2.4 RLC circuit:

In this section, we consider more complex circuits, which contain both an inductor and a capacitor. The result is a second-order differential equation for any voltage or current of interest. Now we need two initial conditions to solve each differential equation.

Such circuits occur routinely in a wide variety of applications, including oscillators and frequency filters. They are also very useful in modelling a number of practical situations, such as automobile suspension systems, temperature controllers, and even the response of an airplane to changes in elevator and aileron positions.

2.4.1 THE SOURCE-FREE PARALLEL CIRCUIT

When a physical capacitor is connected in parallel with an inductor and the capacitor has associated with it a finite resistance, the resulting network can be shown to have an equivalent circuit model like that shown in Fig. 2.28.

The presence of this resistance can be used to model energy loss in the capacitor; over time, all real capacitors will eventually discharge, even if disconnected from a circuit. Energy losses in the physical inductor can also be taken into account by adding an ideal resistor (in series with the ideal inductor). For simplicity, however, we restrict our discussion to the case of an essentially ideal inductor in parallel with a "leaky" capacitor.

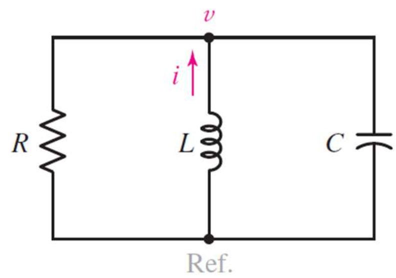


Fig. 2.28: The source-free parallel RLC circuit.

In the following analysis, we will assume that energy may be stored initially in both the inductor and the capacitor; in other words, nonzero initial values of both inductor current and capacitor voltage may be present. With reference to the circuit of Fig. 2.28, we may then write the single nodal equation

$$\frac{v}{R} + \int_{t_0}^{t} v\,dt' - i(t_o) + C\frac{dv}{dt} = 0 \qquad [1]$$

Note that the minus sign is a consequence of the assumed direction for i. We must solve Eq. [1] subject to the initial conditions

$$i(0^+) = I_o \qquad [2]$$

and

$$v(0^+) = V_o \qquad [3]$$

When both sides of Eq. [1] are differentiated once with respect to time, the result is the linear second-order homogeneous differential equation

$$C\frac{d^2v}{dt^2} + \frac{1}{R}\frac{dv}{dt} + \frac{1}{L}v = 0 \qquad [4]$$

whose solution v(t) is the desired natural response.

We will assume a solution based on the exponential form. Thus, we assume

$$v = Ae^{st} \qquad [5]$$

Substituting Eq. [5] in Eq. [4], we obtain

$$CAs^2e^{st} + \frac{A}{R}se^{st} + \frac{A}{L}e^{st} = 0$$

$$Ae^{st}\left(Cs^2 + \frac{1}{R}s + \frac{1}{L}\right) = 0$$

$$Cs^2 + \frac{1}{R}s + \frac{1}{L} = 0 \qquad [6]$$

$$s_1 = -\frac{1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}} \qquad [7]$$

$$s_2 = -\frac{1}{2R} - \sqrt{\left(\frac{1}{2R}\right)^2 - \frac{1}{LC}} \qquad [8]$$

We thus have the general form of the natural response

$$v(t) = A_1e^{s_1t} + A_2e^{s_2t} \qquad [9]$$

where s_1 and s_2 are given by Eqs. [7] and [8]; A_1 and A_2 are two arbitrary constants which are to be selected to satisfy the two specified initial conditions.

The form of the natural response as given in Eq. [9] offers little insight into the nature of the curve we might obtain if v(t) were plotted as a function of time.

Since the exponents, s_1t and s_2t must be dimensionless, s_1 and s_2 must have the unit of some dimensionless quantity "per second". From Eqs. [7] and [8] we therefore see that the units of 1/2RC and 1/√LC must also be s^{-1} (i.e., seconds^{-1}). Units of this type are called frequencies.

Let us define a new term, ω_0 (***resonant frequency***):

$$w_0 = \frac{1}{\sqrt{LC}} \qquad [10]$$

On the other hand, we will call 1/2RC the ***neper frequency***, or the ***exponential damping coefficient***, and represent it by the symbol α (alpha):

$$\alpha = \frac{1}{2RC} \qquad [11]$$

This latter descriptive expression is used because α is a measure of how rapidly the natural response decays or damps out to its steady, final value (usually zero). Finally, s, s1, and s2, which are quantities that will form the basis for some of our later work, are called complex frequencies.

Let us collect these results. The natural response of the parallel RLC circuit is

$$v(t) = A_1e^{s_1t} + A_2e^{s_2t} \qquad [9]$$

Where

$$s_1 = -\alpha + \sqrt{\alpha^2 - w_0^2} \qquad [12]$$

$$s_2 = -\alpha - \sqrt{\alpha^2 - w_0^2} \qquad [13]$$

We note two basic scenarios possible with Eqs. [12] and [13] depending on the relative sizes of α and ω_0 (dictated by the values of R, L, and C). If $\boldsymbol{\alpha > \omega_0}$, s_1 and s_2 will both be real numbers, leading to what is referred to as an overdamped response. In the opposite case, where $\boldsymbol{\alpha < \omega_0}$, both s_1 and s_2 will have nonzero imaginary components, leading to what is known as an underdamped response. Both of these situations are considered separately in the following sections, along with the special case of $\boldsymbol{\alpha = \omega_0}$, which leads to what is called a critically damped response.

Example 2.8: Consider a parallel RLC circuit having an inductance of 10 mH and a capacitance of 100 μF. Determine the resistor values that would lead to overdamped and underdamped responses.
Solution:
We first calculate the resonant frequency of the circuit:

$\omega_0 = 1/\sqrt{LC} = 10^3$ rad/s

An overdamped response will result if $\alpha > \omega_0$; an underdamped response will result if $\alpha < \omega_0$. Thus,

$1/2RC > 10^3$

and so

$R < 5\ \Omega$

leads to an overdamped response; $R > 5\ \Omega$ leads to an underdamped response.

2.4.2 THE OVERDAMPED PARALLEL RLC CIRCUIT

A comparison of Eqs. [10] and [11] shows that α will be greater than ω_0 if $LC > 4R^2C^2$. In this case the radical used in calculating s_1 and s_2 will be real, and both s_1 and s_2 will be real. Moreover, the following inequalities

$$\sqrt{\alpha^2 - w_0^2} < \alpha$$

$$-\alpha - \sqrt{\alpha^2 - w_0^2} < -\alpha + \sqrt{\alpha^2 - w_0^2} < 0$$

may be applied to Eqs. [12] and [13] to show that both s_1 and s_2 are negative real numbers.

The next step is to determine the arbitrary constants A_1 and A_2 in conformance with the initial conditions. We select a parallel RLC circuit with R = 6 Ω, L = 7 H, and, for ease of computation, C = 1/42 F. The initial energy storage is specified by choosing an initial voltage across the circuit v(0) = 0 and an initial inductor current i(0) = 10 A, where v and i are defined in Fig. 2.29.

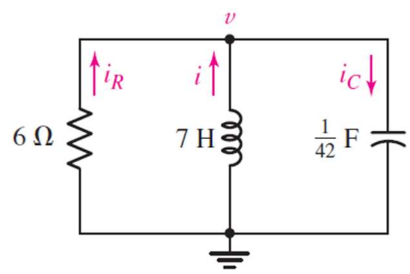


Fig. 2.29

We may easily determine the values of the several parameters

$\alpha = 3.5 \qquad \omega_0 = \sqrt{6}$

$s_1 = -1 \qquad s_2 = -6 \qquad$ (all s^{-1})

and immediately write the general form of the natural response

$$v(t) = A_1 e^{-t} + A_2 e^{-6t} \qquad [14]$$

Only the evaluation of the two constants A_1 and A_2 remains. If we knew the response v(t) at two different values of time, these two values could be substituted in Eq. [14] and A_1 and A_2 easily found. However, we know only one instantaneous value of v(t), v(0) = 0 and, therefore,

$$0 = A_1 + A_2 \qquad [15]$$

We can obtain a second equation relating A_1 and A_2 by taking the derivative of v(t) with respect to time in Eq. [14], determining the initial value of this derivative through the use of the remaining initial condition i (0) = 10, and equating the results. So, taking the derivative of both sides of Eq. [14],

$$\frac{dv(t)}{dt} = -A_1 e^{-t} - 6A_2 e^{-6t}$$

and evaluating the derivative at t = 0,

$$\left.\frac{dv}{dt}\right|_{t=0} = -A_1 - 6A_2$$

$i_C = C dv/dt$

Kirchhoff's current law must hold at any instant in time, as it is based on conservation of electrons. Thus, we may write

$-i_C(0) + i(0) + i_R(0) = 0$

Substituting our expression for capacitor current and dividing by C,

$dv/dt|_{t=0} = i_C(0)/C = (i(0) + i_R(0))/C = i(0)/C = 420$ V/s

since zero initial voltage across the resistor requires zero initial current through it. We thus have our second equation,

$420 = -A_1 - 6A_2$ [16]

and simultaneous solution of Eqs. [15] and [16] provides the two amplitudes $A_1 = 84$ and $A_2 = -84$. Therefore, the final numerical solution for the natural response of this circuit is

$v(t) = 84(e^{-t} - e^{-6t})$ V [17]

Example 2.9: Find an expression for $v_C(t)$ valid for $t > 0$ in the circuit of Fig. 2.30a.

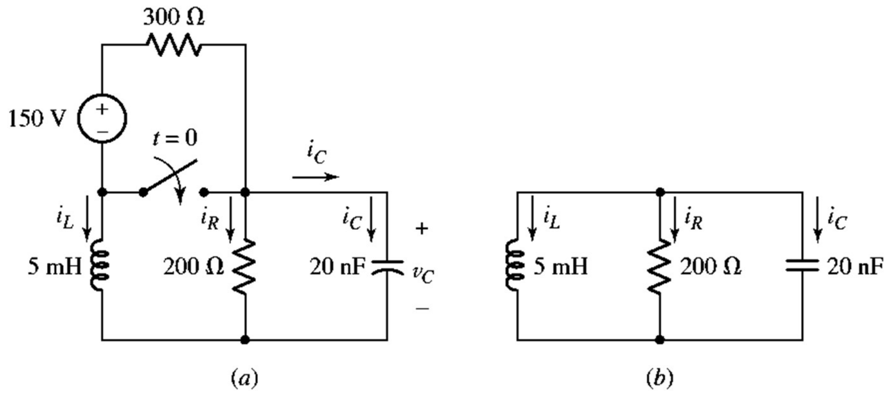


Fig. 2.30

Solution:

After the switch is thrown, the capacitor is left in parallel with a 200 Ω resistor and a 5 mH inductor (Fig. 2.30b). Thus,

$\alpha = 1/2RC = 125{,}000\ s^{-1}$

$\omega_0 = 1/\sqrt{LC} = 100{,}000$ rad/s

$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2} = -50{,}000\ s^{-1}$

$s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2} = -200{,}000\ s^{-1}$.

Since $\alpha > \omega_0$, the circuit is overdamped and so we expect a capacitor voltage of the form

$$v_C(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

From Fig. 2.31a, in which the inductor has been replaced with a short circuit and the capacitor with an open circuit, we see that

$i_L(0^-) = -150/(200 + 300) = -300$ mA

and

$v_C(0^-) = 150*200/(200 + 300) = 60$ V

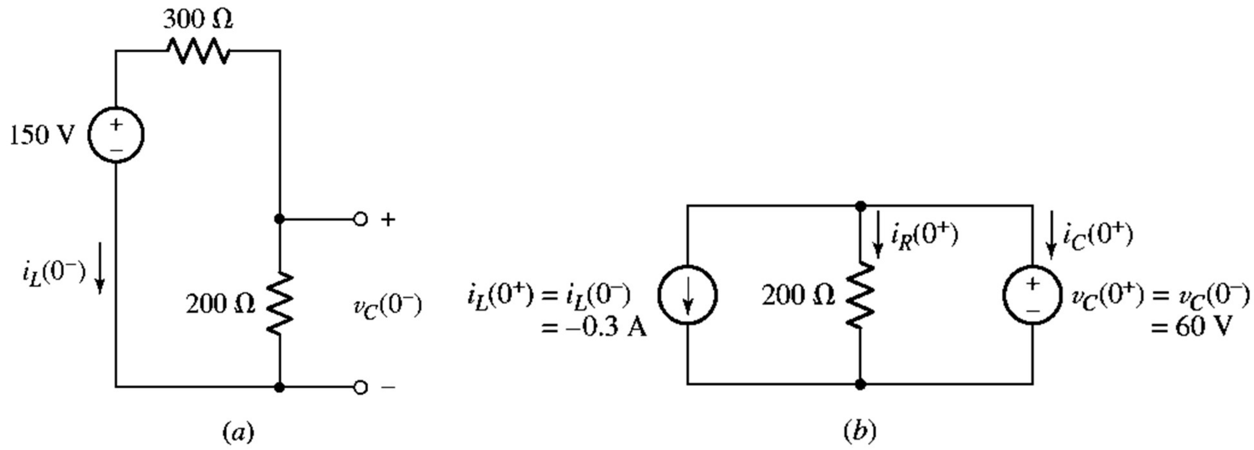


Fig. 2.31

In Fig. 2.31b, we draw the circuit at $t = 0^+$, representing the inductor current and capacitor voltage by ideal sources for simplicity. Since neither can change in zero time, we know that $v_C(0^+) = 60$ V.

We have an equation for the capacitor voltage:

$$v_C(t) = A_1e^{-50{,}000t} + A_2e^{-200{,}000t}.$$

We now know $v_C(0) = 60$ V, but a third equation is still required. Differentiating our capacitor voltage equation, we find

$$dv_C/dt = -50{,}000A_1e^{-50{,}000t} - 200{,}000A_2e^{-200{,}000t}$$

which can be related to the capacitor current as $i_C = C(dv_C/dt)$.

Returning to Fig. 1.31b, KCL yields

$$i_C(0^+) = -i_L(0^+) - i_R(0^+) = 0.3 - [v_C(0^+)/200] = 0$$

Application of our first initial condition yields $v_C(0) = A_1 + A_2 = 60$ and application of our second initial condition yields

$$i_C(0) = -20 \times 10^{-9}(50{,}000A_1 + 200{,}000A_2) = 0$$

Solving, $A_1 = 80$ V and $A_2 = -20$ V, so that

$$v_C(t) = 80e^{-50{,}000t} - 20e^{-200{,}000t} \qquad \text{V}, t > 0$$

At the very least, we can check our solution at $t = 0$, verifying that $v_C(0) = 60$ V. Differentiating and multiplying by 20×10^{-9}, we can also verify that $i_C(0) = 0$. Also, since we have a source-free circuit for $t > 0$, we expect that $v_C(t)$ must eventually decay to zero as t approaches ∞, which our solution does.

H.W.: After being open for a long time, the switch in Fig. 2.32 closes at t = 0. Find (a) $i_L(0^-)$; (b) $v_C(0^-)$; (c) $i_R(0^+)$; (d) $i_C(0^+)$; (e) $v_C(0.2)$.

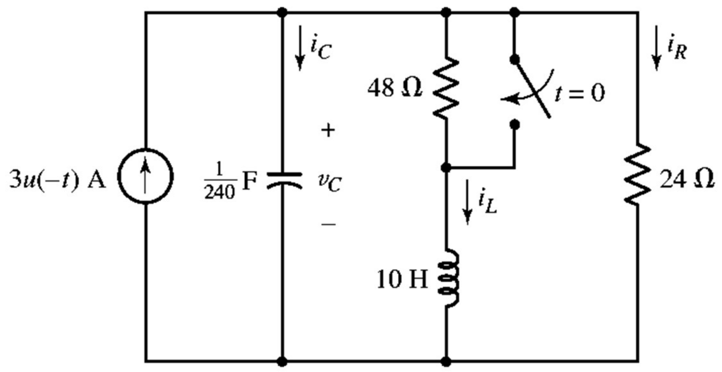


Fig. 2.32.

Graphical Representation of the Overdamped Response

We therefore have a response curve for Eq.[17] which is zero at t = 0, is zero at t =∞, and is never negative; since it is not everywhere zero, it must possess at least one maximum, and this is not a difficult point to determine exactly. We differentiate the response

$$v(t) = 84(e^{-t} - e^{-6t})$$

$$dv/dt = 84(-e^{-t} + 6e^{-6t})$$

set the derivative equal to zero to determine the time tm at which the voltage becomes maximum,

$$0 = -e^{-tm} + 6e^{-6tm}$$

manipulate once,

$$e^{5tm} = 6 \text{ and obtain } tm = 0.358 \text{ s}$$

and $\quad v(tm) = 48.9$ V

A reasonable sketch of the response may be made by plotting the two exponential terms $84e^{-t}$ and $84e^{-6t}$ and then taking their difference. This technique is illustrated by the curves of Fig. 2.33; the two exponentials are shown lightly, and their difference, the total response v(t), is drawn as a coloured line. The curves also verify our previous prediction that the functional behaviour of v(t) for very large t is 84e−t , the exponential term containing the smaller magnitude of s_1 and s_2.

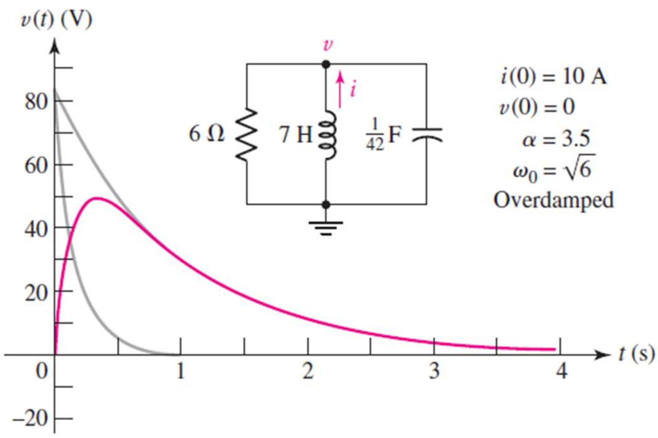


Fig.2.33.

Example 2.10: For t > 0, the capacitor current of a certain source-free parallel RLC circuit is given by $i_C(t) = 2e^{-2t} - 4e^{-t}$ A. Sketch the current in the range 0 < t < 5 s, and determine the settling time.

Solution:

We first sketch the two terms as shown in Fig. 2.34, then subtract them to find $i_C(t)$. The maximum value is clearly $|-2| = 2$ A. We therefore need to find the time at which $|i_C|$ has decreased to 20 mA, or

$$2e^{-2ts} - 4e^{-ts} = -0.02 \qquad [1]$$

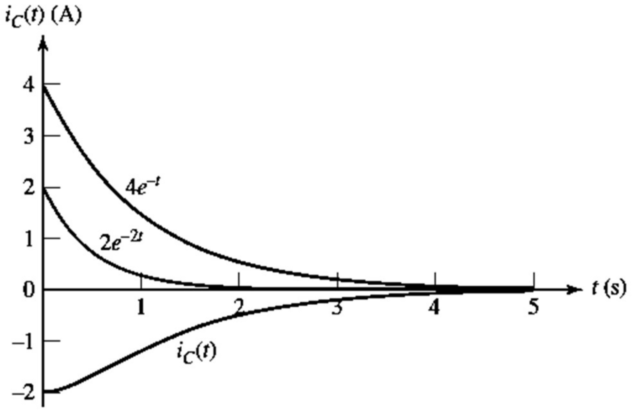


Fig. 2.34

This equation can be solved using an iterative solver routine on a scientific calculator, which returns the solution ts = 5.296 s. If such an option is not available, however, we can approximate Eq. [1] for t ≥ ts as

$$-4e^{-ts} = -0.02 \qquad [2]$$

Solving, $t_s = -\ln(0.02/4) = 5.298$ s

which is reasonably close (better than 0.1% accuracy) to the exact solution.

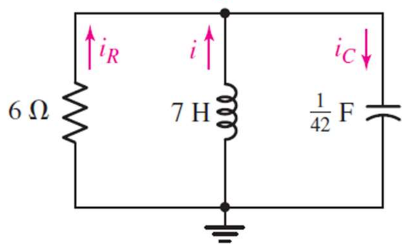
H.W.: (a) Sketch the voltage $v_R(t) = 2e^{-t} - 4e^{-3t}$ V in the range 0 < t < 5 s. (b) Estimate the settling time. (c) Calculate the maximum positive value and the time at which it occurs.

2.4.2 CRITICAL DAMPING

Now let us adjust the element values until α and ω_0 are equal. This is a very special case which is termed ***critical damping***.

Critical damping is achieved when

$$\left.\begin{aligned} \alpha &= \omega_0 \\ or\ LC &= 4R^2C^2 \\ L &= 4R^2C \end{aligned}\right\} \qquad \text{Critical damping}$$



We will select R, increasing its value until critical damping is obtained, and thus leave ω0 unchanged. The necessary value of R is 7√6/2 Ω; L is still 7 H, and C remains 1/42 F. We thus find

$\alpha = \omega_0 = \sqrt{6}\ s^{-1}$

$s_1 = s_2 = -\sqrt{6}\ s^{-1}$

and recall the initial conditions that were specified, v(0) = 0 and i(0) =10 A.

The differential equation of RLC parallel circuit is

$$C\frac{d^2v}{dt^2} + \frac{1}{R}\frac{dv}{dt} + \frac{1}{L}v = 0 \qquad [1]$$

When α = ω0, the differential equation, Eq. [1], becomes

$$\frac{d^2v}{dt^2} + 2\alpha\frac{dv}{dt} + \alpha^2 v = 0$$

The solution of this equation is not a tremendously difficult process, but we will avoid developing it here, since the equation is a standard type found in the usual differential-equation texts. The solution is

$$v = e^{-\alpha t}(A_1 t + A_2) \qquad [2]$$

Let us now complete our numerical example. After we substitute the known value of α in Eq. [2], obtaining

$$v = A_1 t e^{-\sqrt{6}t} + A_2 e^{-\sqrt{6}t}$$

we establish the values of A_1 and A_2 by first imposing the initial condition on v(t) itself, v(0) = 0. Thus, A2 = 0. The second initial condition must be applied to the derivative dv/dt just as in the overdamped case. We therefore differentiate, remembering that A2 = 0:

$$dv/dt = A_1 t(-\sqrt{6})e^{-\sqrt{6}t} + A_1 e^{-\sqrt{6}t}$$

evaluate at t = 0:

$$dv/dt\,|_{t=0} = A_1$$

and express the derivative in terms of the initial capacitor current:

$$dv/dt\,|_{t=0} = i_C(0)/C = (i_R(0)/C) + (i(0)/C)$$

$$A_1 = 420\ V$$

The response is, therefore,

$$v(t) = 420te^{-2.45t} \qquad V$$

Graphical Representation of the Critically Damped Response

$$\lim_{t\to\infty} v(t) = 420 \lim_{t\to\infty} \frac{t}{e^{2.45t}} = 420 \lim_{t\to\infty} \frac{1}{2.45e^{2.45t}} = 0$$

and once again we have a response that begins and ends at zero and has positive values at all other times. A maximum value v_m again occurs at time tm; for our example,

$t_m = 0.408$ s and $v_m = 63.1$ V

This maximum is larger than that obtained in the overdamped case, and is a result of the smaller losses that occur in the larger resistor; the time of the maximum response is slightly later than it was with overdamping. The settling time may also be determined by solving

$$v_m/100 = 420t_s e^{-2.45t_s}$$

for ts (by trial-and-error methods or a calculator’s SOLVE routine):

ts = 3.12 s

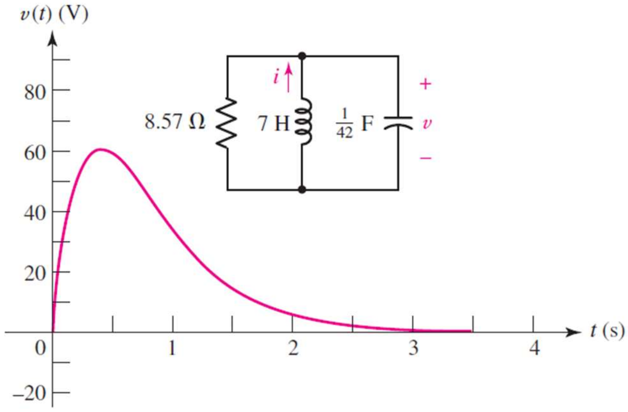


Fig. 2.35.

Example 2.11: Select a value for R_1 such that the circuit of Fig. 2.36 will be characterized by a critically damped response for t > 0, and a value for R_2 such that v(0) = 2 V.

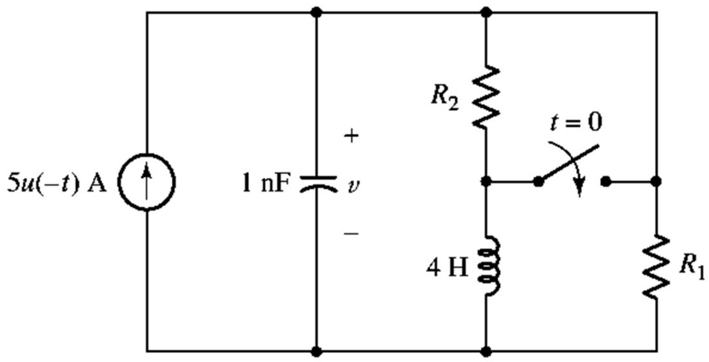


Fig. 2.36

Solution:

We note that at $t = 0^-$, the current source is on, and the inductor can be treated as a short circuit. Thus, $v(0^-)$ appears across R_2, and is given by $v(0^-) = 5R_2$ and a value of 400 mΩ should be selected for R_2 to obtain v(0) = 2 V.

After the switch is thrown, the current source has turned itself off and R_2 is shorted. We are left with a parallel RLC circuit comprised of R_1, a 4 H inductor, and a 1nF capacitor.

We may now calculate (for t > 0)

$$\alpha = 1/2RC = 1/2 \times 10^{-9}R_1$$

and

$$\omega_0 = 1 / \sqrt{LC} = 1 / \sqrt{4 \times 10^{-9}} = 15{,}810 \text{ rad/s}$$

Therefore, to establish a critically damped response in the circuit for t > 0, we need to set R1 = 31.63 kΩ.

H.W.: (a) Choose R_1 in the circuit of Fig. 2.37 so that the response after t = 0 will be critically damped. (b) Now select R_2 to obtain v(0) = 100 V. (c) Find v(t) at t = 1 ms.

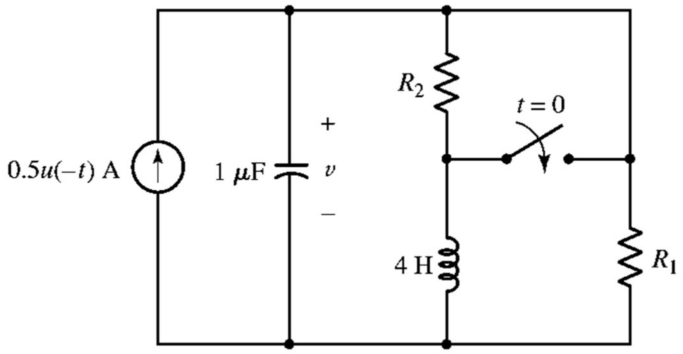


Fig. 2.37.

2.4.3 THE UNDERDAMPED PARALLEL RLC CIRCUIT

The form of the underdamped response

$$v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

where

$$s_{1,2} = -\frac{1}{2R} \pm \sqrt{\left(\frac{1}{2R}\right)^2 - \frac{1}{LC}}$$

and then let

$$\sqrt{\alpha^2 - w_0^2} = \sqrt{-1}\sqrt{w_0^2 - \alpha^2} = j\sqrt{w_0^2 - \alpha^2}$$

We now take the new radical, which is real for the underdamped case, and call it ω_d, the natural resonant frequency:

$$w_d = \sqrt{w_0^2 - \alpha^2}$$

The response may now be written as

$$v(t) = e^{-\alpha t}\left(A_1 e^{jw_d t} + A_2 e^{-jw_d t}\right) \qquad [1]$$

or, in the longer but equivalent form,

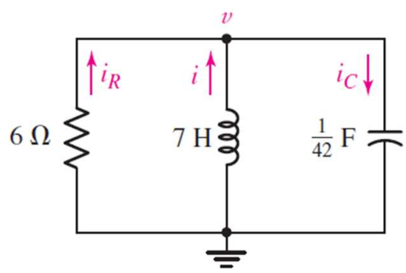
$$v(t) = e^{-\alpha t}\left\{(A_1 + A_2)\left[\frac{e^{jw_d t} + e^{-jw_d t}}{2}\right] + j(A_1 - A_2)\left[\frac{e^{jw_d t} - e^{-jw_d t}}{2j}\right]\right\}$$

$$v(t) = e^{-\alpha t}\{(A_1 + A_2)cos(w_d t) + j(A_1 - A_2)sin(w_d t)\}$$

and the multiplying factors may be assigned new symbols:

$$v(t) = e^{-\alpha t}\{B_1 cos(w_d t) + B_2 sin(w_d t)\} \qquad [2]$$

We return to our simple parallel RLC circuit with R = 6 Ω, C = 1/42 F, and L = 7 H, but now increase the resistance further to 10.5 Ω.



Thus,

$$\alpha = \frac{1}{2RC} = 2 \qquad s^{-1}$$

$$\omega_0 = \frac{1}{\sqrt{LC}} = \sqrt{6} \quad s^{-1}$$

and

$$w_d = \sqrt{w_0^2 - \alpha^2} = \sqrt{2}\, rad/s$$

except for the evaluation of the arbitrary constants, the response is now known:

$$v(t) = e^{-2t}\{B_1 cos(\sqrt{2}t) + B_2 sin(\sqrt{2}t)\}$$

The determination of the two constants proceeds as before. If we still assume that v(0) = 0 and i(0) = 10, then B_1 must be zero. Hence

$$v(t) = e^{-2t} B_2 sin(\sqrt{2}t)$$

The derivative is

$$\frac{dv}{dt} = \sqrt{2}e^{-2t}B_2 cos(\sqrt{2}t) - 2e^{-2t}B_2 sin(\sqrt{2}t)$$

and at t = 0 it becomes

$$\left.\frac{dv}{dt}\right|_{t=0} = \sqrt{2}B_2 = \frac{i_C(0)}{C} = 420$$

where i_C is defined in figure. Therefore,

$$v(t) = 210\sqrt{2}e^{-2t} sin(\sqrt{2}t)$$

Graphical Representation of the Underdamped Response

Returning to our specific numerical problem, differentiation locates the first maximum of v(t),

v_{m1} = 71.8 V at t_{m1} = 0.435 s

The succeeding minimum,

v_{m2} = −0.845 V at t_{m2} = 2.66 s

and so on. The response curve is shown in Fig. 2.38.

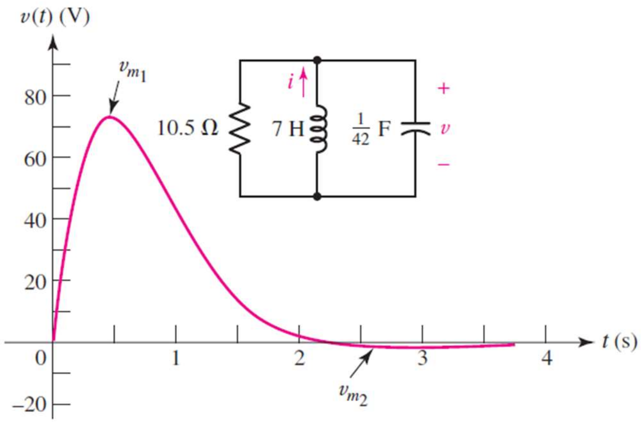


Fig. 2.38

Example 2.12: Determine $i_L(t)$ for the circuit of Fig. 2.39a, and plot the waveform.

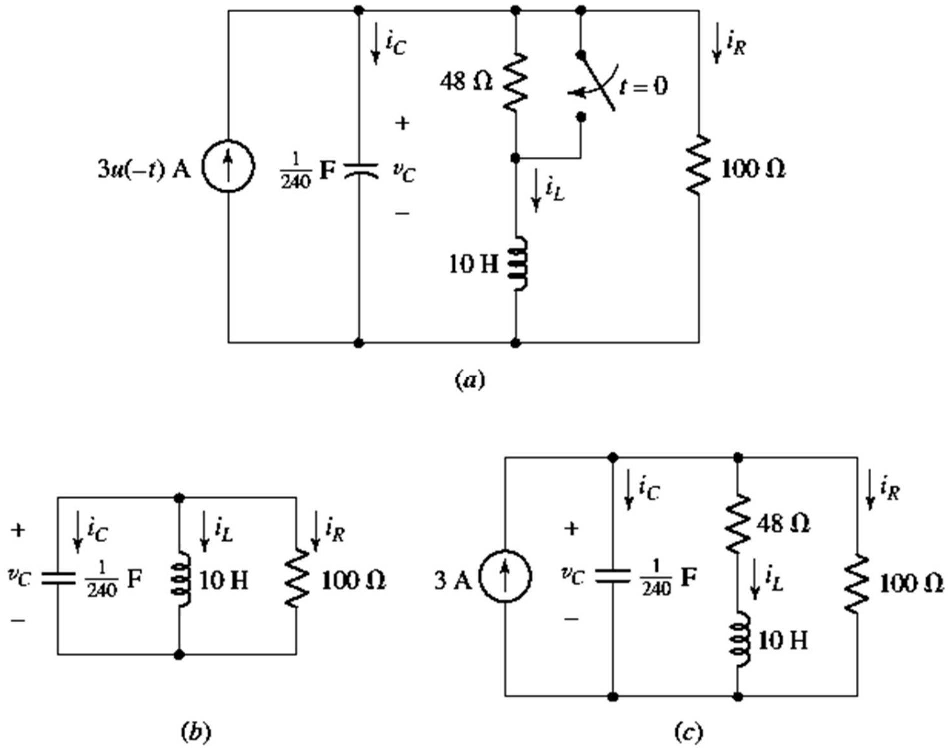


Fig. 2.39

Solution:

At t = 0, both the 3 A source and the 48 Ω resistor are removed, leaving the circuit shown in Fig. 3.39b. Thus, $\alpha = 1.2\ s^{-1}$ and $\omega_0 = 4.899$ rad/s. Since $\alpha < \omega_0$, the circuit is underdamped, and we therefore expect a response of the form

$$i_L(t) = e^{-\alpha t}(B_1 \cos\omega_d t + B_2 \sin\omega_d t) \qquad [1]$$

where $\omega_d = \sqrt{(\omega_0^2 - \alpha^2} = 4.750$ rad/s. The only remaining step is to find B_1 and B_2.

Figure 3.39c shows the circuit as it exists at $t = 0^-$. We may replace the inductor with a short circuit and the capacitor with an open circuit; the result is $v_C(0^-) = 97.30$ V and $i_L(0^-) = 2.027$ A. Since neither quantity can change in zero time, $v_C(0^+) = 97.30$ V and $i_L(0^+) = 2.027$ A.

Substituting $i_L(0) = 2.027$ into Eq. [1] yields $B_1 = 2.027$ A. To determine the other constant, we first differentiate Eq. [1]:

$$di_L/dt = e^{-\alpha t}(-B_1\omega_d \sin\omega_d t + B_2\omega_d \cos\omega_d t) - \alpha e^{-\alpha t}(B_1 \cos\omega_d t + B_2 \sin\omega_d t) \qquad [2]$$

and note that $v_L(t) = L(di_L/dt)$. Referring to the circuit of Fig. 2.39b, we see that $v_L(0^+) = v_C(0^+) = 97.3$ V. Thus, multiplying Eq. [2] by L = 10 H and setting t = 0, we find that

$v_L(0) = 10(B_2\omega_d) - 10\alpha B_1 = 97.3$

Solving, $B_2 = 2.561$ A, so that

$$i_L = e^{-1.2t}(2.027 \cos 4.75t + 2.561 \sin 4.75t) \qquad \text{A}$$

which we have plotted in Fig. 2.40.

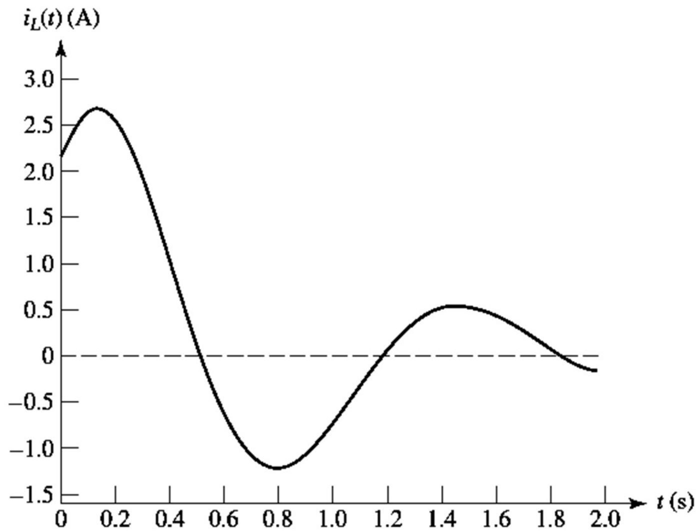


Fig. 2.40

H.W.: The switch in the circuit of Fig. 2.41 has been in the left position for a long time; it is moved to the right at $t = 0$. Find (a) dv/dt at $t = 0^+$; (b) v at $t = 1$ ms; (c) t_0, the first value of t greater than zero at which $v = 0$.

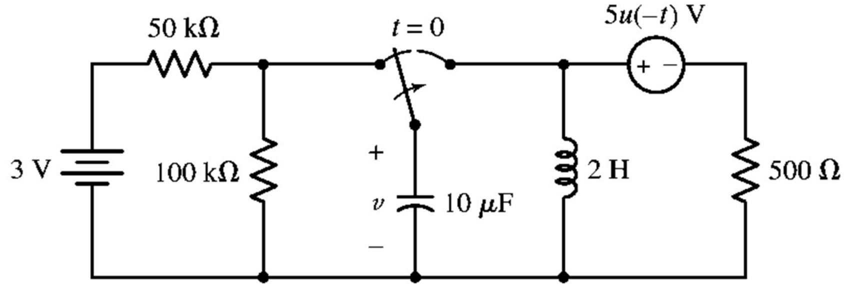


Fig. 2.41

2.4.4 THE SOURCE-FREE SERIES RLC CIRCUIT

Figure 2.42 shows the series circuit. The fundamental integrodifferential equation is

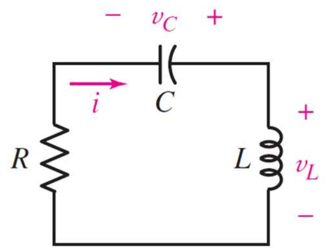


Fig. 2.42.

$$L\frac{di}{dt} + Ri + \int_{t_0}^{t} \frac{1}{C} i\, dt' - v_C(t_0) = 0$$

The respective second-order equations obtained by differentiating these two equations with respect to time are also duals:

$$L\frac{d^2 i}{dt^2} + R\frac{di}{dt} + \frac{1}{C}i = 0 \qquad [1]$$

In terms of the circuit shown in Fig. 2.42, the overdamped response is

$$i(t) = A_1 e^{s_1 t} + A_1 e^{s_1 t}$$

where

$$s_{1,2} = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$

and thus

$$\alpha = \frac{R}{2L}$$

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

The form of the critically damped response is

$$i(t) = e^{-\alpha t}\,(A_1 t + A_2)$$

and the underdamped response may be written

$$i(t) = e^{-\alpha t}\,(B_1 cos\omega_d t + B_2 sin\omega_d t)$$

Where $\omega_d = \sqrt{\omega_0^2 - \alpha^2}$

Type	Condition	Criteria	α	ω_0	Response
Parallel Series	Overdamped	$\alpha > \omega_0$	$\frac{1}{2RC}$ $\frac{R}{2L}$	$\frac{1}{\sqrt{LC}}$	$A_1e^{s_1t} + A_2e^{s_2t}$, where $s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega^2}$
Parallel Series	Critically damped	$\alpha = \omega_0$	$\frac{1}{2RC}$ $\frac{R}{2L}$	$\frac{1}{\sqrt{LC}}$	$e^{-\alpha t}\,(A_1t + A_2)$
Parallel Series	Underdamped	$\alpha < \omega_0$	$\frac{1}{2RC}$ $\frac{R}{2L}$	$\frac{1}{\sqrt{LC}}$	$e^{-\alpha t}(B_1 \cos\omega_d t + B_2 \sin\omega_d t)$, where $\omega_d = \sqrt{\omega_0^2 - \alpha^2}$

Example 2.13: Given the series RLC circuit of Fig. 2.43 in which L = 1 H, R = 2 kΩ, C = 1/401 μF, i(0) = 2 mA, and $v_C(0)$ = 2 V, find and sketch i(t), t > 0.

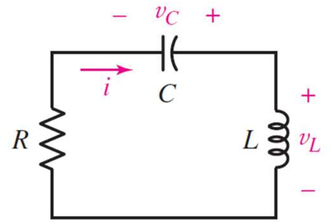


Fig. 2.43

Solution:

We find that $\alpha = \frac{R}{2L}$ = 1000 s^{-1} and $\omega_0 = \frac{1}{\sqrt{LC}}$ = 20,025 rad/s.

This indicates an underdamped response; we therefore calculate the value of ω_d and obtain 20,000 rad/s. Except for the evaluation of the two arbitrary constants, the response is now known:

$$i(t) = e^{-1000}\ (B_1cos20000t + B_2sin20000t)$$

Since we know that i(0) = 2 mA, we may substitute this value into our equation for i (t) to obtain B_1 = 0.002 A

and thus

$$i(t) = e^{-1000t}\ (0.002cos20000t + B_2sin20000t) \qquad \text{A}$$

The remaining initial condition must be applied to the derivative; thus,

$$\frac{di}{dt} = e^{-1000}\ (-40sin20000t + 20000B_2cos20000t - 2cos20000t - 1000B_2sin20000t)$$

and

$$\left.\frac{di}{dt}\right|_{t=0} = 20000B_2 - 2 = \frac{v_L(0)}{L} = \frac{v_C(0) - Ri(0)}{L} = \frac{2 - 2000(0.002)}{1} = -2\ A/s$$

so that B_2 = 0

The desired response is therefore

$$i(t) = 2e^{-1000}\ cos20000t \quad mA$$

A good sketch may be made by first drawing in the two portions of the exponential envelope, $2e^{-1000t}$ and $-2e^{-1000t}$ mA, as shown by the broken lines in Fig. 2.44. The location of the quarter-cycle points of the sinusoidal wave at $20{,}000t = 0, \pi/2, \pi$, etc., or $t = 0.07854k$ ms, $k = 0, 1, 2, \ldots$, by light marks on the time axis then permits the oscillatory curve to be sketched in quickly.

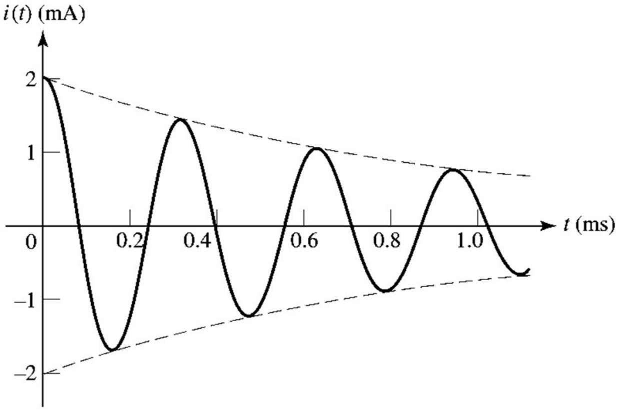


Fig. 2.44.

H.W.: With reference to the circuit shown in Fig. 2.45, find (a) α; (b) ω_0; (c) $i(0^+)$; (d) $di/dt|_{t=0+}$; (e) i (12 ms).

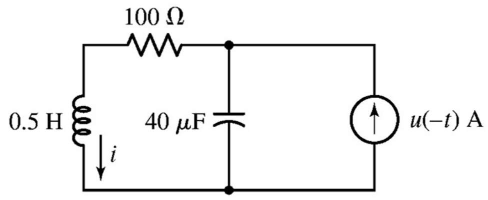


Fig. 2.45.

Example 2.14: Find an expression for $v_C(t)$ in the circuit of Fig. 2.46a, valid for $t > 0$.

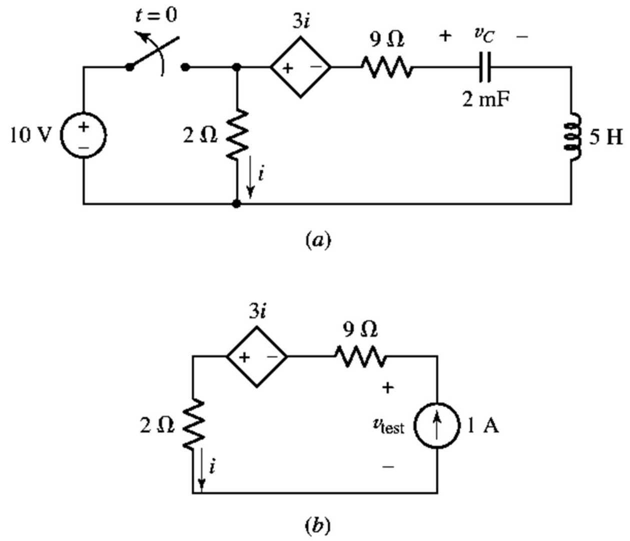


Fig. 2.46.

Solution:
As we are interested only in $v_C(t)$, it is perfectly acceptable to begin by finding the Thévenin equivalent resistance connected in series with the inductor and capacitor at $t = 0^+$. We do this by connecting a 1 A source as shown in Fig. 2.46b, from which we deduce that

$$v_{test} = 11i - 3i = 8i = 8(1) = 8 \text{ V}$$

Thus, Req =8 Ω, so $\alpha = R/2L = 0.8\ s^{-1}$ and $\omega_0 = \frac{1}{\sqrt{LC}} = 10$ rad/s, meaning that we expect an underdamped response with $\omega_d = 9.968$ rad/s and the form

$$v_C(t) = e^{-0.8t}(B_1 \cos 9.968t + B_2 \sin 9.968t) \qquad [1]$$

In considering the circuit at $t = 0^-$, we note that $i_L(0^-) = 0$ due to the presence of the capacitor. By Ohm's law, $i(0^-) = 5$ A, so

$$v_C(0^+) = v_C(0^-) = 10 - 3i = 10 - 15 = -5 \text{ V}$$

This last condition substituted into Eq. [1] yields $B_1 = -5$ V. Taking the derivative of Eq. [1] and evaluating at t = 0 yield

$$dv_C/dt|_{t=0} = -0.8B_1 + 9.968B_2 = 4 + 9.968B_2 \qquad [2]$$

We see from Fig. 2.46a that

$$i = -C dv_C/dt$$

Thus, making use of the fact that $i(0^+) = i_L(0^-) = 0$ in Eq. [2] yields
$B_2 = -0.4013$ V, and we may write

$$v_C(t) = -e^{-0.8t}(5 \cos 9.968t + 0.4013 \sin 9.968t) \quad \text{V} \qquad t > 0$$

H.W.: Find an expression for $i_L(t)$ in the circuit of Fig. 2.47, valid for $t > 0$, if $v_C(0^-) = 10$ V and $i_L(0^-) = 0$. **Note that although it is not helpful to apply Thévenin techniques in this instance, the action of the dependent source links v_C and i_L such that a first-order linear differential equation results**.

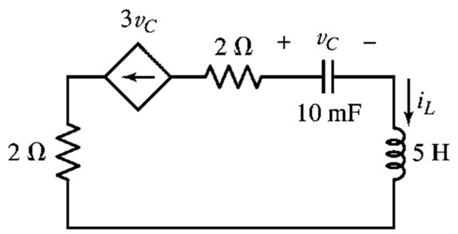


Fig. 2.47.

2.4.5 THE COMPLETE RESPONSE OF THE RLC CIRCUIT

We now consider those RLC circuits in which dc sources are switched into the network and produce forced responses that do not necessarily vanish as time becomes infinite.

The general solution is obtained by the same procedure that was followed for RL and RC circuits. The basic steps are (not necessarily in this order) as follows:

1. Determine the initial conditions.
2. Obtain a numerical value for the forced response.
3. Write the appropriate form of the natural response with the necessary number of arbitrary constants.
4. Add the forced response and natural response to form the complete response.
5. Evaluate the response and its derivative at t = 0, and employ the initial conditions to solve for the values of the unknown constants.

Example 2.15: There are three passive elements in the circuit shown in Fig. 2.48a, and a voltage and a current are defined for each. Find the values of these six quantities at both $t = 0^-$ and $t = 0^+$.

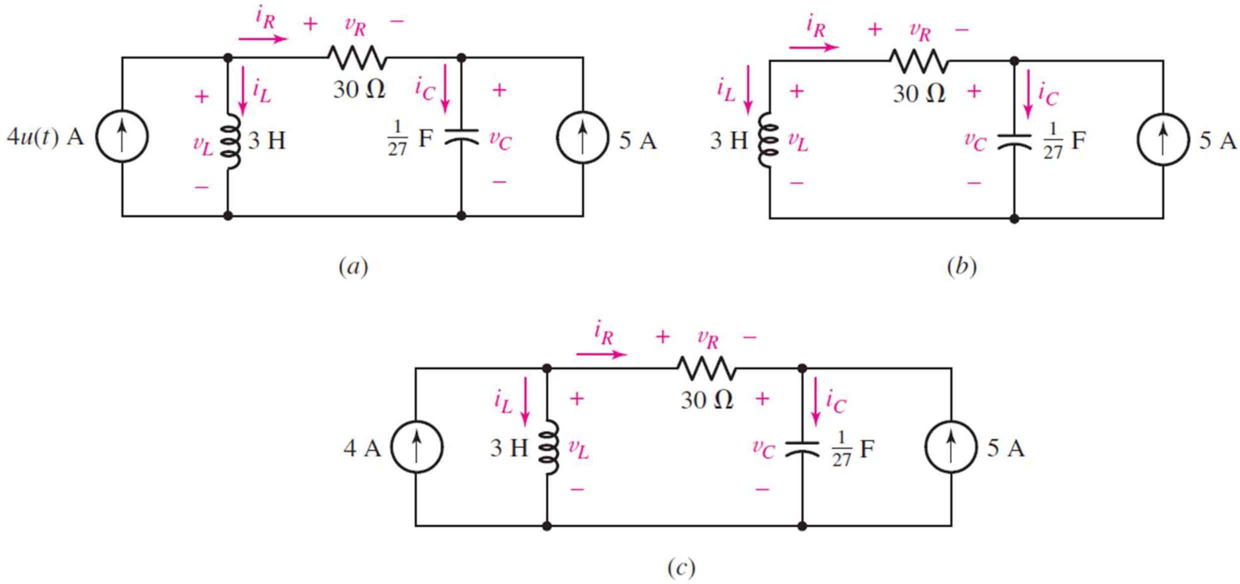


Fig. 2.48.

Solution:

Our object is to find the value of each current and voltage at both t = 0− and t = 0+. Once these quantities are known, the initial values of the derivatives may be found easily.

1. t = 0⁻ At $t = 0^-$, only the right-hand current source is active as depicted in Fig. 2.48b. The circuit is assumed to have been in this state forever, so all currents and voltages are constant. Thus, a dc current through the inductor requires zero voltage across it: $v_L(0^-) = 0$

and a dc voltage across the capacitor ($-v_R$) requires zero current through it: $i_C(0^-) = 0$

We next apply Kirchhoff's current law to the right-hand node to obtain: $i_R(0^-) = -5$ A

which also yields $v_R(0^-) = -150$ V

We may now use Kirchhoff's voltage law around the left-hand mesh, finding: $v_C(0^-) = 150$ V

while KCL enables us to find the inductor current, $i_L(0^-) = 5$ A

2. t = 0⁺ During the interval from $t = 0^-$ to $t = 0^+$, the left-hand current source becomes active and many of the voltage and current values at $t = 0^-$ will change abruptly. The corresponding circuit is shown in Fig. 2.48c. However, we should begin by focusing our attention on those quantities which cannot change, namely, the inductor current and the capacitor voltage. Both of these must remain constant during the switching interval. Thus,

$i_L(0^+) = 5$ A and $v_C(0^+) = 150$ V

Since two currents are now known at the left node, we next obtain

$i_R(0^+) = -1$ A and $v_R(0^+) = -30$ V

so that

$i_C(0^+) = 4$ A and $v_L(0^+) = 120$ V

and we have our six initial values at $t = 0^-$ and six more at $t = 0^+$.
Among these last six values, only the capacitor voltage and the inductor current are unchanged from the $t = 0^-$ values.

H.W.: Let $i_s = 10u(-t) - 20u(t)$ A in Fig. 2.49. Find (a) $i_L(0^-)$ (b) $v_C(0^+)$; (c) $v_R(0^+)$; (d) $i_L(\infty)$; (e) $i_L(0.1\ ms)$.

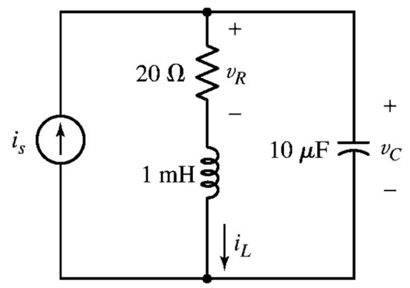


Fig. 2.49.

Example 2.16: Complete the determination of the initial conditions in the circuit of Fig. 2.51, by finding values at $t = 0^+$ for the first derivatives of the three voltage and three current variables defined on the circuit diagram.

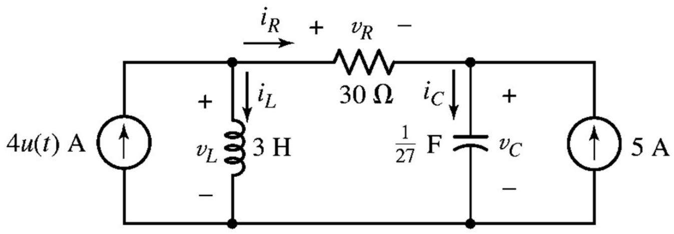


Fig. 2.51.

Solution:
We begin with the two energy storage elements. For the inductor,

$$v_L = L\frac{di_L}{dt}$$

and, specifically,

$$v_L(0^+) = L\frac{di_L}{dt}\bigg|_{t=0^+}$$

Thus,

$$\left.\frac{di_L}{dt}\right|_{t=0^+} = \frac{v_L(0^+)}{L} = \frac{120}{3} = 40 \quad A/s$$

Similarly,

$$\left.\frac{dv_C}{dt}\right|_{t=0^+} = \frac{i_C(0^+)}{C} = \frac{4}{1/27} = 108 \quad V/s$$

The other four derivatives may be determined by realizing that KCL and KVL are both satisfied by the derivatives also. For example, at the left-hand node in Fig. 2.51,

$$4 - i_L - i_R = 0 \qquad t > 0$$

and thus,

$$0 - di_L/dt - di_R/dt = 0 \qquad t > 0$$

and therefore,

$$\left.\frac{di_R}{dt}\right|_{t=0^+} = -40 \quad A/s$$

The three remaining initial values of the derivatives are found to be

$$\left.\frac{dv_R}{dt}\right|_{t=0^+} = -1200 \quad V/s$$

$$\left.\frac{dv_L}{dt}\right|_{t=0^+} = -1092 \quad V/s$$

and

$$\left.\frac{di_C}{dt}\right|_{t=0^+} = -40 \quad A/s$$