Chapter Three: Sinusoidal Steady State Analysis

3.1 Sinusoidal analysis and phasor

Consider a sinusoidally varying voltage

$$v(t) = V_m \sin\omega t$$

shown graphically in Figs. 3.1a and b. The amplitude of the sine wave is V_m, and the argument is ωt. The radian frequency, or angular frequency, is ω. In Fig. 3.1a, V_m sinωt is plotted as a function of the argument ωt, and the periodic nature of the sine wave is evident. The function repeats itself every 2π radians, and its period is therefore 2π radians. In Fig. 3.1b, V_m sinωt is plotted as a function of t and the period is now T. A sine wave having a period T must execute 1/T periods each second; its frequency f is 1/T hertz, abbreviated Hz. Thus,

$$f = \frac{1}{T}$$

and since $\omega T = 2\pi$

we obtain the common relationship between frequency and radian frequency,

$$\omega = 2\pi f$$

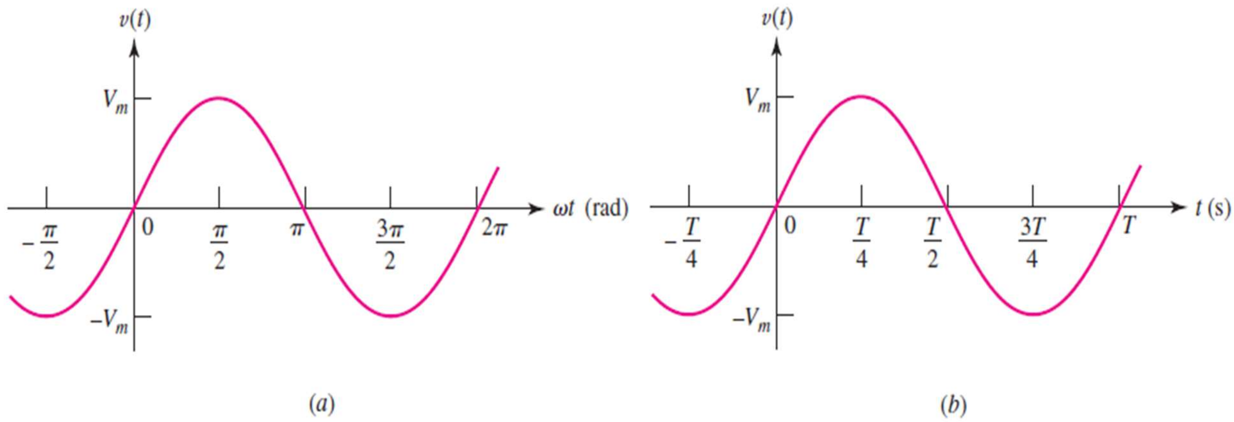


Fig. 3.1: The sinusoidal function v(t) = V_m sin ωt is plotted (a) versus ωt and (b) versus t.

A more general form of the sinusoid,

$$v(t) = V_m \sin(\omega t + \theta) \qquad [1]$$

includes a phase angle θ in its argument. Equation [1] is plotted in Fig. 3.2 as a function of ωt. Since corresponding points on the sinusoid V_m sin(ωt + θ) occur θ rad, or θ/ω seconds, earlier, we say that

- V_m sin(ωt + θ) ***leads*** V_m sinωt by θ rad.
- V_m sin ωt ***lags*** V_m sin(ωt + θ) by θ rad.

In either case, ***leading or lagging***, we say that the sinusoids are ***out of phase***. If the phase angles are ***equal***, the sinusoids are said to be ***in phase***.

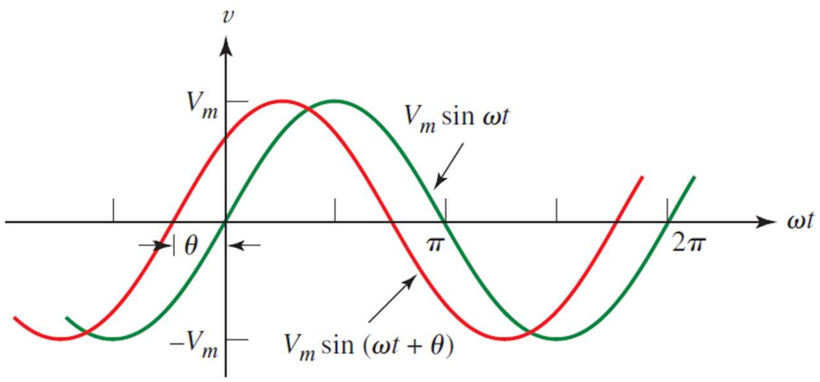


Fig. 3.2: The sine wave $V_m \sin(\omega t + \theta)$ leads $V_m \sin \omega t$ by θ rad.

The complex quantities are usually written in polar form rather than exponential form in order to achieve a slight additional saving of time and effort. For example, a source voltage

$$v(t) = V_m \cos\omega t = V_m \cos(\omega t + 0^\circ)$$

we now represent in complex form as

$$v(t) = V_m \angle 0^\circ$$

and its current response

$$i(t) = I_m \cos(\omega t + \phi)$$

becomes

$$i(t) = I_m \angle\phi$$

This abbreviated complex representation is called a phasor.
A real sinusoidal current

$$i\,(t) = I_m \cos(\omega t + \phi) = Re\{I_m\, e^{j(\omega t+\phi)}\}$$

We then represent the current as a complex quantity by dropping the instruction Re{}, thus adding an imaginary component to the current without affecting the real component; further simplification is achieved by suppressing the factor $e^{j\omega t}$:

$$I = I_m e^{j\phi}$$

and writing the result in polar form:

$$I = I_m \angle\phi$$

The process of returning to the time domain from the frequency domain is exactly the reverse of the previous sequence. Thus, given the phasor voltage

$$V = 115\angle{-45^\circ} \quad \text{volts}$$

and the knowledge that ω = 500 rad/s, we can write the time-domain equivalent directly:

$$v(t) = 115 \cos(500t - 45^\circ) \quad \text{volts}$$

If desired as a sine wave, v(t) could also be written

$$v(t) = 115 \sin(500t + 45^\circ) \quad \text{volts}$$

We can proceed to our simplification of sinusoidal steady-state analysis by establishing the relationship between the phasor voltage and phasor current for each of the three passive elements.

- ***The Resistor***

The resistor provides the simplest case. In the time domain, as indicated by Fig. 3.3a, the defining equation is

$$v(t) = R\, i(t)$$

Now let us apply the complex voltage

$$v(t) = V_m\, e^{j(\omega t+\theta)} = V_m \cos(\omega t + \theta) + jV_m \sin(\omega t + \theta) \qquad [1]$$

and assume the complex current response

$$i(t) = I_m\, e^{j(\omega t+\phi)} = I_m \cos(\omega t + \phi) + jI_m \sin(\omega t + \phi) \qquad [2]$$

so that

$$V_m\, e^{j(\omega t+\theta)} = R\, i(t) = R\, I_m\, e^{j(\omega t+\phi)}$$

Dividing throughout by $e^{j\omega t}$, we find

$$V_m\, e^{j\theta} = R\, I_m\, e^{j\phi}$$

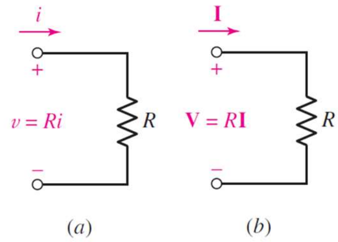


Fig. 3.3.

or, in polar form,

$$V_m \angle\theta = R\, I_m \angle\phi$$

But $V_m \angle\theta$ and $I_m \angle\phi$ merely represent the general voltage and current phasors V and I. Thus,

$$V = R\, I \qquad [3]$$

The voltage-current relationship in phasor form for a resistor has the same form as the relationship between the time-domain voltage and current. The defining equation in phasor form is illustrated in Fig. 3.3b. The angles θ and ϕ are equal, so that the current and voltage are always in phase.

As an example of the use of both the time-domain and frequency domain relationships, let us assume that a voltage of 8 cos(100t − 50°) V is across a 4 Ω resistor. Working in the time domain, we find that the current must be

$$i\,(t) = v(t)\, R = 2 \cos(100t - 50^\circ) \qquad \text{A}$$

The phasor form of the same voltage is 8 ∠−50° V, and therefore

$$I = V\, R = 2 \angle{-50^\circ} \qquad \text{A}$$

- ***The Inductor***

Let us now turn to the inductor. The time-domain representation is shown in Fig. 3.4a, and the defining equation, a time-domain expression, is

$$v(t) = L\, di(t)/dt \qquad [1]$$

After substituting the complex voltage equation and complex current equation in Eq. [1], we have

$$V_m\, e^{j(\omega t+\theta)} = L\, d(I_m\, e^{j(\omega t+\phi)})/dt$$

Taking the indicated derivative:

$$V_m\, e^{j(\omega t+\theta)} = j\omega L\, I_m\, e^{j(\omega t+\phi)}$$

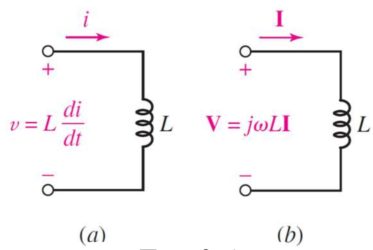


Fig. 3.4

and dividing through by e jωt :

$$V_m\, e^{j\theta} = j\omega L\, I_m\, e^{j\phi}$$

we obtain the desired phasor relationship

$$V = j\omega L\, I \qquad [2]$$

The time-domain differential equation [1] has become the algebraic equation [2] in the frequency domain. The phasor relationship is indicated in Fig. 3.4b. Note that the angle of the factor jωL is exactly +90° and that I must therefore lag V by 90° in an inductor.

- ***<u>The Capacitor</u>***

The final element to consider is the capacitor. The time-domain current voltage relationship is

$i(t) = C\, dv(t)/dt$

The equivalent expression in the frequency domain is obtained once more by letting v(t) and i (t) be the complex quantities, taking the indicated derivative, suppressing $e^{j\omega t}$, and recognizing the phasors V and I. Doing this, we find

$I = j\omega C\, V$ [2]

Thus, I leads V by 90° in a capacitor.

The time-domain and frequency-domain representations are compared in Fig. 3.5a and b.

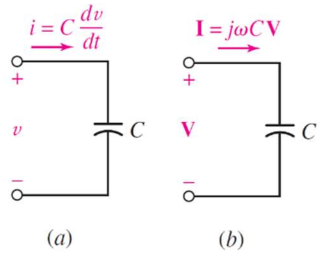
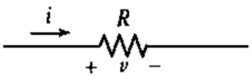
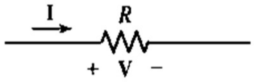
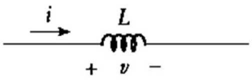
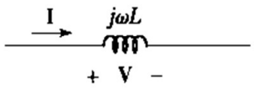
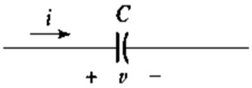
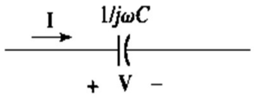


Fig. 3.5.

TABLE 10.1 Comparison of Time-Domain and Frequency-Domain Voltage-Current Expressions

Time Domain		Frequency Domain	
	$v = Ri$	$\mathbf{V} = R\mathbf{I}$	
	$v = L\dfrac{di}{dt}$	$\mathbf{V} = j\omega L\mathbf{I}$	
	$v = \dfrac{1}{C}\displaystyle\int i\, dt$	$\mathbf{V} = \dfrac{1}{j\omega C}\mathbf{I}$	

- ***<u>The impedance</u>***

Of course, we may choose to express impedance in either ***rectangular (Z = R + jX)*** or ***polar (Z = |Z|∠θ)*** form.

In rectangular form, we can see clearly

- ***The real part, which arises only from real resistances (R)***
- ***The imaginary component, termed the reactance, which arises from the energy storage elements (X_L and X_C)***.

*Both resistance and reactance have units of **ohms**, but reactance will always depend upon frequency.* ***<u>An ideal resistor has zero reactance</u>***; ***<u>an ideal inductor or capacitor is purely reactive (i.e., characterized by zero resistance).</u>***

- ***<u>The admittance</u>***

We define this quantity as the admittance Y of a circuit element or passive network, and it is simply the ratio of current to voltage:

- ***The real part of the admittance is the conductance G.***
- ***The imaginary part is the susceptance B***.

*All three quantities (Y, G, and B) are measured in **siemens**.*

$$Y = G + jB = \frac{1}{Z} = \frac{1}{R+jX} \qquad [1]$$

3.2 Mesh and nodal ac analysis

Example 3.1: Find the time-domain node voltages $v_1(t)$ and $v_2(t)$ in the circuit shown in Fig. 3.6.

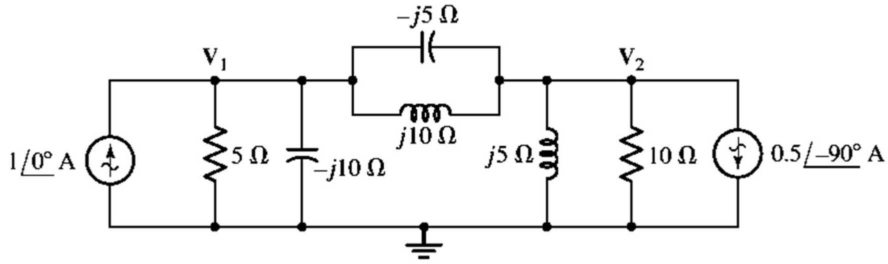


Fig. 3.6.

Solution:

Two current sources are given as phasors, and phasor node voltages V_1 and V_2 are indicated. At the left node we apply KCL, yielding:

$$V_1/5 + V_1/{-j10} + (V_1 - V_2)/{-j5} + (V_1 - V_2)/j10 = 1\angle 0^\circ = 1 + j0$$

At the right node,

$$(V_2 - V_1)/{-j5} + (V_2 - V_1)/j10 + V_2/j5 + V_2/10 = -(0.5\angle -90^\circ) = j0.5$$

Combining terms, we have

$$(0.2 + j0.2)V_1 - j0.1V_2 = 1$$

and

$$-j0.1V_1 + (0.1 - j0.1)V_2 = j0.5$$

These equations are easily solved on most scientific calculators, resulting in $V_1 = 1 - j2$ V and $V_2 = -2 + j4$ V.

The time-domain solutions are obtained by expressing V1 and V2 in polar form:

$$V_1 = 2.24\angle -63.4^\circ$$

$$V_2 = 4.47\angle 116.6^\circ$$

and passing to the time domain:

$$v_1(t) = 2.24\cos(\omega t - 63.4^\circ) \quad \text{V}$$

$$v_2(t) = 4.47\cos(\omega t + 116.6^\circ) \quad \text{V}$$

H.W.: Use nodal analysis on the circuit of Fig. 3.7 to find V_1 and V_2.

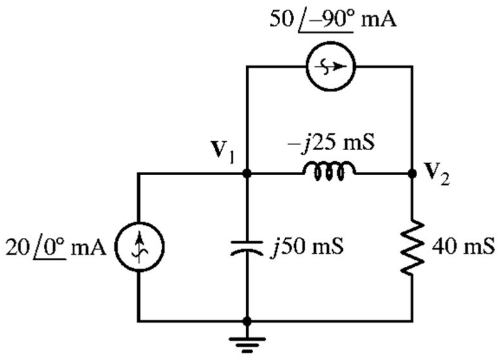


Fig. 3.7.

Example 3.2: Obtain expressions for the time-domain currents i_1 and i_2 in the circuit given as Fig. 3.8a.

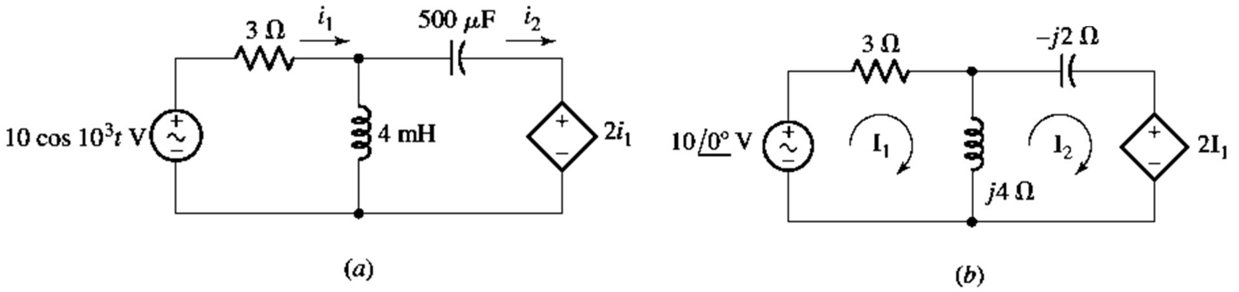


Fig. 3.8.

Solution:

Noting from the left source that $\omega = 10^3$ rad/s, we draw the frequency domain circuit of Fig. 3.8b and assign mesh currents I_1 and I_2.

Around mesh 1,

$$3I_1 + j4(I_1 - I_2) = 10\angle 0^\circ$$

or

$$(3 + j4)I_1 - j4I_2 = 10$$

while mesh 2 leads to

$$j4(I_2 - I_1) - j2I_2 + 2I_1 = 0$$

or

$$(2 - j4)I_1 + j2I_2 = 0$$

Solving,

$$I_1 = (14 + j8)/13 = 1.24\angle 29.7^\circ \quad \text{A}$$

$$I_2 = (20 + j30)/13 = 2.77\angle 56.3^\circ \quad \text{A}$$

Hence,

$$i_1(t) = 1.24 \cos(10^3 t + 29.7^\circ) \quad \text{A}$$

$$i_2(t) = 2.77 \cos(10^3 t + 56.3^\circ) \quad \text{A}$$

H.W.: Use mesh analysis on the circuit of Fig. 3.9 to find I_1 and I_2.

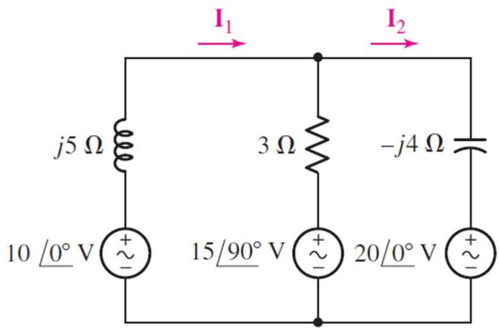


Fig. 3.9.

3.3 Superposition ac analysis

Example 3.3: Use superposition to find V_1 for the circuit of Fig. 3.10.

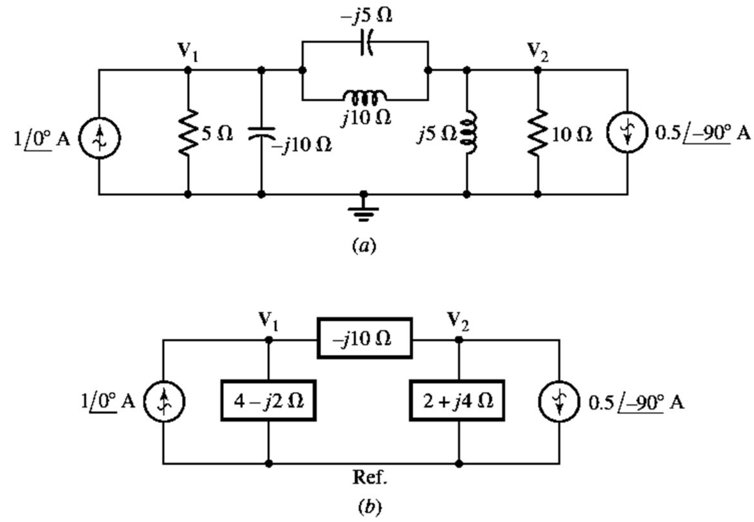


Fig. 3.10.

Solution:

First we redraw the circuit as Fig. 3.10b, where each pair of parallel impedances is replaced by a single equivalent impedance.

To find V_1, we first activate only the left source and find the partial response, V_{1L}. The $1\angle 0^\circ$ source is in parallel with an impedance of $(4 - j2) \,//\, (-j10 + 2 + j4)$ so that

$$V_{1L} = 1\angle 0^\circ\left(\frac{(4-j2)(-j10+2+j4)}{4-j2-j10+2+j4}\right) = \frac{-4-j28}{6-j8} = 2-j2 \qquad \text{V}$$

With only the right source active, current division and Ohm's law yield

$$V_{1R} = -0.5\angle -90^\circ\left(\frac{2+j4}{4-j2-j10+2+j4}\right)(4-j2) = -1 \qquad \text{V}$$

Summing, then,

$$V_1 = V_{1L} + V_{1R} = 2 - j2 - 1 = 1 - j2 \text{ V}$$

H.W.: If superposition is used on the circuit of Fig. 3.11, find V_1 with (a) only the $20\angle 0^\circ$ mA source operating; (b) only the $50\angle -90^\circ$ mA source operating.

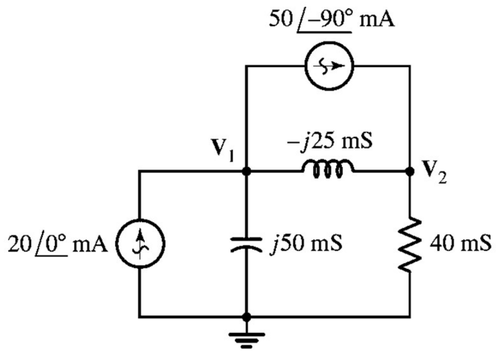


Fig. 3.11.

3.4 Thevenin and Norton ac analysis

Example 3.4: Determine the Thévenin equivalent seen by the −j10 Ω impedance of Fig. 3.12a, and use this to compute V_1.

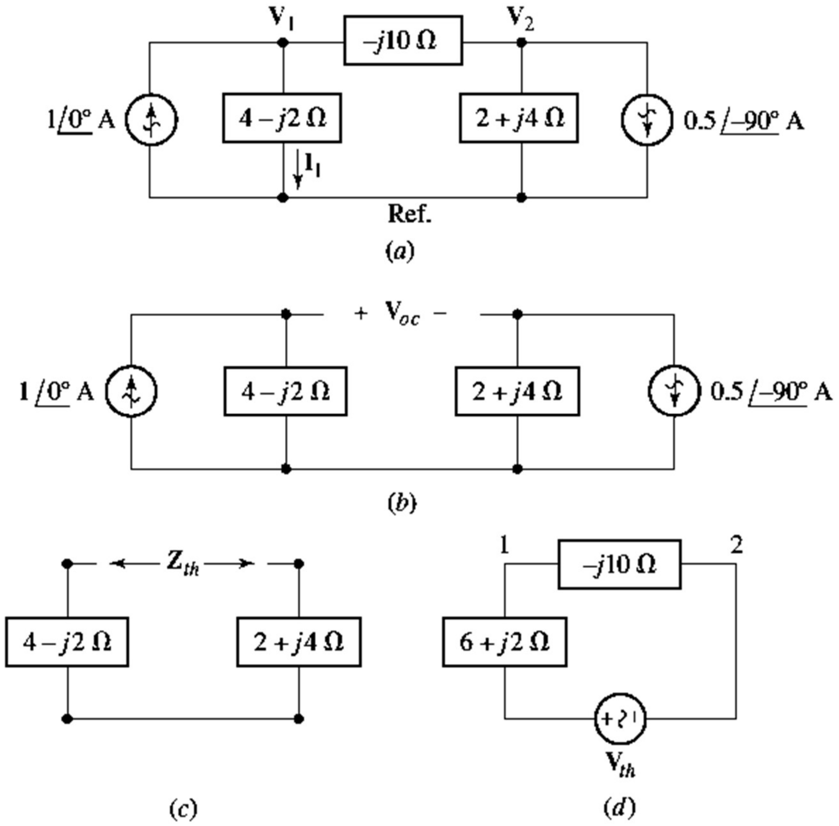


Fig. 3.12: (a) Circuit of Fig. 3.12 b. The Thévenin equivalent seen by the −j10 Ω impedance is desired. (b) V_{oc} is defined. (c) Z_{th} is defined. (d) The circuit is redrawn using the Thévenin equivalent.

Solusion:

The open-circuit voltage, defined in Fig. 3.12b, is

$$V_{oc} = (1\angle 0^\circ)(4 - j2) - (-0.5\angle -90^\circ)(2 + j4)$$
$$= 4 - j2 + 2 - j1 = 6 - j3 \text{ V}$$

The impedance of the inactive circuit of Fig. 3.12c as viewed from the load terminals is simply the sum of the two remaining impedances. Hence,

$$Z_{th} = 6 + j2 \quad \Omega$$

Thus, when we reconnect the circuit as in Fig. 3.12d, the current directed from node 1 toward node 2 through the −j 10 Ω load is

$$I_{12} = (6 - j3)/(6 + j2 - j10) = 0.6 + j0.3 \text{ A}$$

We now know the current flowing through the −j 10 Ω impedance of Fig. 3.12a. Note that we are unable to compute V_1 using the circuit of Fig. 3.12d as the reference node no longer exists. Returning to the original circuit, then, and subtracting the 0.6 + j0.3 A current from the left source current, the downward current through the (4 − j2) Ω branch is found:

$$I_1 = 1 - 0.6 - j0.3 = 0.4 - j0.3 \quad \text{A}$$

and, thus, $V_1 = (0.4 - j0.3)(4 - j2) = 1 - j2 \quad \text{V}$

H.W.: For the circuit of Fig. 3.13, find the (a) open-circuit voltage V_{ab}; (b) downward current in a short circuit between a and b; (c) Thévenin equivalent impedance Zab *in parallel with the current source.*

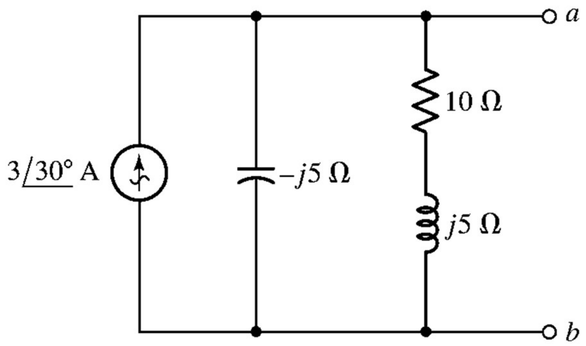


Fig. 3.13

3.5 AC power calculation.

3.5.1 INSTANTANEOUS POWER

The instantaneous power delivered to any device is given by the product of the instantaneous voltage across the device and the instantaneous current through it (the passive sign convention is assumed). The instantaneous power delivered to the entire circuit in the sinusoidal steady state is, therefore,

$$p(t) = v(t)\, i(t) = V_m I_m \cos(\omega t + \phi) \cos \omega t$$

which we will find convenient to rewrite in a form obtained by using the trigonometric identity for the product of two cosine functions. Thus,

$$p(t) = \frac{V_m I_m}{2}[\cos(2\omega t + \phi) + cos\phi] = \frac{V_m I_m}{2} cos\phi + \frac{V_m I_m}{2}\cos(2\omega t + \phi)$$

The last equation possesses several characteristics that are true in general for circuits in the sinusoidal steady state. One term, the first is not a function of time; and a second term is included which has a cyclic variation at twice the applied frequency. Since this term is a cosine wave, and since sine waves and cosine waves have average values which are zero (when averaged over an integral number of periods), this example suggests that the average power is 1/2 Vm Im cos φ; as we will see shortly, this is indeed the case.

3.5.2 AVERAGE POWER

Now let us obtain the general result for the sinusoidal steady state. We assume the general sinusoidal voltage

$$v(t) = V_m \cos(\omega t + \theta)$$

and current

$$i(t) = I_m \cos(\omega t + \phi)$$

associated with the device in question. The instantaneous power is

$$p(t) = V_m I_m \cos(\omega t + \theta) \cos(\omega t + \phi)$$

Again expressing the product of two cosine functions as one-half the sum of the cosine of the difference angle and the cosine of the sum angle,

$$p(t) = \frac{V_m I_m}{2}\cos(\theta - \phi) + \frac{V_m I_m}{2}\cos(2\omega t + \theta + \phi) \qquad [1]$$

we may save ourselves some integration by an inspection of the result.

The first term is a constant, independent of t. The remaining term is a cosine function; p(t) is therefore periodic, and its period is ½T. Note that the period T is associated with the given current and voltage, and not with the power; the power function has a period ½T. However, we may integrate over an interval of T to determine the average value if we wish; it is necessary only that we also divide by T. Our familiarity with cosine and sine waves, however, shows that the average value of either over a period is zero. There is thus no need to integrate Eq. [1] formally; by inspection, the average value of the second term is zero over a period T (or ½T), and the average value of the first term, a constant, must be that constant itself. Thus,

$$P = \frac{V_m I_m}{2}\cos(\theta - \phi) \qquad [2]$$

Example 3.5: Given the time-domain voltage $v = 4\cos(\pi t/6)$ V, find both the average power and an expression for the instantaneous power that result when the corresponding phasor voltage $V = 4\angle 0^\circ$ V is applied across an impedance $Z = 2\angle 60^\circ$ Ω.

Solution:

The phasor current is $V\angle Z = 2\angle{-60^\circ}$ A, and so the average power is

$$P = \tfrac{1}{2}(4)(2)\cos 60^\circ = 2 \quad \text{W}$$

We can write the time-domain voltage,

$$v(t) = 4\cos(\pi t/6) \quad \text{V}$$

and the time-domain current,

$$i(t) = 2\cos(\pi t/6 - 60^\circ) \quad \text{A}$$

The instantaneous power, therefore, is given by their product:

$$p(t) = 8\cos(\pi t/6)\cos(\pi t/6 - 60^\circ)$$
$$= 2 + 4\cos(\pi t/3 - 60^\circ) \quad \text{W}$$

H.W.: Given the phasor voltage $V = 115\sqrt{2}\angle 45^\circ$ V across an impedance $Z = 16.26\angle 19.3^\circ$ Ω, obtain an expression for the instantaneous power, and compute the average power if $\omega = 50$ rad/s.

Note: The phase-angle difference between the current through and the voltage across a pure resistor is zero. Thus,

$$P_R = \tfrac{1}{2} V_m I_m \cos 0 = \tfrac{1}{2} V_m I_m$$

or

$$P_R = \tfrac{1}{2} I^2_m R = V^2_m/2R$$

Note: The average power delivered to any device which is purely reactive (i.e., contains no resistors) must be zero. This is a direct result of the 90° phase difference which must exist between current and voltage; hence, $\cos(\theta - \phi) = \cos\pm 90^\circ = 0$ and $P_X = 0$

Example 3.6: Find the average power absorbed by each of the three passive elements in Fig. 3.14, as well as the average power supplied by each source.

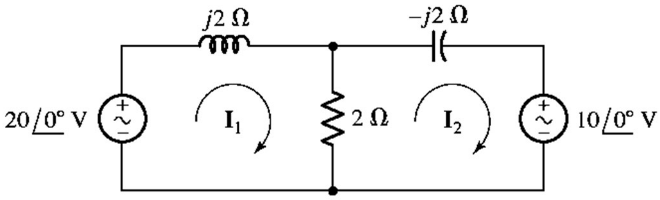


Fig. 3.14.

Solution:

The values of I_1 and I_2 are found by any of several methods, such as mesh analysis, nodal analysis, or superposition. They are

$$I_1 = 5 - j10 = 11.18\angle{-63.43^\circ} \quad A$$

$$I_2 = 5 - j5 = 7.071\angle{-45^\circ} \quad A$$

The downward current through the 2 Ω resistor is

$$I_1 - I_2 = -j5 = 5\angle{-90^\circ} \quad A$$

so that I_m = 5 A, and the average power absorbed by the resistor is found most easily by:

$$P_R = ½\, I^2_m R = ½\,(5^2)2 = 25 \quad W$$

The voltage $20\angle 0^\circ$ V and associated current $I_1 = 11.18\angle{-63.43^\circ}$ A satisfy the active sign convention, and thus the power delivered by this source is

$$P_{left} = ½\,(20)(11.18)\cos[0^\circ - (-63.43^\circ)] = 50 \quad W$$

In a similar manner, we find the power absorbed by the right source using the passive sign convention,

$$P_{right} = ½\,(10)(7.071)\cos(0^\circ + 45^\circ) = 25 \quad W$$

Since 50 = 25 + 25, the power relations check.

H.W.: For the circuit of Fig. 3.15, compute the average power delivered to each of the passive elements. Verify your answer by computing the power delivered by the two sources.

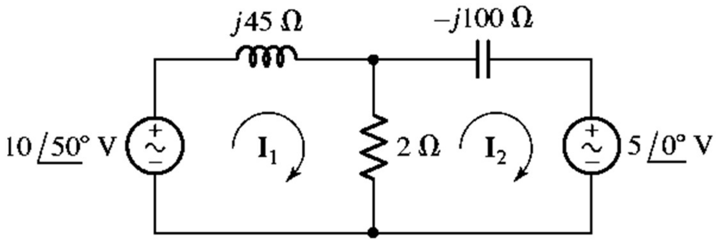
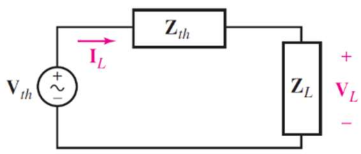


Fig. 3.15

Maximum Power Transfer

An independent voltage source in series with an impedance Z_{th} or an independent current source in parallel with an impedance Z_{th} delivers a maximum average power to that load impedance Z_L which is the conjugate of Z_{th}, or $Z_L = Z^*_{th}$.



Example 3.7: A particular circuit is composed of the series combination of a sinusoidal voltage source 3 cos(100t − 3°) V, a 500 Ω resistor, a 30 mH inductor, and an unknown impedance. If we are assured that the voltage source is delivering maximum average power to the unknown impedance, what is its value?

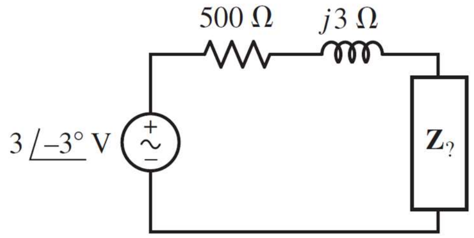


Fig. 3.16

Solution:

The phasor representation of the circuit is sketched in Fig. 3.16. The circuit is easily seen as an unknown impedance $Z_?$ in series with a Thevenin equivalent consisting of the $3\angle{-3^\circ}$ V source and a Thevenin impedance 500 + j3 Ω.

Since the circuit of Fig. 3.16 is already in the form required to employ the maximum average power transfer theorem, we know that maximum average power will be transferred to an impedance equal to the complex conjugate of Z_{th}, or $Z? = Z^*_{th} = 500 - j3\ \Omega$

This impedance can be constructed in several ways, the simplest being a 500 Ω resistor in series with a capacitor having impedance −j3 Ω. Since the operating frequency of the circuit is 100 rad/s, this corresponds to a capacitance of 3.333 mF.

H.W.: If the 30 mH inductor of Example 3.7 is replaced with a 10 µF capacitor, what is the value of the inductive component of the unknown impedance $Z_?$ if it is known that $Z_?$ is absorbing maximum power?

Use of RMS Values to Compute Average Power

The average power delivered to an R ohm resistor by a sinusoidal current is

$$P = \tfrac{1}{2} I^2_m R$$

Since $I_{eff} = I_m/\sqrt{2}$, the average power may be written as

$$P = I_{eff}^2\, R$$

The other power expressions may also be written in terms of effective values:

$$P = V_{eff}\, I_{eff} \cos(\theta - \phi)$$

$$P = V_{eff}^2 / R$$

3.5.3 APPARENT POWER AND POWER FACTOR

The product of the effective values of the voltage and current is not the average power; we define it as the apparent power.

$$P = V_{eff} I_{eff}$$

Dimensionally, apparent power must be measured in the same units as real power, since $\cos(\theta - \varphi)$ is dimensionless; but in order to avoid confusion, the term volt-amperes, or VA, is applied to the apparent power.

Since $\cos(\theta - \varphi)$ cannot have a magnitude greater than unity, the magnitude of the real power can never be greater than the magnitude of the apparent power.

The ratio of the real or average power to the apparent power is called the power factor, symbolized by PF. Hence,

$$PF = \text{average power/apparent power} = V_{eff} I_{eff} \cos(\theta - \phi)/V_{eff} I_{eff} = \cos(\theta - \phi)$$

In the sinusoidal case, the power factor is simply $\cos(\theta - \phi)$, where $(\theta - \phi)$ is the angle by which the voltage leads the current. This relationship is the reason why the angle $(\theta - \phi)$ is often referred to as the PF angle.

For a purely resistive load, the voltage and current are in phase, $(\theta - \phi)$ is zero, and the PF is unity.

Example 3.8: Calculate values for the average power delivered to each of the two loads shown in Fig. 3.17, the apparent power supplied by the source, and the power factor of the combined loads.

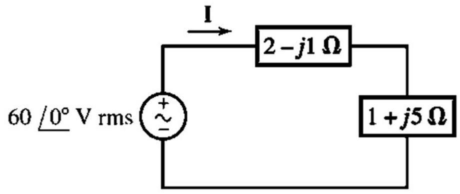


Fig. 3.17

Solution:

We require I_{eff}:

$$I = 60\angle 0^o/(3 + j4) = 12\angle -53.13^\circ \quad \text{A rms}$$

so $I_{eff} = 12$ A rms, and ang $I = -53.13^\circ$.

The average power delivered to the top load is given by

$$P_{upper} = I^2_{eff} R_{top} = (12)^2(2) = 288 \quad \text{W}$$

and the average power delivered to the right load is given by

$$P_{lower} = I^2_{eff} R_{right} = (12)^2(1) = 144 \quad \text{W}$$

The source itself supplies an apparent power of $V_{eff} I_{eff} = (60)(12) = 720$ VA.

Finally, the power factor of the combined loads is found by considering the voltage and current associated with the combined loads.

$$PF = P/V_{eff} I_{eff} = (P_{upper} + P_{lower})/V_{eff} I_{eff} = 432/(60*12) = 0.6 \text{ lagging}$$

H.W.: For the circuit of Fig. 3.18, determine the power factor of the combined loads if $Z_L = 10\ \Omega$.

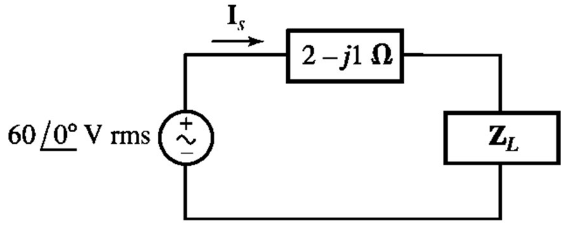


Fig. 3.18

3.5.4 COMPLEX POWER

In this section, we define complex power to allow us to calculate the various contributions to the total power in a clean, efficient fashion. The magnitude of the complex power is simply the apparent power. The real part is the **average power** and—as we are about to see—the imaginary part is a new quantity, termed the **reactive power**, which describes the rate of energy transfer into and out of reactive load components (e.g., inductors and capacitors).

If we first inspect the polar or exponential form of the complex power,

$$S = V_{eff}\, I_{eff}\, e^{\,j(\theta-\phi)}$$

we see that the magnitude of S, $V_{eff}I_{eff}$, is the apparent power. The angle of S, $(\theta - \phi)$, is the PF angle (i.e., the angle by which the voltage leads the current).

In rectangular form, we have

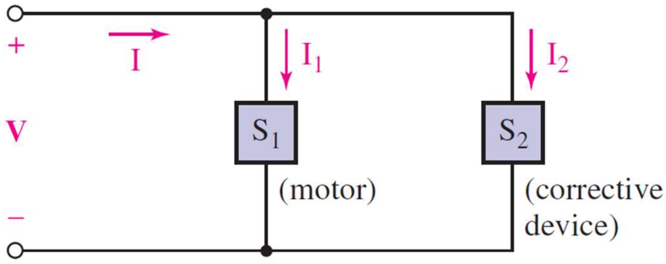
$$S = P + jQ$$

$$P = V_{eff}\, I_{eff} \cos(\theta-\phi) \qquad \text{average power}$$

$$Q = V_{eff}\, I_{eff} \sin(\theta-\phi) \qquad \text{reactive power}$$

Quantity	Symbol	Formula	Units
Average power	P	$V_{eff}I_{eff}\cos(\theta-\phi)$	watt (W)
Reactive power	Q	$V_{eff}I_{eff}\sin(\theta-\phi)$	volt-ampere-reactive (VAR)
Complex power	$\mathbf{S}$	$P + jQ$	
		$V_{eff}I_{eff}\underline{/\theta-\phi}$	volt-ampere (VA)
		$\mathbf{V}_{eff}\mathbf{I}^*_{eff}$	
Apparent power	$\lvert\mathbf{S}\rvert$	$V_{eff}I_{eff}$	volt-ampere (VA)

Example 3.9: An industrial consumer is operating a 50 kW (67.1 hp) induction motor at a lagging PF of 0.8. The source voltage is 230 V rms. In order to obtain lower electrical rates, the customer wishes to raise the PF to 0.95 lagging. Specify a suitable solution.



Solution:

The complex power supplied to the induction motor must have a real part of 50 kW and an angle of $\cos^{-1}(0.8)$, or 36.9°. Hence,

$$S_1 = 50\angle 36.9^\circ/0.8 = 50 + j37.5 \quad \text{kVA}$$

In order to achieve a PF of 0.95, the total complex power must become

$$S = S_1 + S_2 = (50/0.95)\angle\cos^{-1}(0.95) = 50 + j16.43 \quad \text{kVA}$$

Thus, the complex power drawn by the corrective load is

$$S_2 = -j21.07 \quad \text{kVA}$$

The necessary load impedance Z_2 may be found in several simple steps. We select a phase angle of 0° for the voltage source, and therefore the current drawn by Z_2 is

$$I_2^* = S_2/V = -j21{,}070/230 = -j91.6 \quad \text{A}$$

or

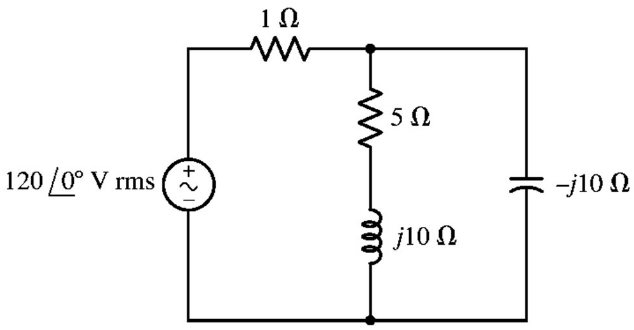
$$I_2 = j91.6 \quad \text{A}$$

Therefore,

$$Z_2 = V/I_2 = 230/j91.6 = -j2.51\ \Omega$$

If the operating frequency is 60 Hz, this load can be provided by a 1056 μF capacitor connected in parallel with the motor. However, its initial cost, maintenance, and depreciation must be covered by the reduction in the electric bill.

H.W.: For the circuit shown in Fig. 3.19, find the complex power absorbed by the (a) 1 Ω resistor; (b) −j10 Ω capacitor; (c) 5 + j10 Ω impedance; (d) source.



3.19

H.W.: A 440 V rms source supplies power to a load Z_L = 10 + j2 Ω through a transmission line having a total resistance of 1.5 Ω. Find (a) the average and apparent power supplied to the load; (b) the average and apparent power lost in the transmission line; (c) the average and apparent power supplied by the source; (d) the power factor at which the source operates.