

6. Power Rule: If r and s are integers with no common factor and $s \neq 0$, then

$$\lim_{x \rightarrow c} (f(x))^{r/s} = L^{r/s}$$

provided that $L^{r/s}$ is a real number. (If s is even, we assume that $L > 0$.)

EXAMPLE 1 Using the Limit Laws

Use the observations $\lim_{x \rightarrow c} k = k$ and $\lim_{x \rightarrow c} x = c$ (Example 8 in Section 2.1) and the properties of limits to find the following limits.

$$(a) \lim_{x \rightarrow c} (x^3 + 4x^2 - 3) \quad (b) \lim_{x \rightarrow c} \frac{x^4 + x^2 - 1}{x^2 + 5} \quad (c) \lim_{x \rightarrow -2} \sqrt{4x^2 - 3}$$

THEOREM 2 Limits of Polynomials Can Be Found by Substitution

If $P(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_0$, then

$$\lim_{x \rightarrow c} P(x) = P(c) = a_n c^n + a_{n-1} c^{n-1} + \dots + a_0.$$

THEOREM 3 Limits of Rational Functions Can Be Found by Substitution If the Limit of the Denominator Is Not Zero

If $P(x)$ and $Q(x)$ are polynomials and $Q(c) \neq 0$, then

$$\lim_{x \rightarrow c} \frac{P(x)}{Q(x)} = \frac{P(c)}{Q(c)}.$$

Examples:

Find the following limits:

$$\begin{aligned} 1) \lim_{x \rightarrow 1} \frac{x^2 + x - 2}{x^2 - x} \\ 2) \lim_{x \rightarrow 0} \frac{\sqrt{x^2 + 100} - 10}{x^2} \end{aligned}$$

THEOREM 4 The Sandwich Theorem

Suppose that $g(x) \leq f(x) \leq h(x)$ for all x in some open interval containing c , except possibly at $x = c$ itself. Suppose also that

$$\lim_{x \rightarrow c} g(x) = \lim_{x \rightarrow c} h(x) = L.$$

Then $\lim_{x \rightarrow c} f(x) = L$.

EXAMPLE 5 Applying the Sandwich Theorem

Given that

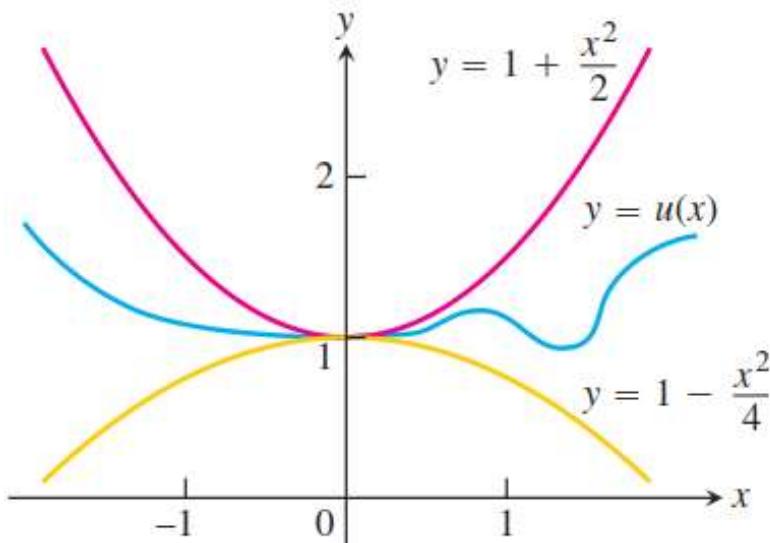
$$1 - \frac{x^2}{4} \leq u(x) \leq 1 + \frac{x^2}{2} \quad \text{for all } x \neq 0,$$

find $\lim_{x \rightarrow 0} u(x)$, no matter how complicated u is.

Solution Since

$$\lim_{x \rightarrow 0} (1 - (x^2/4)) = 1 \quad \text{and} \quad \lim_{x \rightarrow 0} (1 + (x^2/2)) = 1,$$

the Sandwich Theorem implies that $\lim_{x \rightarrow 0} u(x) = 1$ (Figure 2.10).



Example: Using the sandwich theorem and show that:

$$\lim_{\theta \rightarrow 0} \sin \theta = 0.$$

Solution: by the properties of sin function we have

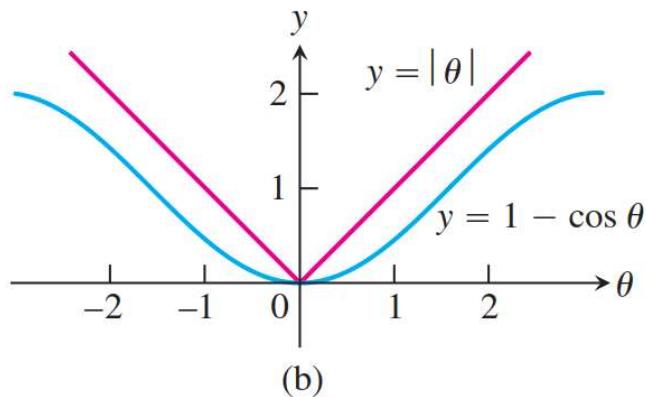
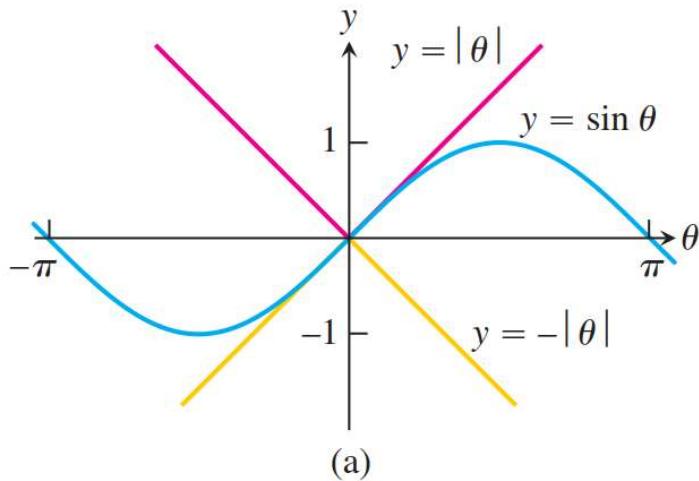
$$-|\theta| \leq \sin \theta \leq |\theta|.$$

Then,

$$\begin{aligned} \lim_{\theta \rightarrow 0} -|\theta| &\leq \lim_{\theta \rightarrow 0} \sin \theta \leq \lim_{\theta \rightarrow 0} |\theta| \\ 0 &\leq \lim_{\theta \rightarrow 0} \sin \theta \leq 0. \end{aligned}$$

Therefore,

$$\lim_{\theta \rightarrow 0} \sin \theta = 0.$$



THEOREM 5 If $f(x) \leq g(x)$ for all x in some open interval containing c , except possibly at $x = c$ itself, and the limits of f and g both exist as x approaches c , then

$$\lim_{x \rightarrow c} f(x) \leq \lim_{x \rightarrow c} g(x).$$

Exercises:

Find the limits in Exercises 1–18.

1. $\lim_{x \rightarrow -7} (2x + 5)$

2. $\lim_{x \rightarrow 12} (10 - 3x)$

3. $\lim_{x \rightarrow 2} (-x^2 + 5x - 2)$

4. $\lim_{x \rightarrow -2} (x^3 - 2x^2 + 4x + 8)$

5. $\lim_{t \rightarrow 6} 8(t - 5)(t - 7)$

6. $\lim_{s \rightarrow 2/3} 3s(2s - 1)$

7. $\lim_{x \rightarrow 2} \frac{x + 3}{x + 6}$

8. $\lim_{x \rightarrow 5} \frac{4}{x - 7}$

9. $\lim_{y \rightarrow -5} \frac{y^2}{5 - y}$

10. $\lim_{y \rightarrow 2} \frac{y + 2}{y^2 + 5y + 6}$

11. $\lim_{x \rightarrow -1} 3(2x - 1)^2$

12. $\lim_{x \rightarrow -4} (x + 3)^{1984}$

13. $\lim_{y \rightarrow -3} (5 - y)^{4/3}$

14. $\lim_{z \rightarrow 0} (2z - 8)^{1/3}$

15. $\lim_{h \rightarrow 0} \frac{3}{\sqrt{3h + 1} + 1}$

16. $\lim_{h \rightarrow 0} \frac{5}{\sqrt{5h + 4} + 2}$

17. $\lim_{h \rightarrow 0} \frac{\sqrt{3h + 1} - 1}{h}$

18. $\lim_{h \rightarrow 0} \frac{\sqrt{5h + 4} - 2}{h}$

Find the limits in Exercises 19–36.

19. $\lim_{x \rightarrow 5} \frac{x - 5}{x^2 - 25}$

20. $\lim_{x \rightarrow -3} \frac{x + 3}{x^2 + 4x + 3}$

21. $\lim_{x \rightarrow -5} \frac{x^2 + 3x - 10}{x + 5}$

22. $\lim_{x \rightarrow 2} \frac{x^2 - 7x + 10}{x - 2}$

23. $\lim_{t \rightarrow 1} \frac{t^2 + t - 2}{t^2 - 1}$

24. $\lim_{t \rightarrow -1} \frac{t^2 + 3t + 2}{t^2 - t - 2}$

25. $\lim_{x \rightarrow -2} \frac{-2x - 4}{x^3 + 2x^2}$

26. $\lim_{y \rightarrow 0} \frac{5y^3 + 8y^2}{3y^4 - 16y^2}$

27. $\lim_{u \rightarrow 1} \frac{u^4 - 1}{u^3 - 1}$

28. $\lim_{v \rightarrow 2} \frac{v^3 - 8}{v^4 - 16}$

29. $\lim_{x \rightarrow 9} \frac{\sqrt{x} - 3}{x - 9}$

30. $\lim_{x \rightarrow 4} \frac{4x - x^2}{2 - \sqrt{x}}$

31. $\lim_{x \rightarrow 1} \frac{x - 1}{\sqrt{x + 3} - 2}$

32. $\lim_{x \rightarrow -1} \frac{\sqrt{x^2 + 8} - 3}{x + 1}$

33. $\lim_{x \rightarrow 2} \frac{\sqrt{x^2 + 12} - 4}{x - 2}$

34. $\lim_{x \rightarrow -2} \frac{x + 2}{\sqrt{x^2 + 5} - 3}$

35. $\lim_{x \rightarrow -3} \frac{2 - \sqrt{x^2 - 5}}{x + 3}$

36. $\lim_{x \rightarrow 4} \frac{4 - x}{5 - \sqrt{x^2 + 9}}$

Using the Sandwich Theorem

49. If $\sqrt{5 - 2x^2} \leq f(x) \leq \sqrt{5 - x^2}$ for $-1 \leq x \leq 1$, find $\lim_{x \rightarrow 0} f(x)$.

50. If $2 - x^2 \leq g(x) \leq 2 \cos x$ for all x , find $\lim_{x \rightarrow 0} g(x)$.

- 51. a.** It can be shown that the inequalities

$$1 - \frac{x^2}{6} < \frac{x \sin x}{2 - 2 \cos x} < 1$$

hold for all values of x close to zero. What, if anything, does this tell you about

$$\lim_{x \rightarrow 0} \frac{x \sin x}{2 - 2 \cos x}?$$

Give reasons for your answer.

- 52. a.** Suppose that the inequalities

$$\frac{1}{2} - \frac{x^2}{24} < \frac{1 - \cos x}{x^2} < \frac{1}{2}$$

hold for values of x close to zero. (They do, as you will see in Section 11.9.) What, if anything, does this tell you about

$$\lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2}?$$

Give reasons for your answer.

- 55.** If $\lim_{x \rightarrow 4} \frac{f(x) - 5}{x - 2} = 1$, find $\lim_{x \rightarrow 4} f(x)$.

- 57. a.** If $\lim_{x \rightarrow 2} \frac{f(x) - 5}{x - 2} = 3$, find $\lim_{x \rightarrow 2} f(x)$.

- b.** If $\lim_{x \rightarrow 2} \frac{f(x) - 5}{x - 2} = 4$, find $\lim_{x \rightarrow 2} f(x)$.

59. Find the following limits

a) $\lim_{x \rightarrow 0} x \sin\left(\frac{1}{x}\right) = 0$

b) $\lim_{x \rightarrow 0} x^2 \cos\left(\frac{1}{x^3}\right) = 0$

More About Limits:

Theorem

$$\lim_{x \rightarrow c} f(x) = L \leftrightarrow \lim_{x \rightarrow c^-} f(x) = L \text{ and } \lim_{x \rightarrow c^+} f(x) = L.$$

Definition: Right-Hand, Left-Hand Limits

We say that $\lim_{x \rightarrow x_0^+} f(x) = L$ (right-hand limit) if

$$\forall \varepsilon > 0, \exists \delta > 0: x_0 < x < x_0 + \delta \Rightarrow |f(x) - L| < \varepsilon.$$

We say that $\lim_{x \rightarrow x_0^-} f(x) = L$ (left-hand limit) if

$$\forall \varepsilon > 0, \exists \delta > 0: x_0 - \delta < x < x_0 \Rightarrow |f(x) - L| < \varepsilon.$$

Theorem

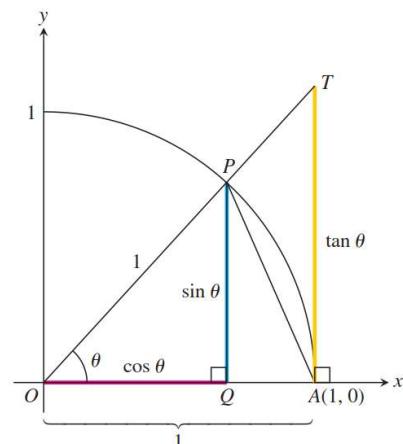
$$\lim_{\theta \rightarrow 0} \frac{\sin(\theta)}{\theta} = 1.$$

Proof:

$$\text{Area } \Delta OAP = \frac{1}{2} \sin \theta$$

$$\text{Area sector } OAP = \frac{\theta}{2}$$

$$\text{Area } \Delta OAT = \frac{1}{2} \tan \theta$$



Now, we have:

$$\begin{aligned}
\frac{1}{2} \sin \theta &< \frac{\theta}{2} < \frac{1}{2} \tan \theta \Rightarrow \sin \theta < \theta < \tan \theta \\
\Rightarrow \{|\sin \theta| &< |\theta| < |\tan \theta|\} \div |\sin \theta| \\
\Rightarrow 1 &< \frac{|\theta|}{|\sin \theta|} < \frac{1}{|\cos \theta|} \\
\Rightarrow 1 &> \frac{|\sin \theta|}{|\theta|} > |\cos \theta| \\
\Rightarrow 1 &> \frac{\sin \theta}{\theta} > \cos \theta \\
\Rightarrow \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} &= 1
\end{aligned}$$

by applying the sandwich theorem.

Examples:

Find the following limits

$$1) \lim_{\theta \rightarrow 0} \frac{\sin \theta}{5\theta} \quad 2) \lim_{\theta \rightarrow 0} \frac{\cos \theta - 1}{\theta}$$

Definition (Limit as $x \rightarrow \pm\infty$)

- 1) We say that $\lim_{x \rightarrow \infty} f(x) = L$ if and only if
 $\forall \varepsilon > 0, \exists N > 0: x > N \Rightarrow |f(x) - L| < \varepsilon.$
- 2) We say that $\lim_{x \rightarrow -\infty} f(x) = L$ if and only if
 $\forall \varepsilon > 0, \exists N < 0: x < N \Rightarrow |f(x) - L| < \varepsilon.$

Example: use the definition and show that

$$\lim_{x \rightarrow \infty} \frac{x}{x+1} = 1$$

Note:

Limits at infinity have properties similar to those of finite limits.

Example:

Find

$$1) \lim_{x \rightarrow \infty} \frac{2x^3 + 3x}{3x^3 + x^2} \quad 2) \lim_{x \rightarrow \infty} \frac{\sin x}{x}$$

Example:

Find the oblique asymptote (if exists) for the function

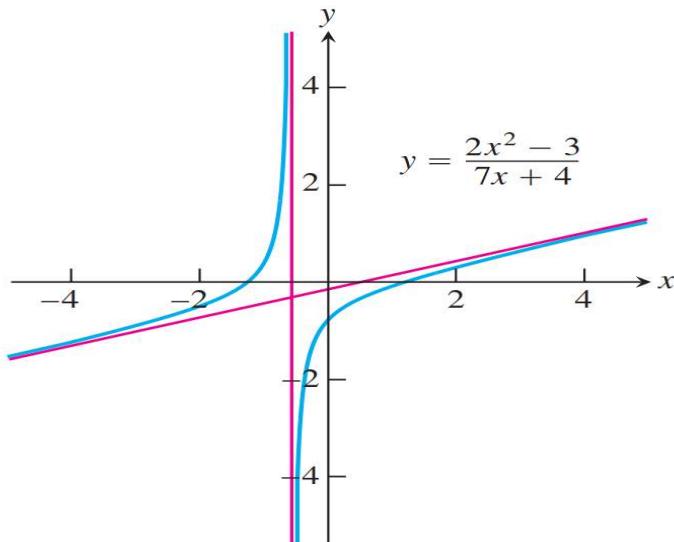
$$f(x) = \frac{2x^2 - 1}{7x + 4}.$$

Solution:

$$\frac{2x^2 - 1}{7x + 4} = \left(\frac{7}{2}x - \frac{8}{49}\right) + \frac{-115}{49(7x + 4)} = g(x) + h(x)$$

$g(x)$ is the linear function, $h(x)$ is the remainder, we have:

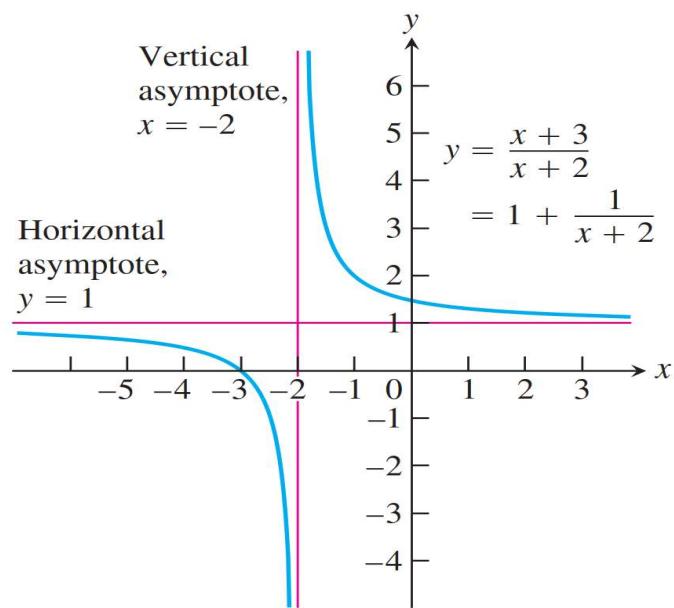
$h(x) \rightarrow 0$ as $x \rightarrow \infty$. Then $y = g(x) = \frac{7}{2}x - \frac{8}{49}$ is oblique asymptote.



Example:

Find the horizontal and vertical asymptotes of the function

$$y = \frac{x+3}{x+2}$$



EXERCISES 2.4 P. 111

7. a. Graph $f(x) = \begin{cases} x^3, & x \neq 1 \\ 0, & x = 1. \end{cases}$

b. Find $\lim_{x \rightarrow 1^-} f(x)$ and $\lim_{x \rightarrow 1^+} f(x)$.

c. Does $\lim_{x \rightarrow 1} f(x)$ exist? If so, what is it? If not, why not?

8. a. Graph $f(x) = \begin{cases} 1 - x^2, & x \neq 1 \\ 2, & x = 1. \end{cases}$

b. Find $\lim_{x \rightarrow 1^+} f(x)$ and $\lim_{x \rightarrow 1^-} f(x)$.

c. Does $\lim_{x \rightarrow 1} f(x)$ exist? If so, what is it? If not, why not?

Find the following limits.

21. $\lim_{\theta \rightarrow 0} \frac{\sin \sqrt{2}\theta}{\sqrt{2}\theta}$

22. $\lim_{t \rightarrow 0} \frac{\sin kt}{t}$ (k constant)

23. $\lim_{y \rightarrow 0} \frac{\sin 3y}{4y}$

24. $\lim_{h \rightarrow 0^-} \frac{h}{\sin 3h}$

25. $\lim_{x \rightarrow 0} \frac{\tan 2x}{x}$

26. $\lim_{t \rightarrow 0} \frac{2t}{\tan t}$

27. $\lim_{x \rightarrow 0} \frac{x \csc 2x}{\cos 5x}$

28. $\lim_{x \rightarrow 0} 6x^2(\cot x)(\csc 2x)$

29. $\lim_{x \rightarrow 0} \frac{x + x \cos x}{\sin x \cos x}$

30. $\lim_{x \rightarrow 0} \frac{x^2 - x + \sin x}{2x}$

31. $\lim_{t \rightarrow 0} \frac{\sin(1 - \cos t)}{1 - \cos t}$

32. $\lim_{h \rightarrow 0} \frac{\sin(\sin h)}{\sin h}$

33. $\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\sin 2\theta}$

34. $\lim_{x \rightarrow 0} \frac{\sin 5x}{\sin 4x}$

35. $\lim_{x \rightarrow 0} \frac{\tan 3x}{\sin 8x}$

36. $\lim_{y \rightarrow 0} \frac{\sin 3y \cot 5y}{y \cot 4y}$

The following limits as $x \rightarrow \infty$ and $x \rightarrow -\infty$