

### Dimensionless analysis

Dimensional analysis is a mathematical technique used to predict physical parameters that influence the flow in fluid mechanics, heat transfer in thermodynamics, and so forth. The analysis involves the fundamental units of dimensions MLT: mass, length, and time. It is helpful in experimental work because it provides a guide to factors that significantly affect the studied phenomena.

Dimensionless analysis is commonly used to determine the relationships between several variables, i.e. to find the force as a function of other variables when an exact functional relationship is unknown. Based on understanding of the problem, we assume a certain functional form.

Before beginning, it is important to define some dimensionless parameters:

**1- Reynolds number:** the ratio of inertia force to viscous force. It is important in all types of fluid dynamics problems

$$Re = \frac{\rho V l}{\mu}$$

**2- Mach number:** the ratio of inertia force to compressibility force. It is important in flows in which the compressibility of the fluid is important.

$$M = \frac{V}{C}$$

Where C is the speed of sound,  $C = \sqrt{\gamma RT}$  for gas, and  $C = \sqrt{\frac{E}{\rho}}$  for liquids, where E is the bulk modulus of compression.

**3- Froude number:** the ratio of inertia force to gravitational force. It is important in flow with a free surface.

$$Fr = \frac{V}{\sqrt{gl}}$$

**4- Weber number:** the ratio of inertia force to surface tension force. It is important in problems in which surface tension is important

$$We = \frac{\rho V^2 l}{\sigma}$$

**5- Euler number:** the ratio of the pressure force to inertia force. It is important in problems in which pressure, or pressure differences, are of interest.

$$Eu = \frac{P}{\rho V^2}$$

**Notes:**

$l$ : is a characteristic length for the system.

**Units/Dimensions**

The defined units are based on the modern MLT system: mass, length, time. All other quantities can be express in terms of these basic units.

For example,

velocity	m/s	L/T
acceleration	m/s <sup>2</sup>	L/T <sup>2</sup>
force	kgm/s <sup>2</sup>	ML/T <sup>2</sup>

Where  $L/T$ ,  $L/T^2$ ,  $ML/T^2$ , etc. are referred to as the derived units. Another system for dimensionless analysis is the FLT system, the force, length, time system. In this case,  $\text{mass} \equiv F/a$ , which makes the units of mass as  $FT^2/L$ , since acceleration has units of  $L/T^2$ .

### Rayleigh Method

An elementary method for finding a functional relationship with respect to a parameter in interest is the Rayleigh Method, and will be illustrated with an example, using the MLT system.

Assume that we are interested in the drag,  $F_D$ , which is a force, on a ship. What exactly is the drag a function of? These variables need to be chosen correctly, though selection of such variables depends largely on one's experience in the topic. It is known that drag depends on

Quantity	Symbol	Dimension
size	$l$	$L$
Viscosity	$\mu$	$m/LT$
Density	$\rho$	$m/L^3$
Velocity	$V$	$L/T$
Gravity	$g$	$L/T^2$

This means that  $F_D = f(l, \rho, \mu, V, g)$  where  $f$  is some function. With the Rayleigh Method, we assume that  $F_D = C l^a \rho^b \mu^c V^d g^e$ , where  $C$  is a dimensionless constant, and  $a, b, c, d$ , and  $e$  are exponents, whose values are not yet known. Note that the dimensions of the left side, force, must equal those on the right side. Here, we use only the three independent dimensions for the variables on the right side: M, L, and T.

#### Step 1: Setting up the equation

Write the equation in terms of dimensions only, i.e. replace the quantities with their respective units. The equation then becomes

$$\frac{ML}{T^2} = (L)^a \left(\frac{M}{L^3}\right)^b \left(\frac{M}{LT}\right)^c \left(\frac{L}{T}\right)^d \left(\frac{L}{T^2}\right)^e$$

On the left side, we have  $M^1 L^1 T^{-2}$ , which is equal to the dimensions on the right side. Therefore, the exponents of the right side must be such that the units are  $M^1 L^1 T^{-2}$

### Step 2: Solving for the exponents

Equate the exponents to each other in terms of their respective fundamental units:

$$M: 1 = b + c \text{ since } M^1 = M^b M^c$$

$$L: 1 = a - 3b - c + d + e \text{ since } L^1 = L^a L^{-3b} L^{-c} L^d L^e$$

$$T: -2 = -c - d - 2e \text{ since } T^{-2} = T^{-c} T^{-d} T^{-2e}$$

It is seen that there are three equations, but 5 unknown variables. This means that a complete solution cannot be obtained. Thus, we choose to solve  $a$ ,  $b$ , and  $d$  in terms of  $c$  and  $e$ . These choices are based on experience. Therefore,

$$\text{From M: } b = 1 - c \quad (i)$$

$$\text{From T: } d = 2 - c - 2e \quad (ii)$$

$$\text{From L: } a = 1 + 3b + c - d - e \quad (iii)$$

Solving (i), (ii), and (iii) simultaneously, we obtain  $a = 2 - c + e$

Substituting the exponents back into the original equation, we obtain

$$F_D = C l^{2+e-c} \rho^{1-c} \mu^c V^{2-c-2e} g^e$$

$$\text{Collecting like exponents together, } FD = C \left(\frac{V^2}{lg}\right)^{-e} \left(\frac{Vl\rho}{\mu}\right)^{-c} \rho l^2 V^2$$

Which means

$$FD = C l^2 l^e l^{-c} \rho \rho^{-c} \mu^c V^2 V^{-c} V^{-2e} g^e$$

For the different exponents,

Terms with exponent of 1:  $C\rho$

Terms with exponent of 2:  $l^2 V^2$

$$\text{Terms with exponent of } e: l^e V^{-2e} g^e = \left(\frac{lg}{V^2}\right)^e = \left(\frac{V^2}{lg}\right)^{-e} \quad (iv)$$

$$\text{Terms with exponent of } c: l^{-c} \rho^{-c} \mu^c V^{-c} = \left(\frac{l\rho V}{\mu}\right)^{-c} \quad (v)$$

The right sides of (iv) and (v) are known as the dimensionless groups.

**Step 3: Determining the dimensionless groups**

Note that  $e$  and  $c$  are unknown. Consider the following cases:

If  $e = 1$  then (iv) becomes  $\left(\frac{lg}{V^2}\right)$

If  $e = -1$  then (iv) becomes  $\left(\frac{V^2}{lg}\right)$

If  $c = 1$  then (v) becomes  $\left(\frac{\mu}{l\rho V}\right)$

If  $c = -1$  then (v) becomes  $\left(\frac{l\rho V}{\mu}\right) = \left(\frac{lV}{\nu}\right)$

Where  $\nu$  is the kinematic viscosity of the fluid. And so on for different exponents. It turns out that:

$$\text{Reynolds number} = Re = \frac{Vl}{\nu}$$

$$\text{Froude number} = Fr = \left(\frac{V^2}{lg}\right)^{\frac{1}{2}} = \frac{V}{\sqrt{lg}}$$

Where  $Re$  and  $Fr$  are the usual notations for the Reynolds and Froude Numbers respectively. Such dimensionless groups keep reoccurring throughout Fluid Mechanics and other fields.

Choosing exponents of  $-1$  for  $c$  and  $-1/2$  for  $e$ , which result in the Reynolds and Froude Numbers respectively, we obtain

$$F_D = g(Fr, Re)\rho l^2 V^2$$

Where  $g(Fr, Re)$  is a dimensionless function,

$$\text{This can also be written as } \frac{F_D}{\rho l^2 V^2} = g(Fr, Re)$$

Which is a dimensionless quantity, and a function of only 2 variables instead of 5. This dimensionless quantity turns out to be the drag coefficient,  $C_D$ .

$$C_D \equiv \frac{F_D}{\rho l^2 V^2}$$

**Notes**

The Rayleigh Method has limitations because of the premise that an exponential relationship exists between the variables.

**The Buckingham  $\pi$  Theorem/Method**

This method will be illustrated by the same example as that for Rayleigh Method, the drag on a ship. Say that we have  $n$  number of quantities (e.g. 6 quantities, which are  $D, l, \rho, \mu, V$ , and  $g$ ) and  $m$  number of dimensions

(e.g. 3 dimensions, which are  $M, L$ , and  $T$ ). These quantities can be reduced to  $(n - m)$  independent dimensionless groups, such as  $Re$  and  $Fr$ . Say that

$$A_1 = f(A_2, A_3, A_4, \dots, A_n).$$

where  $A_x$  are quantities such as drag, length, and so forth, as mentioned under the  $n$  number of quantities, and  $f$  implies the functional relationship between  $A_1$  and the other quantities.

Then re-arranging, we obtain

$$0 = f(A_2, A_3, A_4, \dots, A_n) - A_1 \\ = f(A_1, A_2, A_3, \dots, A_n)$$

Which can be further reduced, using the Buckingham  $\pi$  Theorem, to obtain

$$0 = f(\pi_1, \pi_2, \dots, \pi_{n-m})$$

### Forming $\pi$ Groups

For each  $\pi$  group, take  $m$  of the quantities,  $A_x$ , known as  $m$  repeating variables, and one of the other remaining variables. Note that experience dictates which quantities make the best repeating variables. The  $\pi$  groups, in general form, would then be

$$\pi_1 = A_1^x A_2^y A_3^z A_4$$

$$\pi_2 = A_1^x A_2^y A_3^z A_5$$

□

$$\pi_{n-m} = A_1^x A_2^y A_3^z A_n$$

which are all dimensionless quantities.

### Step 1: Setup $\pi$ groups

For the MLT System,  $m = 3$ , so choose  $A_1$ ,  $A_2$ , and  $A_3$  as the repeating variables. Using the Buckingham  $\pi$  Theorem on the Drag Equation:

$$f(D, l, \rho, \mu, V, g) = 0$$

Where  $m = 3$ ,  $n = 6$ , so there will be  $n - m = 3$   $\pi$  groups. We will select  $\rho$ ,  $V$ , and  $l$  as the repeating variables (RV), leaving the remaining quantities as  $D$ ,  $\mu$ , and  $g$ . Note that if

the analysis does not work out, we could always go back and repeat using new RVs. Thus,

$$\pi_1 = \rho^{x_1} V^{y_1} l^{z_1} F_D$$

$$\pi_2 = \rho^{x_2} V^{y_2} l^{z_2} \mu$$

$$\pi_3 = \rho^{x_3} V^{y_3} l^{z_3} g$$

Which are all dimensionless quantities, i.e. having units of  $M^0 L^0 T^0$

## Step 2: Determine $\pi$ groups

For the first  $\pi$  group,

$$\pi_1 \quad M^0 L^0 T^0 = \left(\frac{M}{L^3}\right)^{x_1} \left(\frac{L}{T}\right)^{y_1} (L)^{z_1} \left(\frac{ML}{T^2}\right)$$

Expanding and collecting like units, we can solve for the exponents:

$$\text{For M: } 0 = x_1 + 1 \Rightarrow x_1 = -1$$

$$\text{For T: } 0 = -y_1 - 2 \Rightarrow y_1 = -2$$

$$\text{For L: } 0 = -3x_1 + y_1 + z_1 + 1 \Rightarrow z_1 = 3(-1) - (-2) - 1 = -2$$

Therefore, we find that the exponents  $x_1$ ,  $y_1$ , and  $z_1$  are -1, -2, and -2 respectively.

This means that the first dimensionless  $\pi$  group,  $\pi_1$ , is

$$\pi_1 = \rho^{-1} V^{-2} l^{-2} F_D = \frac{F_D}{\rho V^2 l^2}$$

For the second  $\pi$  group,

$$\pi_2 \quad M^0 L^0 T^0 = \left(\frac{M}{L^3}\right)^{x_2} \left(\frac{L}{T}\right)^{y_2} (L)^{z_2} \left(\frac{M}{LT}\right)$$

Solving for the exponents,

$$\text{For M: } x_2 + 1 = 0 \Rightarrow x_2 = -1$$

$$\text{For T: } -y_2 - 1 = 0 \Rightarrow y_2 = -1$$

$$\text{For L: } -3x_2 + y_2 + z_2 - 1 = 0 \Rightarrow z_2 = 1 - (-1) + 3(-1) = -1$$

Thus,

$$\pi_2 = \rho^{-1} V^{-1} l^{-1} \mu = \frac{\mu}{\rho V l} = \frac{\nu}{V l}$$

However, we will now invert  $\pi_2$  so that  $\pi_2 = \frac{Vl}{\nu} = \text{Reynolds Number}$

For the third  $\pi$  group,

$$\pi_3 \quad M^0 L^0 T^0 = \left(\frac{M}{L^3}\right)^{x_3} \left(\frac{L}{T}\right)^{y_3} (L)^{z_3} \left(\frac{L}{T^2}\right)$$

Solving for the exponents,

For M:  $x_3 = 0 \Rightarrow x_3 = 0$

For T:  $-y_3 - 2 = 0 \Rightarrow y_3 = -2$

For L:  $-3x_3 + y_3 + z_3 + 1 = 0 \Rightarrow z_3 = -1 - (-2) = 1$

$$\text{Thus, } \pi_3 = \rho^0 V^{-2} l g = \frac{lg}{V^2}$$

Raising it to the power of  $-1/2$ , we get  $\sqrt{\frac{1}{\pi_3}} = \frac{V}{\sqrt{lg}} = \text{Froude Number}$

Thus, the three  $\pi$  groups can be written together as

$$f\left(\frac{F_D}{\rho V^2 l^2}, Re, Fr\right) = 0$$

Finally,

$$\frac{F_D}{\rho V^2 l^2} = f(Re, Fr)$$

Note that this is the same result as obtained with the Rayleigh Method, but with the Buckingham  $\pi$  Method, we did not have to assume a functional dependence.

Notes:

- 1- The repeating variables must represent, as possible as: geometry, fluid properties, external effects
- 2- It is permissible to exponentiate any  $\pi$  group, e.g.  $\pi^{-1}$ ,  $\pi^{1/2}$ ,  $\pi^2$ , etc., to form a new group, as this does not alter the functional form.
- 3- If the problem contains dimensionless variable, this variable can be directly considered as  $\pi$  parameter.



- 4- If the analysis does not work out, we could always go back and repeat using new repeating variables.

Example 6.1: The resistance to motion of a sphere of diameter  $D$  moving at velocity  $V$  in compressible liquid of density  $\rho$  and bulk modulus of compression  $E$ . Show that the dimensionless relationship between these quantities is a function of Mach number.

Example 6.2: The losses  $\frac{\Delta h}{l}$  per unit length of pipe in turbulent flow through smooth pipe depend upon velocity  $V$ , diameter  $D$ , gravity  $g$ , viscosity  $\mu$ , and density  $\rho$ . Determine the dimensionless relationship of these variables.

Example 6.3: A finite cylinder is submerged and exposed to an external viscous flow. The researcher suggests that the resistance to flow  $F$  is a function of the radius  $r$ , velocity  $V$ , the density,  $\rho$ , and the viscosity  $\mu$ . Based on these information construct a relationship of the variables.

Example 6.4: the torque  $T$  of a propeller of a ship is known to be function of the propeller diameter  $D$ , fluid density  $\rho$ , relative velocity of the ship to water  $u$ , rotation speed (rpm)  $N$ , and fluid viscosity  $\mu$ . What is  $T$  a function of?