Chapter Four: Poly-phase Circuits

One possible representation of a three-phase system of voltages is shown in Fig. 4.1. Let us assume that the voltages V_{an}, V_{bn}, and V_{cn} are known:

$$V_{an} = 100\angle 0^\circ \text{ V}$$
$$V_{bn} = 100\angle -120^\circ \text{ V}$$
$$V_{cn} = 100\angle -240^\circ \text{ V}$$

The voltage V_{ab} may be found, with an eye on the subscripts, as

$$V_{ab} = V_{an} + V_{nb} = V_{an} - V_{bn} = 100\angle 0^\circ - 100\angle -120^\circ = 173.2\angle 30^\circ \text{ V}$$

The three given voltages and the construction of the phasor V_{ab} are shown on the phasor diagram of Fig. 4.2.

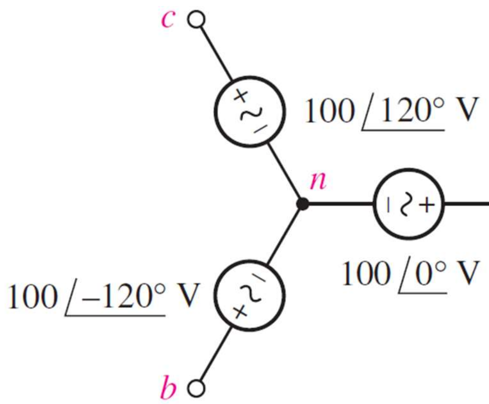


Fig. 4.1 Three phase power supply.

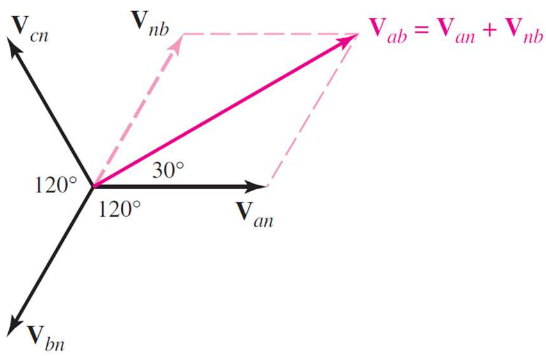


Fig. 4.2 The phasor diagram

4.1 Single-phase three wire system

A single-phase three-wire source is defined as a source having three output terminals, such as a, n, and b in Fig. 4.3a, at which the phasor voltages V_{an} and V_{nb} are equal. The source may therefore be represented by the combination of two identical voltage sources in Fig. 4.3b, $V_{an} = V_{nb} = V_1$.

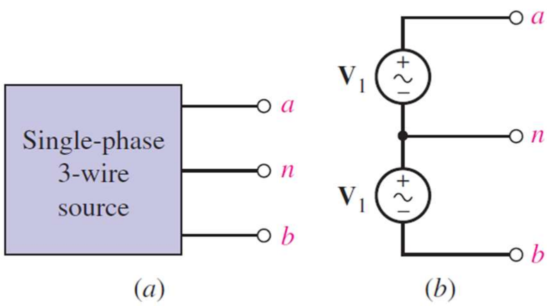


Fig. 4.3 (a) A single-phase three-wire source. (b) The representation of a single-phase three-wire source by two identical voltage sources.

- The name single-phase arises because the voltages V_{an} and V_{nb}, being equal, must have the same phase angle. From another viewpoint, however, the voltages between the outer wires and the central wire, which is usually referred to as the neutral, are exactly 180° out of phase. That is, $V_{an} = -V_{bn}$, and $V_{an} + V_{bn} = 0$.

- Later, we will see that balanced polyphaser systems are characterized by a set of voltages of equal amplitude whose (phasor) sum is zero. From this viewpoint, then, the single-phase three-wire system is really a balanced two-phase system. Two-phase, however, is a term that is traditionally reserved for a relatively unimportant unbalanced system utilizing two voltage sources 90° out of phase.

Let us now consider a single-phase three-wire system that contains identical loads Z_p between each outer wire and the neutral (Fig. 4.4). We first assume that the wires connecting the source to the load are perfect conductors. Since

$$V_{an} = V_{nb}$$

then,

$$I_{aA} = V_{an}/Z_p = I_{Bb} = V_{nb}/Z_p$$

and therefore

$$I_{nN} = I_{Bb} + I_{Aa} = I_{Bb} - I_{aA} = 0$$

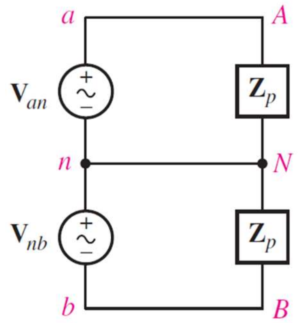


Fig. 4.4

Example 4.1: Analyse the system shown in Fig. 4.5 and determine the power delivered to each of the three loads as well as the power lost in the neutral wire and each of the two lines.

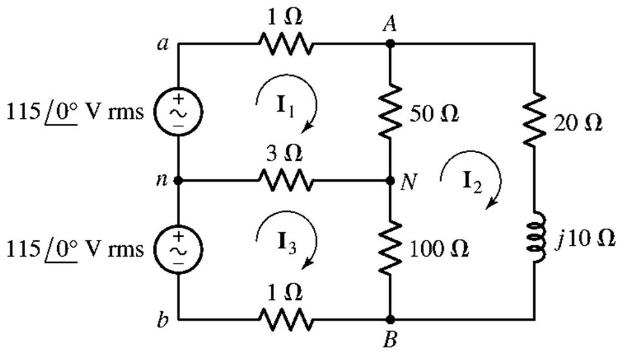


Fig. 4.5

Solution:

The three mesh equations are:

$$-115\underline{/0^\circ} + \mathbf{I}_1 + 50(\mathbf{I}_1 - \mathbf{I}_2) + 3(\mathbf{I}_1 - \mathbf{I}_3) = 0$$
$$(20 + j10)\mathbf{I}_2 + 100(\mathbf{I}_2 - \mathbf{I}_3) + 50(\mathbf{I}_2 - \mathbf{I}_1) = 0$$
$$-115\underline{/0^\circ} + 3(\mathbf{I}_3 - \mathbf{I}_1) + 100(\mathbf{I}_3 - \mathbf{I}_2) + \mathbf{I}_3 = 0$$

which can be rearranged to obtain the following three equations

$$\begin{aligned}
54\mathbf{I}_1 \quad & -50\mathbf{I}_2 & -3\mathbf{I}_3 \quad & = 115\underline{/0^\circ} \\
-50\mathbf{I}_1 \quad & +(170 + j10)\mathbf{I}_2 & -100\mathbf{I}_3 \quad & = 0 \\
-3\mathbf{I}_1 \quad & -100\mathbf{I}_2 & +104\mathbf{I}_3 \quad & = 115\underline{/0^\circ}
\end{aligned}$$

Solving for the phasor currents I_1, I_2, and I_3 using a scientific calculator, we find

$I_1 = 11.24\angle{-19.83^\circ}$ A

$I_2 = 9.389\angle{-24.47^\circ}$ A

$I_3 = 10.37\angle{-21.80^\circ}$ A

The currents in the outer lines are thus

$I_{aA} = I_1 = 11.24\angle{-19.83^\circ}$ A

and

$I_{bB} = -I_3 = 10.37\angle{158.20^\circ}$ A

and the smaller neutral current is

$I_{nN} = I_3 - I_1 = 0.9459\angle{-177.7^\circ}$ A

The average power drawn by each load may thus be determined:

$P_{50} = |I_1 - I_2|^2\,(50) = 206$ W

$P_{100} = |I_3 - I_2|^2\,(100) = 117$ W

$P_{20+j10} = |I_2|^2\,(20) = 1763$ W

The total load power is 2086 W. The loss in each of the wires is next found:

$P_{aA} = |I_1|^2\,(1) = 126$ W

$P_{bB} = |I_3|^2\,(1) = 108$ W

$P_{nN} = |I_{nN}|^2\,(3) = 3$ W

giving a total line loss of 237 W. The wires are evidently quite long; otherwise, the relatively high power loss in the two outer lines would cause a dangerous temperature rise.

 # Verify the solution. Is it reasonable or expected?

The total absorbed power is 206 + 117 + 1763 + 237, or 2323 W, which may be checked by finding the power delivered by each voltage source:

$P_{an} = 115(11.24)\cos 19.83^\circ = 1216$ W

$P_{bn} = 115(10.37)\cos 21.80^\circ = 1107$ W

or a total of 2323 W. The transmission efficiency for the system is

η = total power delivered to load / total power generated = 2086/(2086 + 237)

= 89.8%

H.W.: Modify Fig. 4.5 by adding a 1.5 Ω resistance to each of the two outer lines, and a 2.5 Ω resistance to the neutral wire. Find the average power delivered to each of the three loads.

4.2 3-phase balance and unbalance systems with star and delta connections

4.2.1 THREE-PHASE BALANCE SYSTEM WITH Y-Y CONNECTION

It may be represented by three ideal voltage sources connected in a Y, as shown in Fig. 4.6; terminals a, b, c, and n are available. We will consider only balanced three-phase sources, which may be defined as having

$$|V_{an}| = |V_{bn}| = |V_{cn}|$$

and

$$V_{an} + V_{bn} + V_{cn} = 0$$

These three voltages, each existing between one line and the neutral, are called phase voltages. If we arbitrarily choose V_{an} as the reference, or define

$$V_{an} = V_p\angle 0^{\circ}$$

where we will consistently use V_p to represent the rms amplitude of any of the phase voltages, then the definition of the three-phase source indicates that either

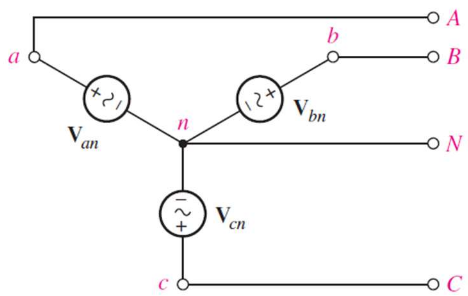


Fig. 4.6.

$$V_{bn} = V_p\angle -120^{\circ} \text{ and } V_{cn} = V_p\angle -240^{\circ}$$

or

$$V_{bn} = V_p\angle 120^{\circ} \text{ and } V_{cn} = V_p\angle 240^{\circ}$$

The former is called **positive phase sequence, or abc phase sequence**, and is shown in Fig. 4.7a; the latter is termed **negative phase sequence, or cba phase sequence**, and is indicated by the phasor diagram of Fig. 4.7b.

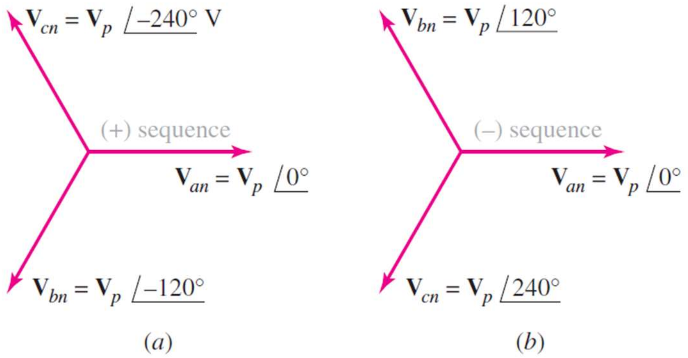


Fig. 4.7: (a) Positive, or abc, phase sequence. (b) Negative, or cba, phase sequence.

Let us next find the line-to-line voltages (often simply called the line voltages) which are present when the phase voltages are those of Fig. 4.7a. It is easiest to do this with the help of a phasor diagram, since the angles are all multiples of 30°. The necessary construction is shown in Fig. 4.8; the results are

$$V_{ab} = \sqrt{3}V_p\angle 30^{\circ}$$
$$V_{bc} = \sqrt{3}V_p\angle -90^{\circ}$$
$$V_{ca} = \sqrt{3}V_p\angle -210^{\circ}$$

Kirchhoff's voltage law requires the sum of these three voltages to be zero; the reader is encouraged to verify this as an exercise. If the rms amplitude of any of the line voltages is denoted by V_L, then one of the important characteristics of the Y-connected three-phase source may be expressed as

$V_L = \sqrt{3}V_p$

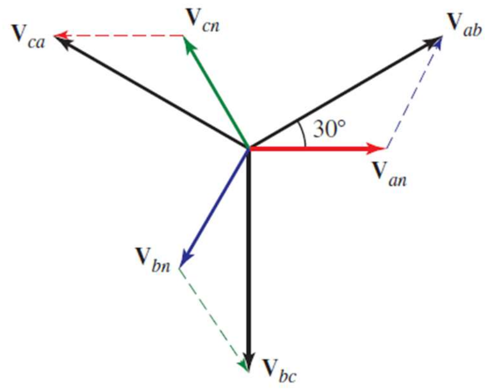


Fig. 4.8: A phasor diagram which is used to determine the line voltages from the given phase voltages. Or, algebraically, $V_{ab} = V_{an} - V_{bn} = V_p\angle 0^\circ - V_p\angle -120^\circ = V_p - V_p\cos(-120^\circ) - jV_p\sin(-120^\circ) = V_p(1 + 1/2 + j\sqrt{3}/2) = \sqrt{3}V_p\angle 30^\circ$.

Note that with positive phase sequence, V_{an} leads V_{bn} and V_{bn} leads V_{cn}, in each case by 120°, and also that V_{ab} leads V_{bc} and V_{bc} leads V_{ca}, again by 120°. The statement is true for negative phase sequence if "lags'' is substituted for "leads".

Now let us connect a balanced Y-connected three-phase load to our source, using three lines and a neutral, as drawn in Fig. 4.9. The load is represented by an impedance Z_p between each line and the neutral. The three line currents are found very easily, since we really have three single phase circuits that possess one common lead:

$I_{aA} = V_{an}/Z_p$

$I_{bB} = V_{bn}/Z_p = V_{an}\angle -120^\circ/Z_p = I_{aA}\angle -120^\circ$

$I_{cC} = I_{aA}\angle -240^\circ$

and therefore

$I_{Nn} = I_{aA} + I_{bB} + I_{cC} = 0$

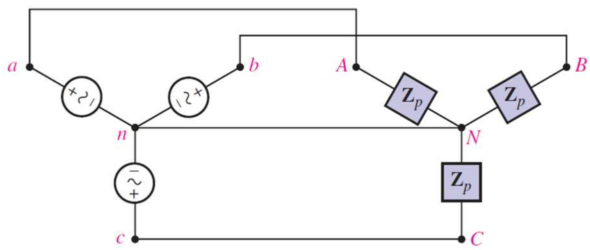


Fig. 4.9.

Example 4.2: For the circuit of Fig. 4.10, find both the phase and line currents, and the phase and line voltages throughout the circuit; then calculate the total power dissipated in the load.

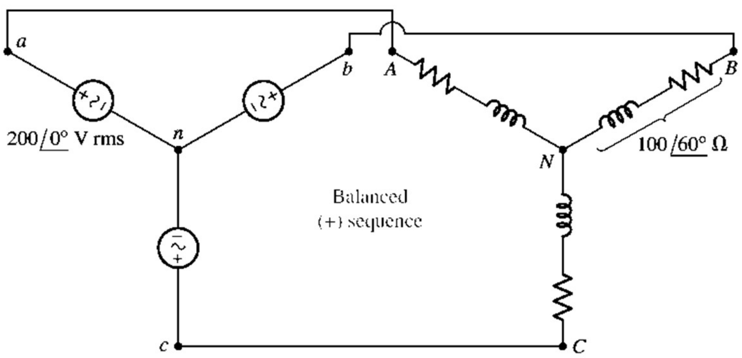


Fig. 4.10.

Solution:

Since one of the source phase voltages is given and we are told to use the positive phase sequence, the three phase voltages are:

$$V_{an} = 200\angle 0^\circ \quad V$$
$$V_{bn} = 200\angle -120^\circ \quad V$$
$$V_{cn} = 200\angle -240^\circ \quad V$$

The line voltage is $200\sqrt{3} = 346$ V; the phase angle of each line voltage can be determined by constructing a phasor diagram, as shown in Fig. 4.11. We find that V_{ab} is $346\angle 30^\circ$ V, $V_{bc} = 346\angle -90^\circ$ V, and $V_{ca} = 346\angle -210^\circ$ V.

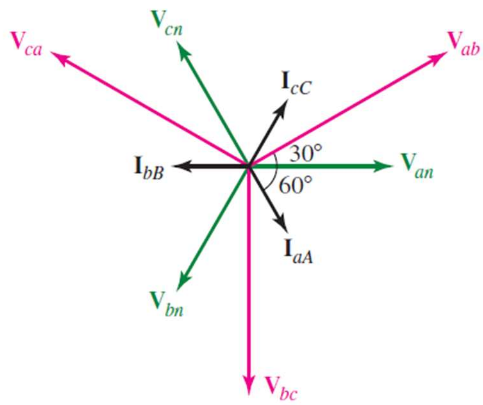


Fig. 4.11.

The line current for phase A is

$$I_{aA} = V_{an}/Z_p = 200\angle 0^\circ/100\angle 60^\circ = 2\angle -60^\circ \text{ A}$$

Since we know this is a balanced three-phase system, we may write the remaining line currents based on I_{aA}:

$$I_{bB} = 2\angle(-60^\circ - 120^\circ) = 2\angle -180^\circ \text{ A}$$
$$I_{cC} = 2\angle(-60^\circ - 240^\circ) = 2\angle -300^\circ \text{ A}$$

Finally, the average power absorbed by phase A is $\text{Re}\{V_{an}I^*_{aA}\}$, or

$$P_{AN} = 200(2)\cos(0^\circ + 60^\circ) = 200 \text{ W}$$

Thus, the total average power drawn by the three-phase load is 600 W.
The phasor diagram for this circuit is shown in Fig. 4.11.

H.W.: A balanced three-phase three-wire system has a Y-connected load. Each phase contains three loads in parallel: $-j100\ \Omega$, $100\ \Omega$, and $50 + j50\ \Omega$. Assume positive phase sequence with $V_{ab} = 400\angle 0^\circ$ V. Find (a) V_{an}; (b) I_{aA}; (c) the total power drawn by the load.

Example 4.3: A balanced three-phase system with a line voltage of 300 V is supplying a balanced Y-connected load with 1200 Wat a leading PF of 0.8. Find the line current and the per-phase load impedance.

Solution:

The phase voltage is $300/\sqrt{3}$ V and the per-phase power is 1200/3 = 400 W. Thus the line current may be found from the power relationship

$$400 = (300/\sqrt{3})(I_L)(0.8)$$

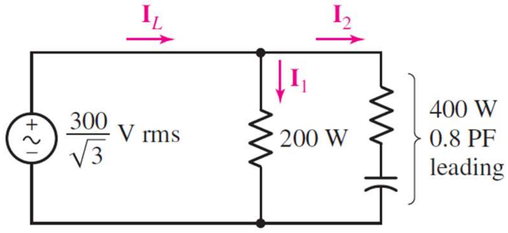
and the line current is therefore 2.89 A. The phase impedance magnitude is given by

$$|Z_p| = V_p/I_L = (300/\sqrt{3})/2.89 = 60\ \Omega$$

Since the PF is 0.8 leading, the impedance phase angle is -36.9°; thus $Z_p = 60\angle -36.9^\circ\ \ \Omega$.

H.W.: A balanced three-phase three-wire system has a line voltage of 500 V. Two balanced Y-connected loads are present. One is a capacitive load with $7 - j2\ \Omega$ per phase, and the other is an inductive load with $4 + j2\ \Omega$ per phase. Find (a) the phase voltage; (b) the line current; (c) the total power drawn by the load; (d) the power factor at which the source is operating.

Example 4.4: A balanced 600 W lighting load is added (in parallel) to the system of Example 4.3. Determine the new line current. We first sketch a suitable per-phase circuit, as shown in Fig. 4.12.



Solution:

The 600 W load is assumed to be a balanced load evenly distributed among the three phases, resulting in an additional 200 W consumed by each phase.

The amplitude of the lighting current (labeled I_1) is determined by

$$200 = (300/\sqrt{3})\ |I_1| \cos 0^\circ$$

so that $|I_1| = 1.155$ A

In a similar way, the amplitude of the capacitive load current (labeled I_2) is found to be unchanged from its previous value, since the voltage across it has remained the same: $|I_2| = 2.89$ A

If we assume that the phase with which we are working has a phase voltage with an angle of 0°, then since $\cos{-1}(0.8) = 36.9^\circ$,

$$I_1 = 1.155\angle 0^\circ \text{ A} \qquad I_2 = 2.89\angle{+36.9^\circ} \text{ A}$$

and the line current is

$$I_L = I_1 + I_2 = 3.87\angle{+26.6^\circ} \text{ A}$$

We can check our results by computing the power generated by this phase of the source

$$P_p = (300/\sqrt{3})\ 3.87 \cos(+26.6^\circ) = 600 \text{ W}$$

which agrees with the fact that the individual phase is known to be supplying 200 W to the new lighting load, as well as 400 W to the original load.

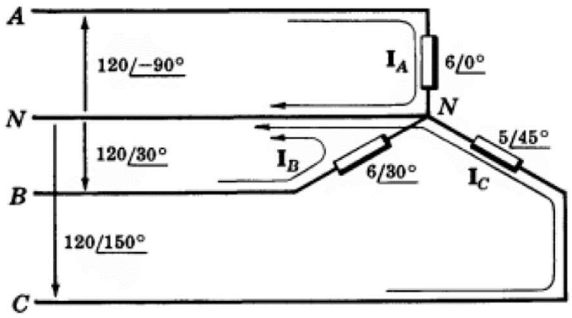
H.W.: Three balanced Y-connected loads are installed on a balanced three-phase four-wire system. Load 1 draws a total power of 6 kW at unity PF, load 2 pulls 10 kVA at PF = 0.96 lagging, and load 3 demands 7 kW at 0.85 lagging. If the phase voltage at the loads is 135 V, if each line has a resistance of 0.1 Ω, and if the neutral has a resistance of 1 Ω, find (a) the total power drawn by the loads; (b) the combined PF of the loads; (c) the total power lost in the four lines; (d) the phase voltage at the source; (e) the power factor at which the source is operating.

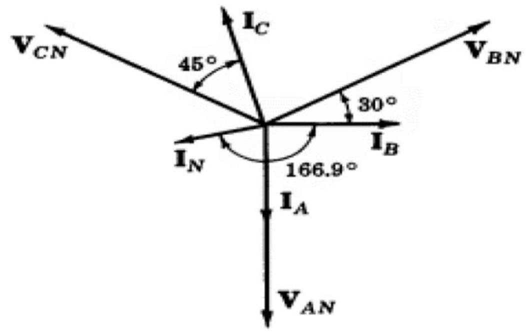
4.4.2 THREE-PHASE UNBALANCE SYSTEM WITH Y-Y CONNECTION

Unbalanced four wires:

1. On a four-wire system, the neutral conductor will carry a current when the load is unbalanced.
2. The voltage across each of the load impedances remains fixed with the same magnitude as the line to neutral voltage.
3. The line currents are unequal and do not have a 120° phase difference.

Example 4.5: A three phase, four wire, 208 volt, CBA system has a Y-Y connected load with $Z_A=6\angle 0^o$, $Z_B=6\angle 30^o$, and $Z_C=5\angle 45^o$. Obtain the three line currents and the neutral current. Draw the phasor diagram.





Solution:

$I_A = V_{AN}/Z_A = 120\angle{-90^o}/6\angle 0^o = 20\angle{-90^o}$

$I_B = V_{BN}/Z_B = 20\angle 0^o$

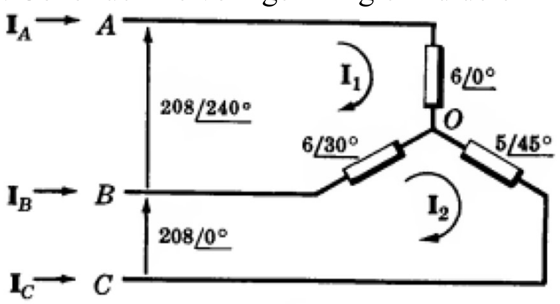
$I_C = V_{CN}/Z_C = 24\angle 105^o$

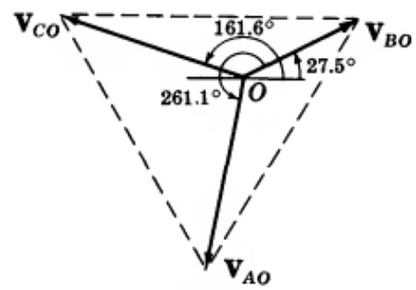
$I_N = -(I_A+I_B+I_C) = -(20\angle{-90^o}+20\angle 0^o+24\angle 105^o) = 14.1\angle{-166.9^o}$

Unbalanced three wires:

1. The common point of the three load impedances is not at the potential of the neutral and is marked "O" instead of N.
2. The voltages across the three impedances can vary considerably from line to neutral magnitude, as shown by the voltage triangle which relates all of the voltages in the circuit.

Example 4.5: A three phase, three wire, 208 volt, CBA system has a Y-Y connected load with $Z_A=6\angle 0^o$, $Z_B=6\angle 30^o$, and $Z_C=5\angle 45^o$. Obtain the line currents and the phasor voltage across each impedance. Construct the voltage triangle and determine the displacement neutral voltage, V_{ON}.





Solution:

$6\angle 0^o\ I_1 + 6\angle 30^o\ (I_1-I_2) = 208\angle 240^o$ [1]

$5\angle 45^o\ I_2 + 6\angle 30^o\ (I_2-I_1) = 208\angle 0^o$ [2]

$I_1 = 23.3\angle 261.1^o$ A $\quad I_2 = 26.5\angle{-63.4^o}$ A

$I_A = I_1 = 23.3\angle 261.1^o$ A

$I_B = I_2 - I_1 = 26.5\angle{-63.4^o} - 23.3\angle 261.1^o = 15.45\angle{-2.5^o}$ A

$I_C = -I_2 = -26.5\angle{-63.4^o}$ A

$V_{AO} = I_A Z_A = 23.3\angle 261.1^o(6\angle 0^o) = 139.8\angle 261.1^o$ V

$V_{BO} = I_B Z_B = 15.45\angle -2.5^o (6\angle 30^o) = 92.7\angle 27.5^o$ V

$V_{CO} = I_C Z_C = 26.5\angle 116.6^o (5\angle 45^o) = 132.5\angle 161.6^o$ V

$V_{ON} = V_{OA} + V_{AN} = -139.8\angle 261.1^o + 120\angle -90^o = 28.1\angle 39.8^o$ V

4.4.3 THREE-PHASE BALANCE SYSTEM WITH Δ-Δ CONNECTION

This type of configuration is very common, and does not possess a neutral connection.

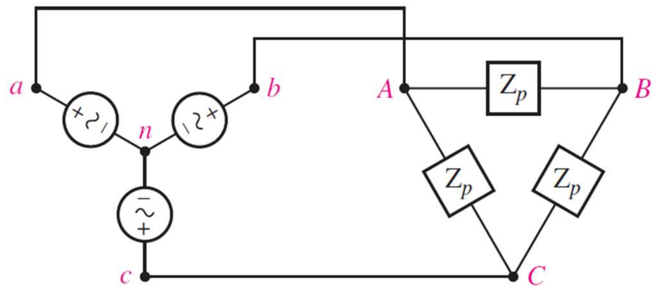


Fig. 4.12: A balanced Δ-connected load is present on a three-wire three-phase system. The source happens to be Y-connected.

Let us consider a balanced Δ-connected load which consists of an impedance Z_p inserted between each pair of lines. With reference to Fig. 4.12, let us assume known line voltages

$$V_L = |V_{ab}| = |V_{bc}| = |V_{ca}|$$

or known phase voltages

$$V_p = |V_{an}| = |V_{bn}| = |V_{cn}|$$

where

$$V_L = \sqrt{3}V_p \text{ and } V_{ab} = \sqrt{3}V_p\angle 30^\circ$$

as we found previously. Because the voltage across each branch of the Δ is known, the phase currents are easily found:

$$I_{AB} = V_{ab}/Z_p$$

$$I_{BC} = V_{bc}/Z_p$$

$$I_{CA} = V_{ca}/Z_p$$

and their differences provide us with the line currents, such as

$$I_{aA} = I_{AB} - I_{CA}$$

Since we are working with a balanced system, the three phase currents are of equal amplitude:

$$I_p = |I_{AB}| = |I_{BC}| = |I_{CA}|$$

The line currents are also equal in amplitude; the symmetry is apparent from the phasor diagram of Fig. 4.13. We thus have

$$I_L = |I_{aA}| = |I_{bB}| = |I_{cC}|$$

and

$$I_L = \sqrt{3}I_p$$

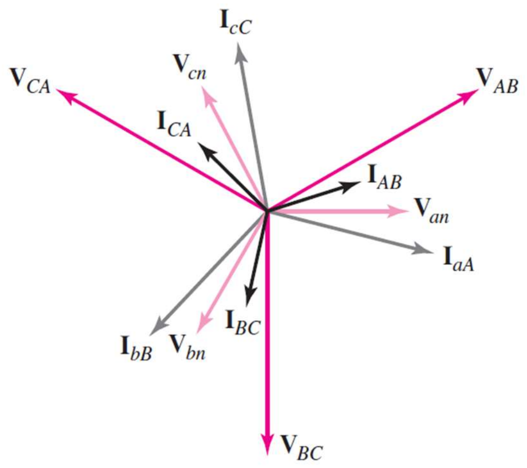


Fig. 4.13: A phasor diagram that could apply to the circuit of Fig. 4.12 if Z_p were an inductive impedance.

Example 4.6: Determine the amplitude of the line current in a three-phase system with a line voltage of 300 V that supplies 1200 W to a Δ-connected load at a lagging PF of 0.8; then find the phase impedance.

Solution:

Let us again consider a single phase. It draws 400 W, 0.8 lagging PF, at a 300 V line voltage. Thus,

$$400 = 300(I_p)(0.8)$$

and

$$I_p = 1.667 \text{ A}$$

and the relationship between phase currents and line currents yields

$$I_L = \sqrt{3}(1.667) = 2.89 \text{ A}$$

Next, the phase angle of the load is $\cos^{-1}(0.8) = 36.9^{\circ}$, and therefore the impedance in each phase must be

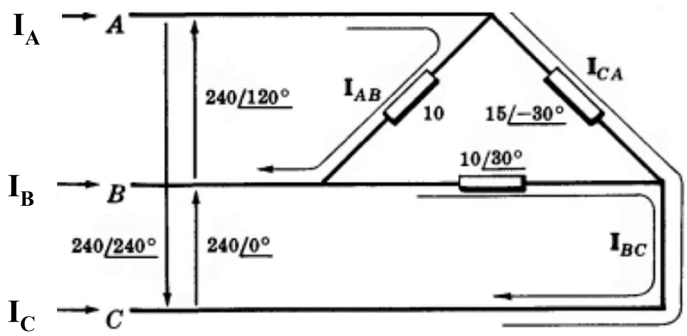
$$Z_p = 300/1.667\angle 36.9^{\circ} = 180\angle 36.9^{\circ} \quad \Omega$$

H.W.: Each phase of a balanced three-phase Δ-connected load consists of a 200 mH inductor in series with the parallel combination of a 5 μF capacitor and a 200 Ω resistance. Assume zero line resistance and a phase voltage of 200 V at ω = 400 rad/s. Find (a) the phase current; (b) the line current; (c) the total power absorbed by the load.

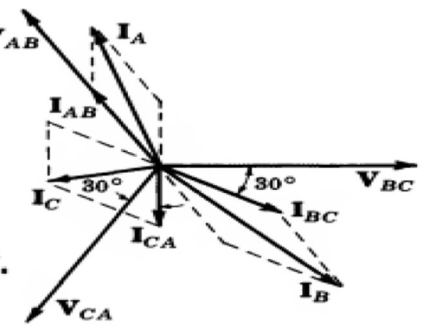
4.4.4 THREE-PHASE UNBALANCE SYSTEM WITH Δ-Δ CONNECTION

The line currents will not be equal nor will they have a 120° phase difference, as was the case with balanced loads.

Example 4.7: A three-phase, three-wire, 240 volt, ABC system has a delta-connected load with $Z_{AB} = 10\angle 0^{o}$, $Z_{BC} = 10\angle 30^{o}$ and $Z_{CA} = 15\angle -30^{o}$. Obtain the three line current and draw the phasor diagram.



g.



Solution:

$I_{AB} = V_{AB}/Z_{AB} = 240\angle 120^{o}/10\angle 120^{o} = 24\angle 120^{o}$ A

$I_{BC} = V_{BC}/Z_{BC} = 24\angle -30^{o}$ A

$I_{CA} = V_{CA}/Z_{CA} = 16\angle 270^{o}$ A

$I_A = I_{AB} + I_{AC} = 24\angle 120^{o} - 16\angle 270^{o} = 38.7\angle 108.1^{o}$ A

$I_B = I_{BA} + I_{BC} = -24\angle 120^{o} + 24\angle -30^{o} = 46.4\angle -45^{o}$ A

$I_C = I_{CA} + I_{CB} = 16\angle 270^{o} - 24\angle -30^{o} = 21.2\angle 190.9^{o}$ A

4.3 power in 3-phase circuits

Power in balanced three-phase loads

For a balanced Y-connected loads:

$P_P = V_P I_L \cos\theta$ Average Phase Power

$P_T = 3 V_P I_L \cos\theta$ Average Total Power

$V_L = \sqrt{3} V_P$

$P_T = \sqrt{3} V_L I_L \cos\theta$ Average Total Power

For a balanced Δ-connected loads:

$P_P = V_L I_P \cos\theta$ Average Phase Power

$P_T = 3 V_L I_P \cos\theta$ Average Total Power

$I_L = \sqrt{3} I_P$

$P_T = \sqrt{3} V_L I_L \cos\theta$ Average Total Power

$P_A = \sqrt{3} V_L I_L$ Apparent Total Power

$Q_T = \sqrt{3} V_L I_L \sin\theta$ Reactive Total Power

$S_T = P_T + Q_T$ Complex Total Power

$$p_t(t) = \sqrt{2} V_p \cos(wt+\phi) \sqrt{2} I_p \sin(wt+\theta) + \sqrt{2} V_p \cos(wt+\phi-120) \sqrt{2} I_p \sin(wt+\theta-120) + \sqrt{2} V_p \cos(wt+\phi+120) \sqrt{2} I_p \sin(wt+\theta+120)$$ Instantaneous Power