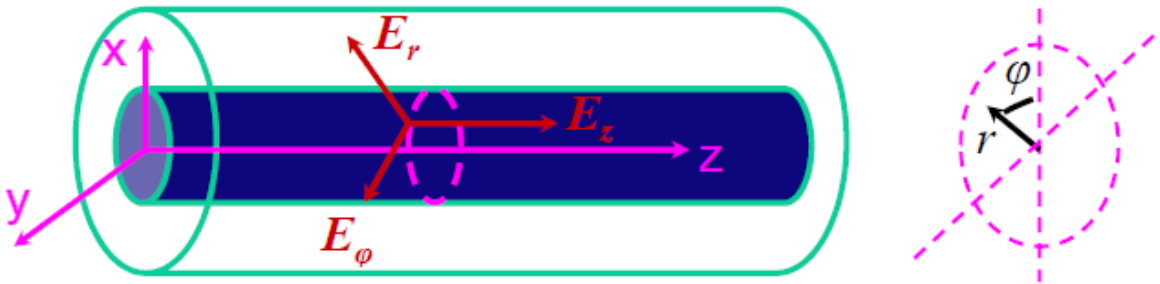➢ Assumption: the cladding radius is infinite

➢ In cylindrical coordinates the Helmholtz equation becomes

$$\frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} + \frac{1}{r^2}\frac{\partial^2 U}{\partial \varphi^2} + \frac{\partial^2 U}{\partial z^2} + n^2 k_0^2 U = 0 \text{ with } \begin{cases} n = n_1 \text{ for } r \le a \\ n = n_2 \text{ for } r > a \\ k_0 = 2\pi/\lambda_0 \end{cases}$$

✓ $U = U(r,\varphi,z) = U(r)U(\varphi)U(z)$

✓ Consider waves travelling in the z-direction ➡ $U(z) = e^{-j\beta z}$

✓ $U(\varphi)$ must be 2π periodic $U(\varphi) = e^{jl\varphi}$, $l = 0, \pm 1, \pm 2 \ldots$ integer

➡

$U(r,\varphi,z) = F(r)e^{-jl\varphi}e^{-j\beta z}$ with $l = 0, \pm 1, \pm 2 \ldots$

Plugging into the Helmoltz Eq. one gets :

$$\frac{d^2 F}{dr^2} + \frac{1}{r}\frac{dF}{dr} + \left(n^2 k_0^2 - \beta^2 - \frac{l^2}{r^2} \right) F = 0 \text{ with } \begin{cases} n = n_1 \text{ for } r \le a \\ n = n_2 \text{ for } r > a \\ k_0 = 2\pi/\lambda_0 \end{cases}$$

One can define an effective index of refraction n_{eff}

such that $\beta = \dfrac{\omega}{c} n_{eff}$, $n_2 < n_{eff} < n_1$

$\beta = k_0 \, n_{eff}$ is the propagation constant

✓ A light wave is guided only if $n_2 k_0 \le \beta \le n_1 k_0$

✓ We introduce

$$\kappa^2 = (n_1 k_0)^2 - \beta^2$$
$$\gamma^2 = \beta^2 - (n_2 k_0)^2$$
$$\kappa^2 + \gamma^2 = k_0^2(n_1^2 - n_2^2) = k_0^2 NA^2 : \text{ constant!}$$

Note : $\kappa^2, \gamma^2 \ge 0$

κ, γ : real

We then get :

$$\frac{d^2F}{dr^2} + \frac{1}{r}\frac{dF}{dr} + \left(\kappa^2 - \frac{l^2}{r^2}\right)F = 0 \text{ for } r \leq a$$

$$\frac{d^2F}{dr^2} + \frac{1}{r}\frac{dF}{dr} - \left(\gamma^2 + \frac{l^2}{r^2}\right)F = 0 \text{ for } r > a$$

The solutions of the equations are of the form :

$$F_l(r) = J_l(\kappa r) \qquad \text{for } \rho \leq a$$

J_l : Bessel function of 1st kind with order l

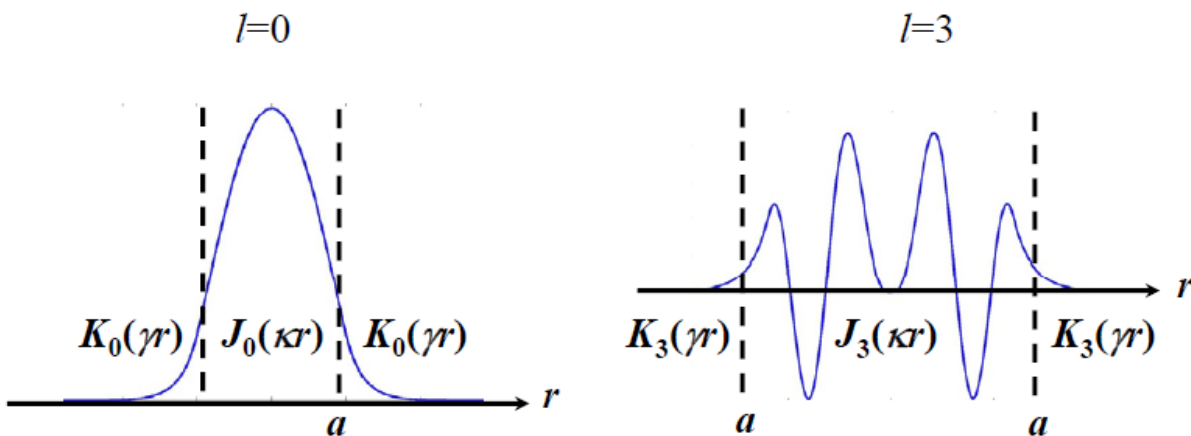$$F_l(r) = K_l(\gamma r) \qquad \text{for } \rho > a$$

K_l : Modified Bessel function of 1st kind with order l

with

$$\kappa^2 = (n_1 k_0)^2 - \beta^2$$

$$\gamma^2 = \beta^2 - (n_2 k_0)^2$$

$$\kappa^2 + \gamma^2 = k_0^2(n_1^2 - n_2^2) = k_0^2 NA^2 : \text{ constant!}$$

l=0   l=3

$$F(r) \propto \begin{cases} J_0(\kappa r) \text{ for } r \leq a \\ K_0(\gamma r) \text{ for } r < a \end{cases}$$

$$F(r) \propto \begin{cases} J_3(\kappa r) \text{ for } r \leq a \\ K_3(\gamma r) \text{ for } r < a \end{cases}$$

Boundary conditions at the core-cladding interface
give a condition on the propagation constant β (characteristics equation)

The propagation constant β can be found by solving
the characteristic equation :

$$\left[\frac{J_l'(K)}{KJ_l(K)} + \frac{K_l'(\Gamma)}{\Gamma K_l(\Gamma)}\right] \times \left[\frac{n_1^2}{n_2^2}\frac{J_l'(K)}{KJ_l(K)} + \frac{K_l'(\Gamma)}{\Gamma K_l(\Gamma)}\right] = \frac{l^2\beta_{lm}^2}{n_2^2 k_0^2}\left[\frac{1}{K^2} + \frac{1}{\Gamma^2}\right]^2$$

with $K = a\kappa$ and $\Gamma = a\gamma$

For each l value there are m solutions for β.

Each value β_{lm} corresponds to a particular fiber mode.

Number of Modes Supported by an Optical Fiber:

Solution of the characteristics equation $U(r,\varphi,z)=F(r)e^{-jl\phi}e^{-j\beta lmz}$ is called a mode, each mode corresponds to a particular electromagnetic field pattern of radiation. The modes are labelled LP_{lm}.

Number of modes M supported by an optical fiber is related to the V parameter (the effective thickness of the optical fiber) defined as:

$$V = ak_0NA = \frac{2\pi a}{\lambda}\sqrt{n_1^2 - n_2^2}$$

Where n_1 , n_2 are refraction index of the core and the cladding, a is core radius, and λ is wavelength of the light propagating through the fiber. Let's remind that the effective thickness of the optical fiber given by equation above is expressed by the same formula as the parameter called the normalized frequency which allow to define a *cut-off frequency*.

Let as suppose M is an increasing function of V.

If $V < 2.405$, $M = 1$ and only the mode LP_{01} propagates: the fiber is said to be Single-Mode. Number of modes well approximated by:

$$M \approx V^2 / 2, \text{ where } V^2 \approx \left(\frac{2\pi a}{\lambda}\right)^2\left(n_1^2 - n_2^2\right)$$

Example: When $2a = 50\ \mu m$, $n_1 = 1.46$, $n_2 = 1.48$ and $\lambda = 1.3\ \mu m$: Calculate the number of modes V.

$$M = V^2/2 = (50\pi/1.3)^2/2\ (1.48^2 - 1.46^2) = \boxed{429.2402}$$

Example: What is the maximum core diameter for *a* fiber if it is to operate in single mode at a wavelength of 1550 *nm*, when the *NA* is 0.12?

From numerical aperture definition *NA*:

$$NA = \sqrt{n_1{}^2 - n_2{}^2}$$

$$V = \left(\frac{2\pi a}{\lambda}\right) \times NA$$

Solving for *a* yields:

$$a = \frac{V\lambda}{2\pi \times NA}$$

For single-mode operation, V must be 2.405 or less. The maximum core diameter occurs when $V = 2.405$. So, plugging into the equation, we get:

$$a_{max} = (2.405)(1550\ nm)/[(2\pi)(0.12)] = \boxed{4.95\ \mu \text{m}}$$

or

$$d_{max} = 2 \times a_{max} = \boxed{9.9\ \mu \text{m}}$$

Example: A manufacturer wishes to make a silica-core, step-index fiber with $V = 75$ and a numerical aperture $NA = 0.30$ to be used 820 *nm*. If $n_1 = 1.458$, what should the core size and cladding index be?

$$a = \frac{V\lambda}{2\pi \times NA}$$

$$a = \frac{(820\ nm)(75)}{2\pi \times (0.3)} = 3.2627 \times 10^4\ nm$$

$$\boxed{a = 32.627\ \mu m}$$

$$n_2 = \sqrt{n_1{}^2 - NA^2}$$

$$n_2 = \sqrt{(1.458)^2 - (0.3)^2}$$

$$\boxed{n_2 = 1.4268}$$

❖Dispersion:

The next important parameter is related to the dispersion phenomena occurring in optical fibers. Generally, dispersion in optical fibers leads to signal degradation. Figure below illustrates the different phenomena leading to signal degradation. The first type of degradation is related to attenuation. However, the attenuation reduces the intensity of signal, but it does not change the time duration of a pulse propagating through the fiber. The pulse remains in its temporal interval.
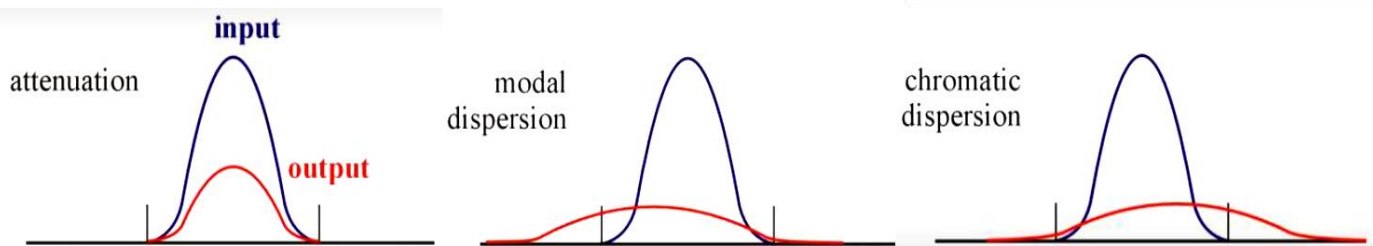
Dispersion, expressed in terms of the symbol Δt, is defined as pulse spreading in an optical fiber. As a pulse of light propagates through a fiber, elements such as numerical aperture, core diameter, refractive index profile, wavelength, and laser linewidth cause the pulse to broaden. This poses a limitation on the overall bandwidth of the fiber as demonstrated in Figure below (*Pulse broadening caused by dispersion*):
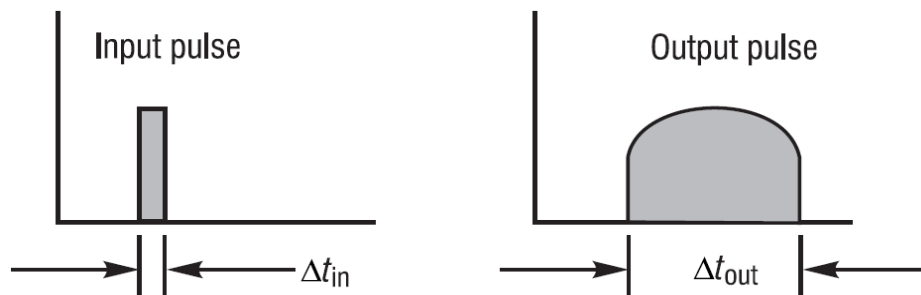
Dispersion Δt can be determined from Equation below:

$$\Delta t = (\Delta t_{out} - \Delta t_{in})^{1/2}$$

and is measured in time, typically nanoseconds or picoseconds. Total dispersion is a function of fiber length. The longer the fiber, the more the dispersion. Equation below gives the total dispersion per unit length.

$$\Delta t_{total} = L \times (\text{Dispersion/km})$$