Parametric Formula for d^2y/dx^2

If the equations $x = f(t)$, $y = g(t)$ define y as a twice-differentiable function of x, then at any point where $dx/dt \neq 0$,

$$\frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt}. \tag{3}$$

EXAMPLE 14 Finding d^2y/dx^2 for a Parametrized Curve

Find d^2y/dx^2 as a function of t if $x = t - t^2,\ y = t - t^3$.

Solution

1. Express $y' = dy/dx$ in terms of t.

$$y' = \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{1 - 3t^2}{1 - 2t}$$

2. Differentiate y' with respect to t.

$$\frac{dy'}{dt} = \frac{d}{dt}\left(\frac{1 - 3t^2}{1 - 2t}\right) = \frac{2 - 6t + 6t^2}{(1 - 2t)^2} \qquad \text{Quotient Rule}$$

3. Divide dy'/dt by dx/dt.

$$\frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt} = \frac{(2 - 6t + 6t^2)/(1 - 2t)^2}{1 - 2t} = \frac{2 - 6t + 6t^2}{(1 - 2t)^3}$$

EXERCISES 3.5 P.201

In Exercises 1–8, given $y = f(u)$ and $u = g(x)$, find $dy/dx = f'(g(x))g'(x)$.

1. $y = 6u - 9, \quad u = (1/2)x^4$ **2.** $y = 2u^3, \quad u = 8x - 1$

3. $y = \sin u, \quad u = 3x + 1$ **4.** $y = \cos u, \quad u = -x/3$

5. $y = \cos u, \quad u = \sin x$ **6.** $y = \sin u, \quad u = x - \cos x$

7. $y = \tan u, \quad u = 10x - 5$ **8.** $y = -\sec u, \quad u = x^2 + 7x$

In Exercises 9–18, write the function in the form $y = f(u)$ and $u = g(x)$. Then find dy/dx as a function of x.

9. $y = (2x + 1)^5$

10. $y = (4 - 3x)^9$

11. $y = \left(1 - \frac{x}{7}\right)^{-7}$

12. $y = \left(\frac{x}{2} - 1\right)^{-10}$

13. $y = \left(\frac{x^2}{8} + x - \frac{1}{x}\right)^4$

14. $y = \left(\frac{x}{5} + \frac{1}{5x}\right)^5$

15. $y = \sec(\tan x)$

16. $y = \cot\left(\pi - \frac{1}{x}\right)$

17. $y = \sin^3 x$

18. $y = 5\cos^{-4} x$

In Exercises 53–58, find the value of $(f \circ g)'$ at the given value of x.

53. $f(u) = u^5 + 1, \quad u = g(x) = \sqrt{x}, \quad x = 1$

54. $f(u) = 1 - \frac{1}{u}, \quad u = g(x) = \frac{1}{1 - x}, \quad x = -1$

55. $f(u) = \cot\frac{\pi u}{10}, \quad u = g(x) = 5\sqrt{x}, \quad x = 1$

56. $f(u) = u + \frac{1}{\cos^2 u}, \quad u = g(x) = \pi x, \quad x = 1/4$

57. $f(u) = \frac{2u}{u^2 + 1}, \quad u = g(x) = 10x^2 + x + 1, \quad x = 0$

58. $f(u) = \left(\frac{u - 1}{u + 1}\right)^2, \quad u = g(x) = \frac{1}{x^2} - 1, \quad x = -1$

In Exercises 87–94, find an equation for the line tangent to the curve at the point defined by the given value of t. Also, find the value of d^2y/dx^2 at this point.

87. $x = 2\cos t, \quad y = 2\sin t, \quad t = \pi/4$

88. $x = \cos t, \quad y = \sqrt{3}\cos t, \quad t = 2\pi/3$

89. $x = t, \quad y = \sqrt{t}, \quad t = 1/4$

90. $x = -\sqrt{t+1}, \quad y = \sqrt{3t}, \quad t = 3$

91. $x = 2t^2 + 3, \quad y = t^4, \quad t = -1$

92. $x = t - \sin t, \quad y = 1 - \cos t, \quad t = \pi/3$

93. $x = \cos t, \quad y = 1 + \sin t, \quad t = \pi/2$

94. $x = \sec^2 t - 1, \quad y = \tan t, \quad t = -\pi/4$

3.6 Implicit Differentiation P.205

Implicit Differentiation

1. Differentiate both sides of the equation with respect to x, treating y as a differentiable function of x.
2. Collect the terms with dy/dx on one side of the equation.
3. Solve for dy/dx.

Example

Show that the point (2, 4) lies on the curve $x^3 + y^3 - 9xy = 0$. Then find the tangent and normal to the curve there (Figure 3.41).

EXAMPLE 5 Finding a Second Derivative Implicitly

Find d^2y/dx^2 if $2x^3 - 3y^2 = 8$.

Related Rates Equations

Suppose we are pumping air into a spherical balloon. Both the volume and radius of the balloon are increasing over time. If V is the volume and r is the radius of the balloon at an instant of time, then

$$V = \frac{4}{3}\pi r^3.$$

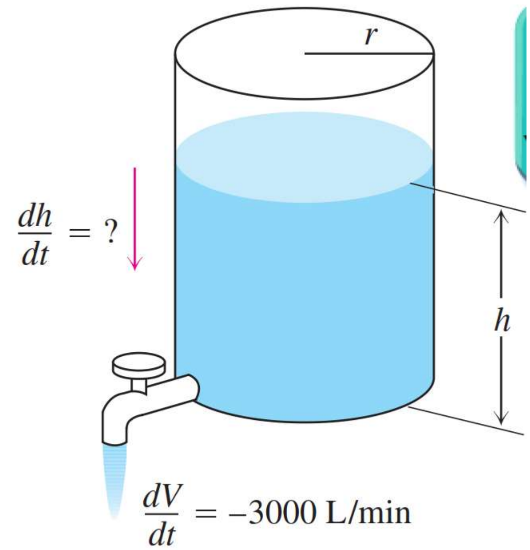
Using the Chain Rule, we differentiate to find the related rates equation

$$\frac{dV}{dt} = \frac{dV}{dr}\frac{dr}{dt} = 4\pi r^2 \frac{dr}{dt}.$$

EXAMPLE 1 Pumping Out a Tank

How rapidly will the fluid level inside a vertical cylindrical tank drop if we pump the fluid out at the rate of 3000 L/min?

solution



$V = 1000\pi r^2 h$ because a cubic meter contains 1000 L.

$$\frac{dV}{dt} = 1000\pi r^2 \frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{-3000}{1000\pi r^2} = -\frac{3}{\pi r^2}$$

EXAMPLE 4 Filling a Conical Tank

Water runs into a conical tank at the rate of 9 ft^3/min. The tank stands point down and has a height of 10 ft and a base radius of 5 ft. How fast is the water level rising when the water is 6 ft deep?