Lectures on
The Hypergeometric Functions

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Syllabus of M. Sc. in Mathematics

Course Name: The Hypergeometric Functions
Credits : 2 units

1. **The Hypergeometric Series:**

   The hypergeometric series, The binomial theorem, Examples, Transformation formulas for ${}_2F_1$ series, Gauss summation formula, Saalschütz's summation formula, The Kummers formula, The Legendres duplication formula, The Gauss multiplication formula, Integral Representation, Mellin transforms.

2. **The Basic Hypergeometric Series:**

   Hypergeometric and basic hypergeometric series, The q-binomial theorem, Heine's transformation formulas for ${}_2\phi_1$ series, Heine's q-analogue of Gauss' summation formula, Jacobi's triple product identity, A q-analogue of Saalschütz's summation formula, The Bailey-Daum summation formula, The q-gamma and q-beta function, The q-integral.

References

[1] G.E. Andrews, *The Theory of Partitions*, Cambridge University Press, Cambridge, 1984.

[2] G. Gasper and M. Rahman, *Basic Hypergeometric Series*, 2nd ed., Cambridge University Press, Cambridge, MA, 2004.

[3] E.D. Rainville, *Special Functions*, The Macmillan Company, New York, 1960.

CHAPTER 1

Infinite Products

1. Introduction. Two topics, infinite products and asymptotic series, which are seldom included in standard courses are treated to some extent in short preliminary chapters.

The variables and parameters encountered are to be considered complex except where it is specifically stipulated that they are real.

Exercises are included not only to present the reader with an opportunity to increase his skill but also to make available a few results for which there seemed to be insufficient space in the text.

A short bibliography is included at the end of the book. All references are given in a form such as Fasenmyer [2], meaning item number two under the listing of references to the work of Sister M. Celine Fasenmyer, or Brafman [1;944], meaning page 944 of item number one under the listing of references to the work of Fred Brafman. In general, specific reference to material a century or more old is omitted. The work of the giants in the field, Euler, Gauss, Legendre, etc., is easily located either in standard treatises or in the collected works of the pertinent mathematician.

2. Definition of an infinite product. The elementary theory of infinite products closely parallels that of infinite series. Given a sequence a_k defined for all positive integral k, consider the finite product

$$(1) \qquad P_n = \prod_{k=1}^{n}(1 + a_k) = (1 + a_1)(1 + a_2)\cdots(1 + a_n).$$

If $\lim_{n \to \infty} P_n$ exists and is equal to $P \neq 0$, we say that the infinite product

$$(2) \qquad \prod_{n=1}^{\infty} (1 + a_n)$$

converges to the value P. If at least one of the factors of the product (2) is zero, if only a finite number of the factors of (2) are zero, and if the infinite product with the zero factors deleted converges to a value $P \neq 0$, we say that the infinite product *converges to zero.*

If the infinite product is not convergent, it is said to be *divergent.* If that divergence is due not to the failure of $\lim_{n \to \infty} P_n$ to exist but to the fact that the limit is zero, the product is said to *diverge to zero.* We make no attempt to treat products with an infinity of zero factors.

The peculiar role which zero plays in multiplication is the reason for the slight difference between the definition of convergence of an infinite product and the analogous definition of convergence of an infinite series.

3. A necessary condition for convergence. The general term of a convergent infinite series must approach zero as the index of summation approaches infinity. A similar result will now be obtained for infinite products.

THEOREM 1. *If* $\prod_{n=1}^{\infty} (1 + a_n)$ *converges,*

$$\lim_{n \to \infty} a_n = 0.$$

Proof: If the product converges to $P \neq 0$,

$$1 = \frac{P}{P} = \frac{\lim_{n \to \infty} \prod_{k=1}^{n} (1 + a_k)}{\lim_{n \to \infty} \prod_{k=1}^{n-1} (1 + a_k)} = \lim_{n \to \infty} (1 + a_n).$$

Hence $\lim_{n \to \infty} a_n = 0$, as desired. If the product converges to zero, remove the zero factors and repeat the argument.

4. The associated series of logarithms. Any product without zero factors has associated with it the series of principal values of the logarithms of the separate factors in the following sense.

THEOREM 2. *If no* $a_n = -1$, $\prod_{n=1}^{\infty}(1 + a_n)$ *and* $\sum_{n=1}^{\infty} \text{Log}\,(1 + a_n)$ *converge or diverge together.*

Proof: Let the partial product and partial sum be indicated as follows:

$$P_n = \prod_{k=1}^{n}(1 + a_k), \qquad S_n = \sum_{k=1}^{n} \text{Log}(1 + a_k).$$

Then* $\exp S_n = P_n$. We know from the theory of complex variables that $\lim_{n\to\infty} \exp S_n = \exp \lim_{n\to\infty} S_n$. Therefore P_n approaches a limit if and only if S_n approaches a limit, and P_n cannot approach zero because the exponential function cannot take on the value zero.

5. Absolute convergence. Assume that the product $\prod_{n=1}^{\infty}(1 + a_n)$ has had its zero factors, if any, deleted. We define *absolute convergence* of the product by utilizing the associated series of logarithms.

The product $\prod_{n=1}^{\infty}(1 + a_n)$, with zero factors deleted, is said to be absolutely convergent if and only if the series $\sum_{n=1}^{\infty} \text{Log}\,(1 + a_n)$ is absolutely convergent.

THEOREM 3. *The product* $\prod_{n=1}^{\infty}(1 + a_n)$, *with zero factors deleted, is absolutely convergent if and only if* $\sum_{n=1}^{\infty} a_n$ *is absolutely convergent.*

Proof: First throw out any a_n's which are zero; they contribute only unit factors in the product and zero terms in the sum and thus have no bearing on convergence.

We know that if either the series or the product in the theorem converges, $\lim_{n\to\infty} a_n = 0$. Let us then consider n large enough, $n > n_0$, so that $|a_n| < \frac{1}{2}$ for all $n > n_0$. We may now write

$$\frac{\text{Log}\,(1 + a_n)}{a_n} = \sum_{k=0}^{\infty} \frac{(-1)^k a_n{}^k}{k + 1}, \tag{1}$$

from which it follows that

$$\left|\frac{\text{Log}\,(1 + a_n)}{a_n} - 1\right| \leq \sum_{k=1}^{\infty} \frac{|a_n|^k}{k+1} < \sum_{k=1}^{\infty} \frac{1}{2^{k+1}} = \frac{1}{2}.$$

*We make frequent use of the common notation $\exp u = e^u$.

Thus we have

$$\frac{1}{2} < \left| \frac{\operatorname{Log}(1 + a_n)}{a_n} \right| < \frac{3}{2},$$

from which

$$\left| \frac{\operatorname{Log}(1 + a_n)}{a_n} \right| < \frac{3}{2} \qquad \text{and} \qquad \left| \frac{a_n}{\operatorname{Log}(1 + a_n)} \right| < 2.$$

By the comparison test it follows that the absolute convergence of either of $\sum_{n=1}^{\infty} \operatorname{Log}(1 + a_n)$ or $\sum_{n=1}^{\infty} a_n$ implies the absolute convergence of the other. We then use the definition of absolute convergence of the product to complete the proof of Theorem 3.

Because of Theorem 2 it follows at once that an infinite product which is absolutely convergent is also convergent.

Example (*a*): Show that the following product converges and find its value:

$$\prod_{n=1}^{\infty} \left[1 + \frac{1}{(n + 1)(n + 3)} \right].$$

The series of positive numbers

$$\sum_{n=1}^{\infty} \frac{1}{(n + 1)(n + 3)}$$

is known to be convergent. It can easily be tested by the polynomial test or by comparison with the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$. Hence our product is absolutely convergent by Theorem 3.

The partial products are often useful in evaluating an infinite product. When the following method is employed, there is no need for the separate testing for convergence made in the preceding paragraph. Consider the partial products

$$P_n = \prod_{k=1}^{n} \left[1 + \frac{1}{(k + 1)(k + 3)} \right] = \prod_{k=1}^{n} \frac{(k + 2)^2}{(k + 1)(k + 3)}$$

$$= \frac{[3 \cdot 4 \cdot 5 \cdots (n + 2)]^2}{[2 \cdot 3 \cdot 4 \cdots (n + 1)][4 \cdot 5 \cdot 6 \cdots (n + 3)]} = \frac{n + 2}{2} \cdot \frac{3}{n + 3}.$$

At once $\lim_{n \to \infty} P_n = \frac{3}{2}$, from which we conclude both that the infinite product converges and that its value is $\frac{3}{2}$.

EXAMPLE (*b*): Show that if z is not a negative integer,

$$\lim_{n\to\infty} \frac{(n-1)!\ n^z}{(z+1)(z+2)(z+3)\cdots(z+n-1)}$$

exists.

We shall form an infinite product for which the expression

$$P_n = \frac{(n-1)!\ n^z}{(z+1)(z+2)(z+3)\cdots(z+n-1)}$$

is a partial product, prove that the infinite product converges, and thus conclude that $\lim_{n\to\infty} P_n$ exists.

Write

$$\begin{aligned} P_{n+1} &= \frac{n!\ (n+1)^z}{(z+1)(z+2)\cdots(z+n)} \\ &= \frac{n!}{(z+1)(z+2)\cdots(z+n)} \cdot \frac{2^z}{1^z} \cdot \frac{3^z}{2^z} \cdot \frac{4^z}{3^z} \cdots \frac{(n+1)^z}{n^z} \\ &= \prod_{k=1}^{n} \left[\frac{k}{z+k} \cdot \frac{(k+1)^z}{k^z} \right] = \prod_{k=1}^{n} \left[\left(1+\frac{z}{k}\right)^{-1} \left(1+\frac{1}{k}\right)^z \right]. \end{aligned}$$

Consider now the product*

$$\prod_{n=1}^{\infty} \left[\left(1+\frac{z}{n}\right)^{-1} \left(1+\frac{1}{n}\right)^z \right]. \tag{2}$$

Since

$$\lim_{n\to\infty} n^2 \left[\left(1+\frac{z}{n}\right)^{-1} \left(1+\frac{1}{n}\right)^z - 1 \right]$$

$$= \lim_{\beta\to 0} \frac{(1+z\beta)^{-1}(1+\beta)^z - 1}{\beta^2} = \lim_{\beta\to 0} \frac{(1+\beta)^z - 1 - z\beta}{\beta^2}$$

$$= \lim_{\beta\to 0} \frac{z[(1+\beta)^{z-1} - 1]}{2\beta} = \lim_{\beta\to 0} \frac{z(z-1)(1+\beta)^{z-2}}{2} = \tfrac{1}{2} z(z-1),$$

we conclude with the aid of the comparison test and the convergence of $\sum_{n=1}^{\infty} \frac{1}{n^2}$ that the product (2) converges. Therefore $\lim_{n\to\infty} P_n$ exists.

6. Uniform convergence. Let the factors in the product $\prod_{n=1}^{\infty} [1 + a_n(z)]$ be dependent upon a complex variable z. Let R

*We shall find in Chapter 2 that this product has the value $z\Gamma(z)$.

be a closed region in the z-plane. If the product converges in such a way that, given any $\epsilon > 0$, there exists an n_0 independent of z for all z in R such that

$$\left| \prod_{k=1}^{n_0+p} [1 + a_k(z)] - \prod_{k=1}^{n_0} [1 + a_k(z)] \right| < \epsilon$$

for all positive integral p, we say that the product $\prod_{n=1}^{\infty} [1 + a_n(z)]$ is *uniformly convergent* in the region R.

Again the convergence properties parallel those of infinite series. We need a Weierstrass M-test.

THEOREM 4. *If there exist positive constants M_n such that $\sum_{n=1}^{\infty} M_n$ is convergent and $|a_n(z)| < M_n$ for all z in the closed region R, the product $\prod_{n=1}^{\infty} [1 + a_n(z)]$ is uniformly convergent in R.*

Proof: Since $\sum_{n=1}^{\infty} M_n$ is convergent and $M_n > 0$, $\prod_{n=1}^{\infty} (1 + M_n)$ is convergent and $\lim_{n\to\infty} \prod_{k=1}^{n} (1 + M_k)$ exists. Therefore, given any $\epsilon > 0$, there exists an n_0 such that

$$\prod_{k=1}^{n_0+p} (1 + M_k) - \prod_{k=1}^{n_0} (1 + M_k) < \epsilon$$

for all positive integers p. For all z in R, each $a_k(z)$ is such that $|a_k(z)| < M_k$. Hence

$$\left| \prod_{k=1}^{n_0+p} [1 + a_k(z)] - \prod_{k=1}^{n_0} [1 + a_k(z)] \right|$$

$$= \left| \prod_{k=1}^{n_0} [1 + a_k(z)] \right| \cdot \left| \prod_{k=n_0+1}^{n_0+p} [1 + a_k(z)] - 1 \right|$$

$$< \prod_{k=1}^{n_0} (1 + M_k) \left[\prod_{k=n_0+1}^{n_0+p} (1 + M_k) - 1 \right]$$

$$< \prod_{k=1}^{n_0+p} (1 + M_k) - \prod_{k=1}^{n_0} (1 + M_k) < \epsilon,$$

which was to be proved.

EXERCISES

1. Show that the following product converges, and find its value:

$$\prod_{n=1}^{\infty}\left[1+\frac{6}{(n+1)(2n+9)}\right].$$

Ans. $\dfrac{21}{8}$.

2. Show that $\displaystyle\prod_{n=2}^{\infty}\left(1-\frac{1}{n^2}\right)=\frac{1}{2}$.

3. Show that $\displaystyle\prod_{n=2}^{\infty}\left(1-\frac{1}{n}\right)$ diverges to zero.

4. Investigate the product $\displaystyle\prod_{n=0}^{\infty}(1+z^{2^n})$ in $|z|<1$.

Ans. Abs. conv. to $\dfrac{1}{1-z}$.

5. Show that $\displaystyle\prod_{n=1}^{\infty}\exp\left(\frac{1}{n}\right)$ diverges.

6. Show that $\displaystyle\prod_{n=1}^{\infty}\exp\left(-\frac{1}{n}\right)$ diverges to zero.

7. Test $\displaystyle\prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2}\right)$.

Ans. Abs. conv. for all finite z.

8. Show that $\displaystyle\prod_{n=1}^{\infty}\left[1+\frac{(-1)^{n+1}}{n}\right]$ converges to unity.

9. Test for convergence: $\displaystyle\prod_{n=2}^{\infty}\left(1-\frac{1}{n^p}\right)$ for real $p\neq 0$.

Ans. Conv. for $p>1$· div. for $p\leqq 1$.

10. Show that $\displaystyle\prod_{n=1}^{\infty}\frac{\sin(z/n)}{z/n}$ is absolutely convergent for all finite z with the usual convention at $z=0$. *Hint:* Show first that

$$\lim_{n\to\infty} n^2\left[\frac{\sin(z/n)}{z/n}-1\right]=-\frac{z^2}{6}.$$

11. Show that if c is not a negative integer,

$$\prod_{n=1}^{\infty}\left[\left(1-\frac{z}{c+n}\right)\exp\left(\frac{z}{n}\right)\right]$$

is absolutely convergent for all finite z. *Hint*: Show first that

$$\lim_{n\to\infty} n^2\left[\left(1-\frac{z}{c+n}\right)\exp\left(\frac{z}{n}\right)-1\right]=z\left(c-\frac{1}{2}z\right).$$

CHAPTER 2

The Gamma and Beta Functions

7. The Euler or Mascheroni constant γ. At times we need to use the constant γ, defined by

$$\gamma = \lim_{n \to \infty} (H_n - \operatorname{Log} n), \tag{1}$$

in which, as usual,

$$H_n = \sum_{k=1}^{n} \frac{1}{k}. \tag{2}$$

We shall prove that γ exists and that $0 \leqq \gamma < 1$. Actually $\gamma = 0.5772$, approximately.

Let $A_n = H_n - \operatorname{Log} n$. Then the A_n form a decreasing sequence because

$$\begin{aligned} A_{n+1} - A_n &= H_{n+1} - H_n - \operatorname{Log}(n+1) + \operatorname{Log} n \\ &= \frac{1}{n+1} + \operatorname{Log}\frac{n}{n+1} = \frac{1}{n+1} + \operatorname{Log}\left(1 - \frac{1}{n+1}\right) \\ &= -\sum_{k=1}^{\infty} \frac{1}{(k+1)(n+1)^{k+1}} < 0. \end{aligned}$$

Furthermore, since $1/t$ decreases steadily as t increases,

$$\frac{1}{k} < \int_{k-1}^{k} \frac{dt}{t} < \frac{1}{k-1}, \qquad k \geqq 2. \tag{3}$$

We sum the inequalities (3) from $k = 2$ to $k = n$ and thus obtain

$$H_n - 1 < \int_1^2 \frac{dt}{t} + \int_2^3 \frac{dt}{t} + \cdots + \int_{n-1}^n \frac{dt}{t} < H_{n-1},$$

or

$$H_n - 1 < \operatorname{Log} n < H_{n-1},$$

from which it follows that

$$-1 < -H_n + \operatorname{Log} n < -\frac{1}{n},$$

or

$$1 > A_n > \frac{1}{n}.$$

Thus we see that the A_n decrease steadily, are all positive, and are less than unity. It follows that γ exists and is non-negative and less than unity.

8. The Gamma function. We follow Weierstrass in defining the function $\Gamma(z)$ by

$$(1) \qquad \frac{1}{\Gamma(z)} = ze^{\gamma z} \prod_{n=1}^{\infty} \left[\left(1 + \frac{z}{n}\right) \exp\left(-\frac{z}{n}\right) \right],$$

in which γ is the Euler constant of Section 7. The product in (1) is absolutely convergent for all finite z as was seen in Ex. 11, page 7, the special case $c = 0$ and z replaced by $(-z)$. That the product is also uniformly convergent in any closed region in the z-plane is easily shown by employing the associated series of logarithms.

We shall see in Section 15 that the function $\Gamma(z)$ defined by (1) is identical with that defined by Euler's integral; that is,

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1}\, dt, \qquad \operatorname{Re}(z) > 0.$$

The right member of (1) is analytic for all finite z. Its only zeros are simple ones at $z = 0$ and at each negative integer. We may therefore conclude that

(a) $\Gamma(z)$ is analytic except at $z =$ nonpositive integers and $z = \infty$;

(b) $\Gamma(z)$ has a simple pole at $z =$ each nonpositive integer, $z = 0, -1, -2, -3, \cdots$;

(c) $\Gamma(z)$ has an essential singularity at $z = \infty$, a point of condensation of poles;

(d) $\Gamma(z)$ is never zero [because $1/\Gamma(z)$ has no poles].

9. A series for $\Gamma'(z)/\Gamma(z)$. By taking logarithms of each member of equation (1) of Section 8, we obtain

$$\log \Gamma(z) = -\operatorname{Log} z - \gamma z - \sum_{n=1}^{\infty}\left[\operatorname{Log}\left(1 + \frac{z}{n}\right) - \frac{z}{n}\right].$$

Term-by-term differentiation of the members of the foregoing equation yields

$$\frac{\Gamma'(z)}{\Gamma(z)} = -\frac{1}{z} - \gamma - \sum_{n=1}^{\infty}\left(\frac{1}{z+n} - \frac{1}{n}\right),$$

or

$$\frac{\Gamma'(z)}{\Gamma(z)} = -\gamma - \frac{1}{z} + \sum_{n=1}^{\infty}\frac{z}{n(z+n)}, \tag{1}$$

the series on the right being absolutely and uniformly convergent in any closed region excluding the singular points of $\Gamma(z)$, a result easily deduced by using the Weierstrass M-test and the convergence of $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

10. Evaluation of $\Gamma(1)$ and $\Gamma'(1)$. In the Weierstrass definition of $\Gamma(z)$ put $z = 1$ to get

$$\begin{aligned}
\frac{1}{\Gamma(1)} &= e^{\gamma}\prod_{n=1}^{\infty}\left[\left(1 + \frac{1}{n}\right)\exp\left(-\frac{1}{n}\right)\right] \\
&= e^{\gamma}\lim_{n\to\infty}\prod_{k=1}^{n}\left[\frac{k+1}{k}\exp\left(-\frac{1}{k}\right)\right] \\
&= e^{\gamma}\lim_{n\to\infty}(n+1)\exp(-H_n) \\
&= e^{\gamma}\lim_{n\to\infty}(n+1)\exp(-\gamma - \operatorname{Log} n - \epsilon_n),
\end{aligned}$$

in which $\epsilon_n \to 0$ as $n \to \infty$. It follows that

$$\frac{1}{\Gamma(1)} = e^{\gamma}\lim_{n\to\infty}\frac{n+1}{n}e^{-\gamma} = 1,$$

so that $\Gamma(1) = 1$.

We know from the series for $\Gamma'(z)/\Gamma(z)$ obtained in Section 9 that

$$\frac{\Gamma'(1)}{\Gamma(1)} = -\gamma - 1 + \sum_{n=1}^{\infty}\frac{1}{n(n+1)},$$

so that

$$\Gamma'(1) = -\gamma - 1 + \sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+1}\right)$$

$$= -\gamma - 1 + \operatorname*{Lim}_{n\to\infty}\left(1 - \frac{1}{n+1}\right),$$

since the series involved telescopes. Thus we find that $\Gamma'(1) = -\gamma$.

11. The Euler product for $\Gamma(z)$. From the Weierstrass product definition of $\Gamma(z)$ we obtain

$$z\Gamma(z) = \frac{\exp(-\gamma z)}{\prod_{n=1}^{\infty}\left[\left(1 + \frac{z}{n}\right)\exp\left(-\frac{z}{n}\right)\right]},$$

so that

$$(1) \qquad z\Gamma(z) = \exp(-\gamma z) \operatorname*{Lim}_{n\to\infty}\prod_{k=1}^{n}\left[\left(1 + \frac{z}{k}\right)^{-1}\exp\left(\frac{z}{k}\right)\right].$$

But

$$\gamma = \operatorname*{Lim}_{n\to\infty}(H_n - \operatorname{Log} n) = \operatorname*{Lim}_{n\to\infty}[H_n - \operatorname{Log}(n+1)]$$

$$= \operatorname*{Lim}_{n\to\infty}\left[H_n - \sum_{k=1}^{n}\operatorname{Log}\frac{k+1}{k}\right].$$

Hence

$$\exp(-\gamma z) = \operatorname*{Lim}_{n\to\infty}\exp\left[-zH_n + z\sum_{k=1}^{n}\operatorname{Log}\frac{k+1}{k}\right]$$

$$= \operatorname*{Lim}_{n\to\infty}\prod_{k=1}^{n}\left[\left(\frac{k+1}{k}\right)^{z}\exp\left(-\frac{z}{k}\right)\right].$$

Therefore (1) can be written

$$z\Gamma(z) = \operatorname*{Lim}_{n\to\infty}\prod_{k=1}^{n}\left[\left(1 + \frac{1}{k}\right)^{z}\exp\left(-\frac{z}{k}\right)\left(1 + \frac{z}{k}\right)^{-1}\exp\left(\frac{z}{k}\right)\right],$$

from which it follows that

$$(2) \qquad \Gamma(z) = \frac{1}{z}\prod_{n=1}^{\infty}\left[\left(1 + \frac{1}{n}\right)^{z}\left(1 + \frac{z}{n}\right)^{-1}\right],$$

which is Euler's product for $\Gamma(z)$. Note that for real $x > 0$, $\Gamma(x) > 0$.

Refer now to Example (*b*), page 5, to conclude that

$$(3) \qquad \Gamma(z) = \operatorname*{Lim}_{n\to\infty}\frac{(n-1)!\, n^{z}}{z(z+1)(z+2)\cdots(z+n-1)}.$$

It will be of value to us later to note that, since

$$\lim_{n\to\infty} \frac{(n+1)^z}{n^z} = 1,$$

we can equally well write the result (3) in the form

$$\Gamma(z) = \lim_{n\to\infty} \frac{n!\, n^z}{z(z+1)(z+2)\cdots(z+n)}. \tag{4}$$

12. The difference equation $\Gamma(z+1) = z\Gamma(z)$. From Euler's product for $\Gamma(z)$ we obtain

$$\frac{\Gamma(z+1)}{\Gamma(z)} = \frac{z}{z+1} \frac{\prod_{n=1}^{\infty}\left[\left(1+\frac{1}{n}\right)^{z+1}\left(1+\frac{z+1}{n}\right)^{-1}\right]}{\prod_{n=1}^{\infty}\left[\left(1+\frac{1}{n}\right)^{z}\left(1+\frac{z}{n}\right)^{-1}\right]}$$

$$= \frac{z}{z+1} \prod_{n=1}^{\infty}\left[\left(1+\frac{1}{n}\right)\left(1+\frac{z}{n}\right)\left(1+\frac{z+1}{n}\right)^{-1}\right]$$

$$= \frac{z}{z+1} \lim_{n\to\infty} \prod_{k=1}^{n}\left(\frac{k+1}{k} \cdot \frac{k+z}{k+z+1}\right)$$

$$= \frac{z}{z+1} \lim_{n\to\infty} \frac{n+1}{1} \cdot \frac{1+z}{n+z+1} = z.$$

Therefore

$$\Gamma(z+1) = z\Gamma(z) \tag{1}$$

for all finite z except for the poles of $\Gamma(z)$.

If $z = m$, a positive integer, iterated use of the equation (1) yields $\Gamma(m+1) = m!$. Since $\Gamma(1) = 1$, this is another of the many reasons we define $0! = 1$.

13. The order symbols o and O. Let R be a region in the complex z-plane. If and only if

$$\lim_{z\to c \text{ in } R} \frac{f(z)}{g(z)} = 0,$$

we write

$$f(z) = o[g(z)], \qquad \text{as } z \to c \text{ in } R.$$

If and only if $\left|\dfrac{f(z)}{g(z)}\right|$ is bounded as $z \to c$ in R, we write

$$f(z) = O[g(z)], \qquad \text{as } z \to c \text{ in } R.$$

It is common practice to omit the qualifying expressions such as "$z \to c$ in R" whenever the surrounding text is deemed to make such qualification unnecessary to a trained reader. The point $z = c$ may on occasion be the point at infinity. Also, the symbols o and O are sometimes used when the variable z is real, the approach is along the real axis, and even when z takes on only integral values.

EXAMPLE (*a*): Since $\lim\limits_{z \to 0} \dfrac{\sin^2 z}{z} = 0$, we may write

$$\sin^2 z = o(z), \qquad \text{as } z \to 0,$$

noting that in this instance the manner of approach is immaterial.

EXAMPLE (*b*): For real x, $|\cos x| \leqq 1$, from which it is easy to conclude that

$$\cos x - 4x = O(x), \qquad \text{as } x \to \infty,\ x \text{ real}.$$

EXAMPLE (*c*): In Chapter 3 we shall show that if

$$s_n(x) = \sum_{k=0}^{n} k!\, x^k,$$

$$\left| \int_0^\infty \frac{e^{-t}\, dt}{1 - xt} - s_n(x) \right| \leqq (n+1)!\, |x|^{n+1}, \qquad \text{for } \operatorname{Re}(x) \leqq 0.$$

From the preceding inequality we may conclude that, for fixed n,

$$\int_0^\infty \frac{e^{-t}\, dt}{1 - xt} - s_n(x) = o(x^n), \qquad \text{as } x \to 0 \text{ in } \operatorname{Re}(x) \leqq 0.$$

14. Evaluation of certain infinite products. The Weierstrass infinite product for $\Gamma(z)$ yields a simple evaluation of all infinite products whose factors are rational functions of the index n. The most general such product must take the form

$$(1) \qquad P = \prod_{n=1}^{\infty} \frac{(n + a_1)(n + a_2) \cdots (n + a_s)}{(n + b_1)(n + b_2) \cdots (n + b_s)}$$

$$= \prod_{n=1}^{\infty} \frac{\prod\limits_{k=1}^{s} \left(1 + \dfrac{a_k}{n}\right)}{\prod\limits_{k=1}^{s} \left(1 + \dfrac{b_k}{n}\right)}$$

because convergence requires that the nth factor approach unity as $n \to \infty$, which in turn forces the numerator and denominator polynomials to be of the same degree and to have equal leading coefficients. Now the nth factor in the right member of (1) may be put in the form

$$1 + \frac{1}{n}\left(\sum_{k=1}^{s} a_k - \sum_{k=1}^{s} b_k\right) + O\left(\frac{1}{n^2}\right),$$

so that we must also insist, to obtain convergence, that

$$\sum_{k=1}^{s} a_k = \sum_{k=1}^{s} b_k. \tag{2}$$

If (2) is not satisfied, the product in (1) diverges; we get absolute convergence or no convergence.

We now have an absolutely convergent product (1) in which the a's and b's satisfy the condition (2).

Since

$$\exp\left(\frac{1}{n}\sum_{k=1}^{s} a_k\right) = \exp\left(\frac{1}{n}\sum_{k=1}^{s} b_k\right),$$

we may, without changing the value of the product (1), insert the appropriate exponential factors to write

$$P = \prod_{n=1}^{\infty} \frac{\prod_{k=1}^{s}\left[\left(1 + \frac{a_k}{n}\right)\exp\left(-\frac{a_k}{n}\right)\right]}{\prod_{k=1}^{s}\left[\left(1 + \frac{b_k}{n}\right)\exp\left(-\frac{b_k}{n}\right)\right]}. \tag{3}$$

The Weierstrass product, page 9, for $1/\Gamma(z)$ yields

$$\prod_{n=1}^{\infty}\left[\left(1 + \frac{z}{n}\right)\exp\left(-\frac{z}{n}\right)\right] = \frac{1}{z\exp(\gamma z)\,\Gamma(z)} = \frac{1}{\Gamma(z+1)\exp(\gamma z)}.$$

Thus we obtain from (3) the result

$$\begin{aligned} P &= \prod_{k=1}^{s} \frac{\Gamma(1 + b_k)\exp(\gamma b_k)}{\Gamma(1 + a_k)\exp(\gamma a_k)} \\ &= \exp\left[\gamma\left(\sum_{k=1}^{s} b_k - \sum_{k=1}^{s} a_k\right)\right]\prod_{k=1}^{s} \frac{\Gamma(1 + b_k)}{\Gamma(1 + a_k)} \\ &= \prod_{k=1}^{s} \frac{\Gamma(1 + b_k)}{\Gamma(1 + a_k)}. \end{aligned}$$

THEOREM 5. *If* $\sum_{k=1}^{s} a_k = \sum_{k=1}^{s} b_k$, *and if no* a_k *or* b_k *is a negative integer,*

$$\prod_{n=1}^{\infty} \frac{(n+a_1)(n+a_2)\cdots(n+a_s)}{(n+b_1)(n+b_2)\cdots(n+b_s)} = \frac{\Gamma(1+b_1)\Gamma(1+b_2)\cdots\Gamma(1+b_s)}{\Gamma(1+a_1)\Gamma(1+a_2)\cdots\Gamma(1+a_s)}.$$

If one or more of the a_k is a negative integer, the product on the left is zero, which agrees with the existence of one or more poles of the denominator factors on the right.

EXAMPLE: Evaluate

$$\prod_{n=1}^{\infty} \frac{(c-a+n-1)(c-b+n-1)}{(c+n-1)(c-a-b+n-1)}.$$

Since $(c-a-1)+(c-b-1)=(c-1)+(c-a-b-1)$, we may employ Theorem 5 if no one of the quantities c, $c-a$, $c-b$, $c-a-b$ is either zero or a negative integer. With those restrictions we obtain

$$\prod_{n=1}^{\infty} \frac{(c-a+n-1)(c-b+n-1)}{(c+n-1)(c-a-b+n-1)} = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}. \tag{4}$$

15. Euler's integral for $\Gamma(z)$. Elementary treatments of the Gamma function are usually based on an integral definition. Theorem 6 connects the function $\Gamma(z)$ defined by the Weierstrass product with that defined by Euler's integral.

THEOREM 6. *If* $\text{Re}(z) > 0$,

$$\Gamma(z) = \int_0^\infty e^{-t}t^{z-1}\,dt. \tag{1}$$

We shall establish four lemmas intended to break the proof of Theorem 6 into simple parts.

Lemma 1. If $0 \leqq \alpha < 1$, $1+\alpha \leqq \exp(\alpha) \leqq (1-\alpha)^{-1}$.

Proof: Compare the three series

$$1+\alpha = 1+\alpha, \qquad \exp(\alpha) = 1+\alpha+\sum_{n=2}^{\infty}\frac{\alpha^n}{n!}, \qquad (1-\alpha)^{-1} = 1+\alpha+\sum_{n=2}^{\infty}\alpha^n.$$

Lemma 2. If $0 \leqq \alpha < 1$, $(1-\alpha)^n \geqq 1-n\alpha$, *for n a positive integer.*

Proof: For $n=1$, $1-\alpha = 1-1\cdot\alpha$, as desired. Next assume that

$$(1-\alpha)^k \geqq 1-k\alpha,$$

and multiply each member by $(1-\alpha)$ to obtain

$$(1-\alpha)^{k+1} \geqq (1-\alpha)(1-k\alpha) = 1-(k+1)\alpha+k\alpha^2,$$

so that

$$(1 - \alpha)^{k+1} \geqq 1 - (k + 1)\alpha.$$

Lemma 2 now follows by induction.

Lemma 3. If $0 \leqq t < n$, *n a positive integer,*

$$0 \leqq e^{-t} - \left(1 - \frac{t}{n}\right)^n \leqq \frac{t^2 e^{-t}}{n}.$$

Proof: Use $\alpha = t/n$ in Lemma 1 to get

$$1 + \frac{t}{n} \leqq \exp\left(\frac{t}{n}\right) \leqq \left(1 - \frac{t}{n}\right)^{-1}$$

from which

$$\left(1 + \frac{t}{n}\right)^n \leqq e^t \leqq \left(1 - \frac{t}{n}\right)^{-n} \tag{2}$$

or

$$\left(1 + \frac{t}{n}\right)^{-n} \geqq e^{-t} \geqq \left(1 - \frac{t}{n}\right)^n,$$

so that

$$e^{-t} - \left(1 - \frac{t}{n}\right)^n \geqq 0. \tag{3}$$

But also

$$e^{-t} - \left(1 - \frac{t}{n}\right)^n = e^{-t}\left[1 - e^t\left(1 - \frac{t}{n}\right)^n\right]$$

and, by (2), $e^t \geqq \left(1 + \frac{t}{n}\right)^n$. Hence

$$e^{-t} - \left(1 - \frac{t}{n}\right)^n \leqq e^{-t}\left[1 - \left(1 - \frac{t^2}{n^2}\right)^n\right]. \tag{4}$$

Now Lemma 2 with $\alpha = t^2/n^2$ yields

$$\left(1 - \frac{t^2}{n^2}\right)^n \geqq 1 - \frac{t^2}{n}$$

which may be used in (4) to obtain

$$e^{-t} - \left(1 - \frac{t}{n}\right)^n \leqq e^{-t}\left[1 - 1 + \frac{t^2}{n}\right] = \frac{t^2 e^{-t}}{n}. \tag{5}$$

The inequalities (3) and (5) constitute the result stated in Lemma 3.

Lemma 4. If n is integral and $\text{Re}(z) > 0$,

$$\Gamma(z) = \lim_{n\to\infty} \int_0^n \left(1 - \frac{t}{n}\right)^n t^{z-1}\, dt. \tag{6}$$

Proof: In the integral on the right in (6) put $t = n\beta$ and thus obtain

$$\int_0^n \left(1 - \frac{t}{n}\right)^n t^{z-1}\, dt = n^z \int_0^1 (1 - \beta)^n \beta^{z-1}\, d\beta. \tag{7}$$

An integration by parts gives us the reduction formula

$$\int_0^1 (1 - \beta)^n \beta^{z-1}\, d\beta = \frac{n}{z}\int_0^1 (1 - \beta)^{n-1}\beta^z\, d\beta,$$

iteration of which yields

$$\begin{aligned}\int_0^1 (1 - \beta)^n \beta^{z-1}\, d\beta &= \frac{n(n-1)(n-2)\cdots 1}{z(z+1)(z+2)\cdots(z+n-1)}\int_0^1 \beta^{z+n-1}\, d\beta \\ &= \frac{n!}{z(z+1)(z+2)\cdots(z+n)}.\end{aligned}$$

Now (7) becomes

$$\int_0^n \left(1 - \frac{t}{n}\right)^n t^{z-1}\, dt = \frac{n!n^z}{z(z+1)(z+2)\cdots(z+n)}$$

so that

$$\lim_{n\to\infty}\int_0^n \left(1 - \frac{t}{n}\right)^n t^{z-1}\, dt = \lim_{n\to\infty}\frac{n!n^z}{z(z+1)\cdots(z+n)} = \Gamma(z)$$

by equation (4), page 12.

We are now in a position to prove Theorem 6, which states that

$$\Gamma(z) = \int_0^\infty e^{-t}t^{z-1}\, dt, \qquad \text{Re}(z) > 0. \tag{8}$$

The integral on the right in (8) converges for $\text{Re}(z) > 0$. With the aid of Lemma 4, write

$$\begin{aligned}\int_0^\infty e^{-t}t^{z-1}\, dt - \Gamma(z) &= \lim_{n\to\infty}\left[\int_0^\infty e^{-t}t^{z-1}\, dt - \int_0^n \left(1 - \frac{t}{n}\right)^n t^{z-1}\, dt\right] \\ &= \lim_{n\to\infty}\left[\int_0^n \left\{e^{-t} - \left(1 - \frac{t}{n}\right)^n\right\} t^{z-1}\, dt + \int_n^\infty e^{-t}t^{z-1}\, dt\right].\end{aligned}$$

From the convergence of the integral on the right in (8) it follows that

$$\lim_{n\to\infty} \int_n^\infty e^{-t}t^{z-1}\,dt = 0.$$

Hence

$$(9)\quad \int_0^\infty e^{-t}t^{z-1}\,dt - \Gamma(z) = \lim_{n\to\infty}\int_0^n \left[e^{-t} - \left(1 - \frac{t}{n}\right)^n\right] t^{z-1}\,dt.$$

But, by Lemma 3 and the fact that $|t^z| = t^{\mathrm{Re}(z)}$,

$$\left|\int_0^n \left[e^{-t} - \left(1 - \frac{t}{n}\right)^n\right] t^{z-1}\,dt\right| \leqq \int_0^n \frac{t^2e^{-t}}{n} \cdot t^{\mathrm{Re}(z)-1}\,dt$$

$$\leqq \frac{1}{n}\int_0^n e^{-t}t^{\mathrm{Re}(z)+1}\,dt.$$

Now $\int_0^\infty e^{-t}t^{\mathrm{Re}(z)+1}\,dt$ converges, so $\int_0^n e^{-t}t^{\mathrm{Re}(z)+1}\,dt$ is bounded. Therefore

$$\lim_{n\to\infty}\int_0^n \left[e^{-t} - \left(1 - \frac{t}{n}\right)^n\right] t^{z-1}\,dt = 0,$$

and we may conclude from equation (9) that (8) is valid.

16. The Beta function. We define the Beta function $B(p, q)$ by

$$(1)\quad B(p,q) = \int_0^1 t^{p-1}(1-t)^{q-1}\,dt, \qquad \mathrm{Re}(p) > 0,\ \mathrm{Re}(q) > 0.$$

Another useful form for this function can be obtained by putting $t = \sin^2\varphi$, thus arriving at

$$(2)\quad B(p,q) = 2\int_0^{\frac{1}{2}\pi} \sin^{2p-1}\varphi\,\cos^{2q-1}\varphi\,d\varphi, \qquad \mathrm{Re}(p) > 0,\ \mathrm{Re}(q) > 0.$$

The Beta function is intimately related to the Gamma function. Consider the product

$$(3)\qquad \Gamma(p)\,\Gamma(q) = \int_0^\infty e^{-t}t^{p-1}\,dt \cdot \int_0^\infty e^{-v}v^{q-1}\,dv.$$

In (3) use $t = x^2$ and $v = y^2$ to obtain

$$\Gamma(p)\,\Gamma(q) = 4\int_0^\infty \exp(-x^2)x^{2p-1}\,dx \cdot \int_0^\infty \exp(-y^2)y^{2q-1}\,dy,$$

$$\Gamma(p)\Gamma(q) = 4\int_0^\infty\int_0^\infty \exp(-x^2 - y^2)x^{2p-1}y^{2q-1}\,dx\,dy.$$

Next turn to polar coordinates for the iterated integration over the first quadrant in the xy-plane. Using $x = r\cos\theta$, $y = r\sin\theta$, we may write

$$\Gamma(p)\Gamma(q) = 4\int_0^\infty\int_0^{\frac{1}{2}\pi} \exp(-r^2)r^{2p+2q-2}\cos^{2p-1}\theta\,\sin^{2q-1}\theta\; r\,d\theta\,dr$$

$$= 2\int_0^\infty \exp(-r^2)r^{2p+2q-1}dr \cdot 2\int_0^{\frac{1}{2}\pi}\cos^{2p-1}\theta\,\sin^{2q-1}\theta\,d\theta.$$

Now put $r = \sqrt{t}$ and $\theta = \frac{1}{2}\pi - \varphi$ to obtain

$$\Gamma(p)\Gamma(q) = \int_0^\infty e^{-t}t^{p+q-1}\,dt \cdot 2\int_0^{\frac{1}{2}\pi}\sin^{2p-1}\varphi\,\cos^{2q-1}\varphi\,d\varphi,$$

from which it follows that

$$\Gamma(p)\Gamma(q) = \Gamma(p+q)B(p,q).$$

THEOREM 7. *If* $\text{Re}(p) > 0$ *and* $\text{Re}(q) > 0$,

$$B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}. \tag{4}$$

By (4), $B(p,q) = B(q,p)$, a result just as easily obtained directly from (1) or (2).

Equations (2) and (4) yield a generalization of Wallis' formula of elementary calculus. In (2) put $2p - 1 = m$, $2q - 1 = n$, and use (4) to write

$$\int_0^{\frac{1}{2}\pi}\sin^m\varphi\,\cos^n\varphi\,d\varphi = \frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{2\Gamma\left(\frac{m+n+2}{2}\right)}, \tag{5}$$

valid for $\text{Re}(m) > -1$, $\text{Re}(n) > -1$.

17. The value of $\Gamma(z)\Gamma(1-z)$. The important relation (4) of Section 16 suggests that the product of two Gamma functions whose arguments have the sum unity may possess some pleasant property, since if $p + q = 1$, $\Gamma(p+q) = \Gamma(1) = 1$.

If z is such that $0 < \text{Re}(z) < 1$, both z and $(1 - z)$ have real part positive, and we may use (4) of Section 16 to write

$$\Gamma(z)\Gamma(1-z) = B(z, 1-z) = \int_0^1 t^{z-1}(1-t)^{-z}\,dt$$
$$= \int_0^1 \left(\frac{t}{1-t}\right)^z \frac{dt}{t}.$$

Now put $t/(1-t) = y$ to arrive at

(1) $$\Gamma(z)\Gamma(1-z) = \int_0^\infty \frac{y^{z-1}\,dy}{1+y}, \qquad 0 < \mathrm{Re}(z) < 1.$$

The integral on the right in (1) can be evaluated with the aid of contour integration in an α-plane where $\mathrm{Re}(\alpha) = y$. The contour

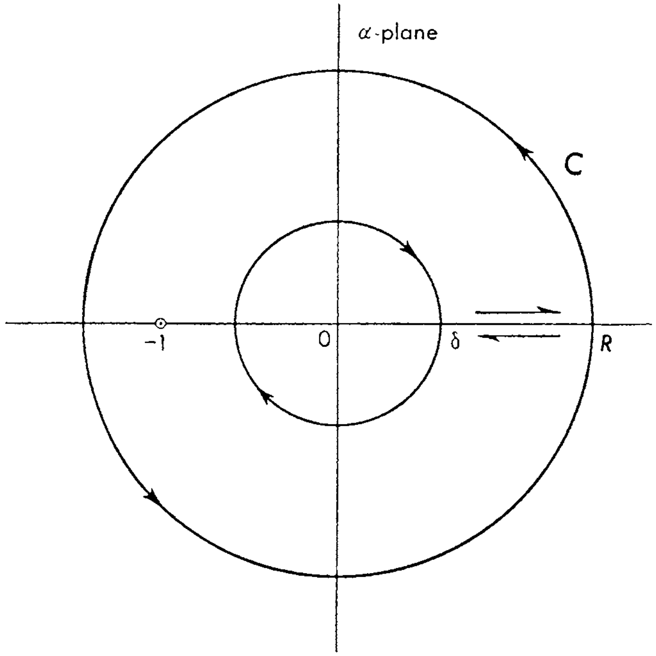

Figure 1

C in Figure 1 encircles a single simple pole $\alpha = -1$ of the integrand in

$$\int_C \frac{\alpha^{z-1}\,d\alpha}{1+\alpha}$$

so that the residue theory at once yields

(2) $$\int_C \frac{\alpha^{z-1}\,d\alpha}{1+\alpha} = 2\pi i(-1)^{z-1} = 2\pi i \exp[\pi i(z-1)].$$

The integral on the left in (2) may be split into four parts, as indicated in the figure. In detail we use

(a) $\alpha = Re^{i\theta}$, θ from 0 to 2π;
(b) $\alpha = ye^{2\pi i}$, y from R to δ;
(c) $\alpha = \delta e^{i\theta}$, θ from 2π to 0;
(d) $\alpha = ye^{0i}$, y from δ to R.

Thus (2) can be written in the form

$$(3)\quad \int_0^{2\pi}\frac{iR^z\exp(iz\theta)\,d\theta}{1+R\exp(i\theta)}+\int_R^{\delta}\frac{y^{z-1}\exp(2\pi iz)\,dy}{1+y}$$
$$+\int_{2\pi}^{0}\frac{i\delta^z\exp(iz\theta)\,d\theta}{1+\delta\exp(i\theta)}+\int_\delta^R\frac{y^{z-1}\exp(0iz)\,dy}{1+y}=2\pi i\exp[\pi i(z-1)].$$

Now let $\delta\to 0$ and $R\to\infty$ and use $0<\text{Re}(z)<1$ to conclude that the first and third integrals on the left in (3) approach zero. Then the limiting form of (3) is

$$\exp(2\pi iz)\int_\infty^0\frac{y^{z-1}\,dy}{1+y}+\int_0^\infty\frac{y^{z-1}\,dy}{1+y}=-\,2\pi i\exp(\pi iz),$$

from which we obtain

$$\int_0^\infty\frac{y^{z-1}\,dy}{1+y}=\frac{2\pi i\exp(\pi iz)}{\exp(2\pi iz)-1}=\frac{2\pi i}{\exp(\pi iz)-\exp(-\pi iz)}.$$

We have thus shown that, for $0<\text{Re}(z)<1$,

$$(4)\qquad \Gamma(z)\Gamma(1-z)=\int_0^\infty\frac{y^{z-1}\,dy}{1+y}=\frac{\pi}{\sin\pi z}.$$

But each member of (4) is analytic for all nonintegral z, and the theory of analytic continuation permits us to come to the useful conclusion of Theorem 8.

THEOREM 8. *If z is nonintegral,*

$$\Gamma(z)\Gamma(1-z)=\frac{\pi}{\sin\pi z}.$$

Our first, and extremely simple, application of Theorem 8 is the evaluation of $\Gamma(\frac{1}{2})$. Use $z=\frac{1}{2}$ to get

$$\Gamma(\tfrac{1}{2})\Gamma(\tfrac{1}{2})=\pi,$$

which, since $\Gamma(\frac{1}{2})>0$, yields

$$(5)\qquad \Gamma(\tfrac{1}{2})=\sqrt{\pi}.$$

18. The factorial function. Throughout this book we make frequent use of the common notation

$$(1) \qquad \begin{aligned}(\alpha)_n &= \prod_{k=1}^{n}(\alpha + k - 1) \\ &= \alpha(\alpha+1)(\alpha+2)\cdots(\alpha+n-1), \qquad n \geqq 1, \\ (\alpha)_0 &= 1, \qquad \alpha \neq 0.\end{aligned}$$

The function $(\alpha)_n$ is called the *factorial function.* It is an immediate generalization of the elementary factorial, since $n! = (1)_n$.

In manipulations with $(\alpha)_n$ it is important to keep in mind that $(\alpha)_n$ is a product of n factors, starting with α and with each factor larger by unity than the preceding factor.

Lemma 5. $(\alpha)_{2n} = 2^{2n}\left(\frac{\alpha}{2}\right)_n\left(\frac{\alpha+1}{2}\right)_n.$

Proof: In the product

$$(\alpha)_{2n} = \alpha(\alpha+1)(\alpha+2)(\alpha+3)\cdots(\alpha+2n-1),$$

group alternate factors, factor 2 out of each factor on the right, and thus conclude that

$$(\alpha)_{2n} = 2^{2n}\left(\frac{\alpha}{2}\right)_n\left(\frac{\alpha+1}{2}\right)_n.$$

Lemma 6. If k is a positive integer and n a non-negative integer,

$$(2) \qquad (\alpha)_{kn} = k^{nk}\left(\frac{\alpha}{k}\right)_n\left(\frac{\alpha+1}{k}\right)_n\cdots\left(\frac{\alpha+k-1}{k}\right)_n.$$

The proof of Lemma 6 is like that of Lemma 5 except that the factors of $(\alpha)_{kn}$ are grouped into k sets of n factors each, and then k is factored out of each factor to obtain (2).

Other properties of $(\alpha)_n$ will be introduced when needed, particularly in series manipulations involving functions of hypergeometric character. At present we are concerned only with the relation of $(\alpha)_n$ to the Gamma function.

We know that $\Gamma(1+z) = z\Gamma(z)$. It follows that, for n a positive integer,

$$\begin{aligned}\Gamma(\alpha+n) &= (\alpha+n-1)\Gamma(\alpha+n-1) \\ &= (\alpha+n-1)(\alpha+n-2)\Gamma(\alpha+n-2) \\ &= \cdots \\ &= (\alpha+n-1)(\alpha+n-2)\cdots\alpha\Gamma(\alpha).\end{aligned}$$

Theorem 9. *If α is neither zero nor a negative integer,*

$$(3) \qquad (\alpha)_n = \frac{\Gamma(\alpha + n)}{\Gamma(\alpha)}.$$

We have already had, in equation (3), page 11, the result

$$\Gamma(z) = \lim_{n\to\infty} \frac{(n-1)!\, n^z}{z(z+1)(z+2)\cdots(z+n-1)},$$

which can now be written in the form

$$(4) \qquad \Gamma(z) = \lim_{n\to\infty} \frac{(n-1)!\, n^z}{(z)_n}.$$

Equation (4), reinterpreted in the light of Theorem 9, yields a result of value to us in the subsequent two sections.

Lemma 7. If n is integral and z is not a negative integer,

$$(5) \qquad \lim_{n\to\infty} \frac{(n-1)!\, n^z}{\Gamma(z+n)} = 1.$$

19. Legendre's duplication formula. Let us turn to Lemma 5, page 22, and use $\alpha = 2z$. We thus obtain

$$(2z)_{2n} = 2^{2n}(z)_n(z+\tfrac{1}{2})_n.$$

In view of Theorem 9 we may rewrite the above as

$$\frac{\Gamma(2z+2n)}{\Gamma(2z)} = \frac{2^{2n}\Gamma(z+n)\Gamma(z+\frac{1}{2}+n)}{\Gamma(z)\Gamma(z+\frac{1}{2})},$$

or

$$\frac{\Gamma(2z)}{\Gamma(z)\Gamma(z+\frac{1}{2})} = \frac{\Gamma(2z+2n)}{2^{2n}\Gamma(z+n)\Gamma(z+\frac{1}{2}+n)},$$

which, since the left member is independent of n, also implies

$$(1) \qquad \frac{\Gamma(2z)}{\Gamma(z)\Gamma(z+\frac{1}{2})} = \lim_{n\to\infty} \frac{\Gamma(2z+2n)}{2^{2n}\Gamma(z+n)\Gamma(z+\frac{1}{2}+n)}.$$

We next insert in the right member of (1) the appropriate factors to permit us to make use of the result in Lemma 7. From (1) we write

$$\frac{\Gamma(2z)}{\Gamma(z)\Gamma(z+\frac{1}{2})}$$

$$= \lim_{n\to\infty} \frac{\Gamma(2z+2n)}{(2n-1)!(2n)^{2z}} \cdot \frac{(n-1)!n^z}{\Gamma(z+n)} \cdot \frac{(n-1)!n^{z+\frac{1}{2}}}{\Gamma(z+\frac{1}{2}+n)} \cdot \frac{2^{2z}(2n-1)!}{2^{2n}n^{\frac{1}{2}}[(n-1)!]^2},$$

which, because of Lemma 7, becomes

$$\frac{\Gamma(2z)}{\Gamma(z)\,\Gamma(z+\frac{1}{2})} = \lim_{n\to\infty} \frac{2^{2z}(2n-1)!}{2^{2n}n^{\frac{1}{2}}[(n-1)!]^2}.$$

It follows that

$$\frac{\Gamma(2z)}{2^{2z}\Gamma(z)\,\Gamma(z+\frac{1}{2})} = c,$$

in which c is independent of z. To evaluate c we use $z = \frac{1}{2}$ and find that

$$c = \frac{\Gamma(1)}{2\,\Gamma(\frac{1}{2})\,\Gamma(1)} = \frac{1}{2\sqrt{\pi}}.$$

We have thus discovered an expression for $\Gamma(2z)$ in terms of $\Gamma(z)$ and $\Gamma(z+\frac{1}{2})$. It is *Legendre's duplication formula,*

(2) $$\sqrt{\pi}\,\Gamma(2z) = 2^{2z-1}\Gamma(z)\,\Gamma(z+\tfrac{1}{2}).$$

20. Gauss' multiplication theorem. Following the technique used to discover and prove Legendre's duplication formula, we readily move on to a theorem of Gauss involving the product of k Gamma functions.

Lemma 6, page 22, can be written

$$(\alpha)_{nk} = k^{nk}\prod_{s=1}^{k}\left(\frac{\alpha+s-1}{k}\right)_n$$

and by Theorem 9, page 23, $(\alpha)_n = \Gamma(\alpha+n)/\Gamma(\alpha)$. We thus obtain

(1) $$\frac{\Gamma(\alpha+nk)}{\Gamma(\alpha)} = k^{nk}\prod_{s=1}^{k}\frac{\Gamma\left(\frac{\alpha+s-1}{k}+n\right)}{\Gamma\left(\frac{\alpha+s-1}{k}\right)}.$$

In (1) put $\alpha = kz$ and rearrange the members of the equation to arrive at

(2) $$\frac{\Gamma(kz)}{\prod_{s=1}^{k}\Gamma\left(z+\frac{s-1}{k}\right)} = \frac{\Gamma(kz+kn)}{k^{nk}\prod_{s=1}^{k}\Gamma\left(z+n+\frac{s-1}{k}\right)}$$

$$= \lim_{n\to\infty}\frac{\Gamma(kz+kn)}{k^{nk}\prod_{s=1}^{k}\Gamma\left(z+n+\frac{s-1}{k}\right)}.$$

By Lemma 7, page 23, we know that

$$\lim_{n\to\infty} \frac{(n-1)!\, n^{z+\frac{s-1}{k}}}{\Gamma\left(z+n+\dfrac{s-1}{k}\right)} = 1,$$

and

$$\lim_{n\to\infty} \frac{(nk-1)!\,(nk)^{kz}}{\Gamma(kz+kn)} = 1.$$

We now use the foregoing two limits in conjunction with (2) to obtain

$$\frac{\Gamma(kz)}{\prod_{s=1}^{k} \Gamma\left(z+\dfrac{s-1}{k}\right)}$$

$$= \lim_{n\to\infty} \frac{\Gamma(kz+kn)}{(nk-1)!(nk)^{kz}} \cdot \frac{(nk)^{kz}(nk-1)!}{k^{nk}} \prod_{s=1}^{k} \frac{(n-1)!n^{z+\frac{s-1}{k}}}{\Gamma\left(z+n+\dfrac{s-1}{k}\right)} \cdot \frac{1}{(n-1)!n^{z+\frac{s-1}{k}}}$$

$$= \lim_{n\to\infty} \frac{(nk)^{kz}(nk-1)!}{k^{nk}} \prod_{s=1}^{k} \frac{1}{(n-1)!\, n^{z+\frac{s-1}{k}}}$$

$$= \lim_{n\to\infty} \frac{(nk)^{kz}(nk-1)!}{k^{nk}[(n-1)!]^k n^{kz+\frac{1}{2}(k-1)}}.$$

Therefore,

$$\frac{\Gamma(kz)}{k^{kz}\prod_{s=1}^{k} \Gamma\left(z+\dfrac{s-1}{k}\right)} = c,$$

in which c is independent of z. To determine c, we put $z = 1/k$, use the fact that $\Gamma(1) = 1$, and obtain

$$\frac{1}{kc} = \prod_{s=1}^{k-1} \Gamma\left(\frac{s}{k}\right) = \prod_{s=1}^{k-1} \Gamma\left(\frac{k-s}{k}\right).$$

Then

$$\frac{1}{k^2c^2} = \prod_{s=1}^{k-1} \Gamma\left(\frac{s}{k}\right)\Gamma\left(\frac{k-s}{k}\right) = \prod_{s=1}^{k-1} \frac{\pi}{\sin\dfrac{\pi s}{k}},$$

or

(3) $$k^2c^2\pi^{k-1} = \prod_{s=1}^{k-1} \sin\frac{\pi s}{k}.$$

We can obtain c once we know the value of the product on the right in (3).

Lemma 8. If $k \geqq 2, \prod_{s=1}^{k-1} \sin \frac{\pi s}{k} = \frac{k}{2^{k-1}}.$

Proof: Let $\alpha = \exp(2\pi i/k)$ be a primitive kth root of unity. Then for all x,

$$x^k - 1 = (x - 1) \prod_{s=1}^{k-1} (x - \alpha^s),$$

from which, by differentiation of both members, we get

$$kx^{k-1} = \prod_{s=1}^{k-1} (x - \alpha^s) + (x - 1)g(x), \tag{4}$$

in which $g(x)$ is a polynomial in x. Put $x = 1$ in (4) to obtain

$$k = \prod_{s=1}^{k-1} (1 - \alpha^s).$$

But

$$1 - \alpha^s = 1 - \exp\left(\frac{2\pi i s}{k}\right) = -\exp\left(\frac{\pi i s}{k}\right)\left[\exp\left(\frac{\pi i s}{k}\right) - \exp\left(\frac{-\pi i s}{k}\right)\right]$$

$$= -2i \exp\left(\frac{\pi i s}{k}\right) \sin \frac{\pi s}{k}.$$

Hence

$$k = (-2i)^{k-1} \exp[\tfrac{1}{2}\pi i(k-1)] \prod_{s=1}^{k-1} \sin \frac{\pi s}{k} = 2^{k-1} \prod_{s=1}^{k-1} \sin \frac{\pi s}{k},$$

which yields the desired result.

With the aid of Lemma 8, equation (3) can be written

$$k^2 c^2 \pi^{k-1} = \frac{k}{2^{k-1}}.$$

The constant c is positive because the Gamma function is positive for positive argument. Hence

$$c = (2\pi)^{-\frac{1}{2}(k-1)} k^{-\frac{1}{2}}.$$

This completes the proof of the Gauss multiplication theorem.

THEOREM 10 $\prod_{s=1}^{k} \Gamma\left(z + \frac{s-1}{k}\right) = (2\pi)^{\frac{1}{2}(k-1)} k^{\frac{1}{2}-kz} \Gamma(kz).$

21. A summation formula due to Euler. Let

$$P(x) = x - [x] - \tfrac{1}{2},$$

in which $[x]$ means the greatest integer $\leqq x$, a notation also used frequently in later chapters. The function $P(x)$ is periodic,

$$P(x + 1) = P(x),$$

and is represented graphically in Figure 2.

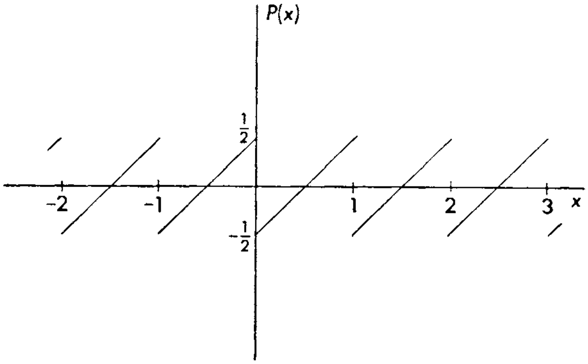

Figure 2

Euler employed $P(x)$ in obtaining some useful summation formulas, of which we use only that in Theorem 11.

THEOREM 11. *If $f'(x)$ is continuous for $x \geqq 0$,*

$$\sum_{k=0}^{n} f(k) = \int_0^n f(x)\,dx + \tfrac{1}{2}f(0) + \tfrac{1}{2}f(n) + \int_0^n P(x)f'(x)\,dx,$$

in which $P(x) = x - [x] - \frac{1}{2}$.

Proof. First write

$$\int_0^n P(x)f'(x)\,dx = \sum_{k=1}^{n} \int_{k-1}^{k} P(x)f'(x)\,dx.$$

Now

$$\int_{k-1}^{k} P(x)f'(x)\,dx = \int_{k-1}^{k} (x - k + \tfrac{1}{2})f'(x)\,dx,$$

and we integrate by parts to obtain

$$\int_{k-1}^{k} P(x)f'(x)\,dx = \Big[(x - k + \tfrac{1}{2})f(x)\Big]_{k-1}^{k} - \int_{k-1}^{k} f(x)\,dx$$

$$= \tfrac{1}{2}f(k) + \tfrac{1}{2}f(k - 1) - \int_{k-1}^{k} f(x)\,dx.$$

We may therefore write

$$\int_0^n P(x)f'(x)\,dx = \tfrac{1}{2}\sum_{k=1}^{n} f(k) + \tfrac{1}{2}\sum_{k=1}^{n} f(k-1) - \sum_{k=1}^{n}\int_{k-1}^{k} f(x)\,dx$$

$$= \tfrac{1}{2}\sum_{k=1}^{n} f(k) + \tfrac{1}{2}\sum_{k=0}^{n-1} f(k) - \int_0^n f(x)\,dx$$

$$= \sum_{k=0}^{n} f(k) - \tfrac{1}{2}f(0) - \tfrac{1}{2}f(n) - \int_0^n f(x)\,dx,$$

which is a simple rearrangement of the result in Theorem 11.

Lemma 9. For $|\arg z| \leqq \pi - \delta$, *where* $\delta > 0$,

$$\sum_{k=0}^{n} \operatorname{Log}(z+k) = (z + n + \tfrac{1}{2}) \operatorname{Log}(z+n)$$
$$- n - (z - \tfrac{1}{2}) \operatorname{Log} z + \int_0^n \frac{P(x)\,dx}{z+x}.$$

Proof: Lemma 9 follows at once by applying Theorem 11 to the function $f(x) = \operatorname{Log}(z+x)$.

Let us next turn to the result

$$\Gamma(z) = \lim_{n\to\infty} \frac{(n-1)!\, n^z}{(z)_n}, \tag{1}$$

established on page 23. In (1) put $z = \frac{1}{2}$ and shift from n to $(n+1)$ to obtain

$$\lim_{n\to\infty} \frac{n!\,(n+1)^{\frac{1}{2}}}{(\frac{1}{2})_{n+1}} = \Gamma(\tfrac{1}{2}),$$

which may be put in the form

$$\lim_{n\to\infty} \frac{n!\,(n+1)^{\frac{1}{2}}n!\,2^{2n}}{(\frac{1}{2})(\frac{3}{2})_n(1)_n 2^{2n}} = \sqrt{\pi}.$$

Now Lemma 6, page 22, yields

$$2^{2n}\left(\frac{3}{2}\right)_n (1)_n = (2)_{2n} = (2n+1)!.$$

Therefore we have

$$\lim_{n\to\infty} \frac{2^{2n+1}[(n+1)!]^2(n+1)^{-\frac{3}{2}}}{(2n+1)!} = \sqrt{\pi}. \tag{2}$$

It is legitimate to take logarithms of each member of (2) and thus write

$$(3)\quad \lim_{n\to\infty}\Big[(2n+1)\operatorname{Log} 2 - \tfrac{3}{2}\operatorname{Log}(1+n) + 2\sum_{k=0}^{n}\operatorname{Log}(1+k) - \sum_{k=0}^{2n}\operatorname{Log}(1+k)\Big] = \tfrac{1}{2}\operatorname{Log}\pi.$$

We shall apply the formula of Lemma 9 to the sums involved on the left in equation (3). The choice $z = 1$ in the result in Lemma 9 yields

$$(4)\quad \sum_{k=0}^{n}\operatorname{Log}(1+k) = \left(\frac{3}{2}+n\right)\operatorname{Log}(1+n) - n + \int_0^n \frac{P(x)\,dx}{1+x}.$$

By Lemma 9, with $z = 1$ and n replaced by $2n$, we get

$$\sum_{k=0}^{2n}\operatorname{Log}(1+k) = \left(\frac{3}{2}+2n\right)\operatorname{Log}(1+2n) - 2n + \int_0^{2n} \frac{P(x)\,dx}{1+x}.$$

Equation (3) can now be put in the form

$$\lim_{n\to\infty}\left[(2n+1)\operatorname{Log} 2 + \left(2n+\frac{3}{2}\right)\operatorname{Log}\frac{1+n}{1+2n} + 2\int_0^n \frac{P(x)\,dx}{1+x} - \int_0^{2n}\frac{P(x)\,dx}{1+x}\right] = \frac{1}{2}\operatorname{Log}\pi,$$

which leads* to

$$\lim_{n\to\infty}\left[-\frac{1}{2}\operatorname{Log} 2 + \left(2n+\frac{3}{2}\right)\operatorname{Log}\frac{2+2n}{1+2n}\right] + \int_0^\infty \frac{P(x)\,dx}{1+x} = \frac{1}{2}\operatorname{Log}\pi.$$

But

$$\lim_{n\to\infty}\left[\left(2n+\frac{3}{2}\right)\operatorname{Log}\frac{2+2n}{1+2n}\right] = 1.$$

Therefore we arrive at the evaluation

$$(5)\quad \int_0^\infty \frac{P(x)\,dx}{1+x} = -1 + \frac{1}{2}\operatorname{Log}(2\pi).$$

22. The behavior of log $\Gamma(z)$ for large $|z|$. From formula (3), page 11, it follows that

$$\Gamma(z) = \lim_{n\to\infty}\frac{(n+1)!\,(n+1)^{z-1}}{(z)_{n+1}},$$

*For proof of convergence of $\int_0^\infty \frac{P(x)\,dx}{1+x}$, see the exercises at the end of this chapter.

and so also that

$$(1)\quad \log \Gamma(z) = \lim_{n\to\infty}\left[\sum_{k=0}^{n} \operatorname{Log}(1+k) + (z-1)\operatorname{Log}(1+n) - \sum_{k=0}^{n}\operatorname{Log}(z+k)\right].$$

Using equation (4) and Lemma 9 of the preceding section, we may now conclude that, if $|\arg(z)| \leqq \pi - \delta,\ \delta > 0$,

$$(2)\quad \log \Gamma(z) = \lim_{n\to\infty}\left[\left(z+n+\frac{1}{2}\right)\{\operatorname{Log}(1+n) - \operatorname{Log}(z+n)\} + \left(z-\frac{1}{2}\right)\operatorname{Log} z + \int_0^n \frac{P(x)\,dx}{1+x} - \int_0^n \frac{P(x)\,dx}{z+x}\right].$$

The elementary limit

$$\lim_{n\to\infty}\left[\left(z+n+\frac{1}{2}\right)\{\operatorname{Log}(1+n) - \operatorname{Log}(z+n)\}\right] = 1 - z,$$

together with equation (5) of Section 21, permits us to put (2) in the form

$$\log \Gamma(z) = 1 - z + \left(z - \frac{1}{2}\right)\operatorname{Log} z - 1 + \frac{1}{2}\operatorname{Log}(2\pi) - \int_0^\infty \frac{P(x)\,dx}{z+x}.$$

Theorem 12. *If* $|\arg(z)| \leqq \pi - \delta$, *where* $\delta > 0$,

$$(3)\quad \log \Gamma(z) = \left(z - \frac{1}{2}\right)\operatorname{Log} z - z + \frac{1}{2}\operatorname{Log}(2\pi) - \int_0^\infty \frac{P(x)\,dx}{z+x},$$

in which $P(x) = x - [x] - \frac{1}{2}$, *as in Section 21.*

Let us next consider the integral on the right in (3). Since

$$\int P(x)\,dx = \tfrac{1}{2}P^2(x) + c,$$

we may use $c = -\frac{1}{24}$ and integrate by parts to find that

$$\int_0^\infty \frac{P(x)\,dx}{z+x} = \frac{1}{2}\left[\frac{P^2(x) - \frac{1}{12}}{z+x}\right]_0^\infty + \frac{1}{2}\int_0^\infty \frac{[P^2(x) - \frac{1}{12}]\,dx}{(z+x)^2}$$

$$= -\frac{1}{12z} + \frac{1}{2}\int_0^\infty \frac{[P^2(x) - \frac{1}{12}]\,dx}{(z+x)^2}.$$

Now the maximum value of $[P^2(x) - \frac{1}{12}]$ is $\frac{1}{6}$ and, in the region $|\arg z| \leqq \pi - \delta,\ \delta > 0$,

$$|z + x|^2 \geqq x^2 + |z|^2, \qquad \text{for } \operatorname{Re}(z) \geqq 0,$$

$$|z + x|^2 \geqq [x + \mathrm{Re}(z)]^2 + |z|^2 \sin^2 \delta, \qquad \text{for } \mathrm{Re}(z) < 0.$$

It follows that

$$\int_0^\infty \frac{[P^2(x) - \frac{1}{12}]\, dx}{(z + x)^2} = O\left(\frac{1}{|z|}\right),$$

as $|z| \to \infty$ in $|\arg z| \leqq \pi - \delta$, $\delta > 0$.

We have shown that as $|z| \to \infty$ in $|\arg z| \leqq \pi - \delta$, $\delta > 0$,

$$\text{(4)} \qquad \log \Gamma(z) = (z - \tfrac{1}{2}) \operatorname{Log} z - z + \tfrac{1}{2} \operatorname{Log}(2\pi) + o(1).$$

Indeed we showed a little more than that, but (4) is itself more precise than is needed later in this book.

From (4) we obtain at once the actual result to be employed in Chapter 5.

THEOREM 13. *As* $|z| \to \infty$ *in the region where* $|\arg z| \leqq \pi - \delta$ *and* $|\arg(z + a)| \leqq \pi - \delta$, $\delta > 0$,

$$\text{(5)} \qquad \log \Gamma(z + a) = (z + a - \tfrac{1}{2}) \operatorname{Log} z - z + O(1).$$

EXERCISES

1. Start with $\dfrac{\Gamma'(z)}{\Gamma(z)} = -\gamma - \dfrac{1}{z} - \displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{z + n} - \frac{1}{n}\right)$,

prove that

$$\frac{2\Gamma'(2z)}{\Gamma(2z)} - \frac{\Gamma'(z)}{\Gamma(z)} - \frac{\Gamma'(z + \frac{1}{2})}{\Gamma(z + \frac{1}{2})} = 2 \operatorname{Log} 2,$$

and thus derive Legendre's duplication formula, page 24.

2. Show that $\Gamma'(\frac{1}{2}) = -(\gamma + 2 \operatorname{Log} 2)\sqrt{\pi}$.

3. Use Euler's integral form $\Gamma(z) = \displaystyle\int_0^\infty e^{-t}t^{z-1}\, dt$ to show that $\Gamma(z + 1) = z\Gamma(z)$.

4. Show that $\Gamma(z) = \displaystyle\lim_{n \to \infty} n^z B(z, n)$.

5. Derive the following properties of the Beta function:
   (a) $pB(p, q + 1) = qB(p + 1, q)$;
   (b) $B(p, q) = B(p + 1, q) + B(p, q + 1)$;
   (c) $(p + q)B(p, q + 1) = qB(p, q)$;
   (d) $B(p, q)B(p + q, r) = B(q, r)B(q + r, p)$.

6. Show that for positive integral n, $B(p, n + 1) = n!/(p)_{n+1}$.

7. Evaluate $\displaystyle\int_{-1}^{1} (1 + x)^{p-1}(1 - x)^{q-1}\, dx$.

Ans. $2^{p+q-1}B(p, q)$.

8. Show that for $0 \leqq k \leqq n$

$$(\alpha)_{n-k} = \frac{(-1)^k(\alpha)_n}{(1-\alpha-n)_k}.$$

Note particularly the special case $\alpha = 1$.

9. Show that if α is not an integer,

$$\frac{\Gamma(1-\alpha-n)}{\Gamma(1-\alpha)} = \frac{(-1)^n}{(\alpha)_n}.$$

In Exs. 10–14, the function $P(x)$ is that of Section 21.

10. Evaluate $\int_0^x P(y)\,dy$. *Ans.* $\frac{1}{2}P^2(x) - \frac{1}{8}$.

11. Use integration by parts and the result of Ex. 10 to show that

$$\left| \int_n^\infty \frac{P(x)\,dx}{1+x} \right| \leqq \frac{1}{8(1+n)}.$$

12. With the aid of Ex. 11 prove the convergence of $\int_0^\infty \frac{P(x)\,dx}{1+x}$.

13. Show that

$$\int_0^\infty \frac{P(x)\,dx}{1+x} = \sum_{n=0}^\infty \int_n^{n+1} \frac{P(x)\,dx}{1+x} = \sum_{n=0}^\infty \int_0^1 \frac{(y-\frac{1}{2})\,dy}{1+n+y}.$$

Then prove that

$$\lim_{n\to\infty} n^2 \int_0^1 \frac{(y-\frac{1}{2})\,dy}{1+n+y} = -\frac{1}{12}$$

and thus conclude that $\int_0^\infty \frac{P(x)\,dx}{1+x}$ is convergent.

14. Apply Theorem 11, page 27, to the function $f(x) = (1+x)^{-1}$; let $n \to \infty$ and thus conclude that

$$\gamma = \tfrac{1}{2} - \int_1^\infty y^{-2}P(y)\,dy.$$

15. Use the relation $\Gamma(z)\Gamma(1-z) = \pi/\sin \pi z$ and the elementary result

$$\sin x \sin y = \tfrac{1}{2}\,[\cos(x-y) - \cos(x+y)]$$

to prove that

$$1 - \frac{\Gamma(c)\Gamma(1-c)\Gamma(c-a-b)\Gamma(a+b+1-c)}{\Gamma(c-a)\Gamma(a+1-c)\Gamma(c-b)\Gamma(b+1-c)}$$
$$= \frac{\Gamma(2-c)\Gamma(c-1)\Gamma(c-a-b)\Gamma(a+b+1-c)}{\Gamma(a)\Gamma(1-a)\Gamma(b)\Gamma(1-b)}.$$

CHAPTER 3

Asymptotic Series

23. Definition of an asymptotic expansion. Let us first recall the sense in which a convergent power series expansion represents the function being expanded. When a function $F(z)$, analytic at $z = 0$, is expanded in a power series about $z = 0$, we write

$$F(z) = \sum_{n=0}^{\infty} c_n z^n, \qquad |z| < r. \tag{1}$$

Define a partial sum of the series by

$$S_n(z) = \sum_{k=0}^{n} c_k z^k.$$

Then the series on the right in (1) represents $F(z)$ in the sense that

$$\lim_{n \to \infty} [F(z) - S_n(z)] = 0 \tag{2}$$

for each z in the region $|z| < r$. That is, for each fixed z the series in (1) can be made to approximate $F(z)$ as closely as desired by taking a sufficiently large number of terms of the series.

We now define an *asymptotic power series representation* of a function $f(z)$ as $z \to 0$ in some region R. We write

$$f(z) \sim \sum_{n=0}^{\infty} a_n z^n, \qquad z \to 0 \text{ in } R, \tag{3}$$

if and only if

$$\lim_{z \to 0 \text{ in } R} \frac{|f(z) - s_n(z)|}{|z|^n} = 0, \tag{4}$$

for each fixed n, with

$$s_n(z) = \sum_{k=0}^{n} a_k z^k. \tag{5}$$

By employing the order symbol defined in Section 13, we may write the condition (4) in the form

$$f(z) - s_n(z) = o(z^n), \qquad \text{as } z \to 0 \text{ in } R. \tag{6}$$

Here we see that the series in (3) represents the function $f(z)$ in the sense that for each fixed n, the sum of the terms out to the term $a_n z^n$ can be made to approximate $f(z)$ more closely than $|z|^n$ approximates zero, in the sense of (4), by choosing z sufficiently close to zero in the region R.

It is particularly noteworthy that in the definition of an asymptotic expansion, there is no requirement that the series converge. Indeed some authors include the additional restriction that the series in (3) diverge. Most asymptotic expansions do diverge, but it seems artificial to insist upon that behavior.

Asymptotic series are of great value in many computations. They play an important role in the solution of linear differential equations about irregular singular points. Such series were used by astronomers more than a century ago, long before the pertinent mathematical theory was developed.

Example: Show that

$$\int_0^\infty \frac{e^{-t}\,dt}{1 - xt} \sim \sum_{n=0}^{\infty} n!x^n, \qquad x \to 0 \text{ in } \operatorname{Re}(x) \leqq 0. \tag{7}$$

Let us put

$$s_n(x) = \sum_{k=0}^{n} k!x^k.$$

In the region $\operatorname{Re}(x) \leqq 0$, the integral on the left in (7) is absolutely and uniformly convergent. To see this, note that $t \geqq 0$ so that $\operatorname{Re}(1 - xt) \geqq 1$. Hence $|1 - xt| \geqq 1$, and we have

$$\left| \int_0^\infty \frac{e^{-t}\,dt}{1 - xt} \right| \leqq \int_0^\infty e^{-t}\,dt = 1.$$

For k a non-negative integer,

$$\int_0^\infty e^{-t}t^k\,dt = \Gamma(k+1) = k!. \tag{8}$$

Hence

$$\begin{aligned}\int_0^\infty \frac{e^{-t}\,dt}{1-xt} - s_n(x) &= \int_0^\infty \frac{e^{-t}\,dt}{1-xt} - \sum_{k=0}^{n}\int_0^\infty e^{-t}t^kx^k\,dt \\ &= \int_0^\infty e^{-t}\left[\frac{1}{1-xt} - \sum_{k=0}^{n}(xt)^k\right]dt.\end{aligned}$$

From elementary algebra we have

$$\sum_{k=0}^{n} r^k = \frac{1-r^{n+1}}{1-r}, \qquad r \neq 1.$$

Therefore

$$\int_0^\infty \frac{e^{-t}\,dt}{1-xt} - s_n(x) = \int_0^\infty \frac{e^{-t}(xt)^{n+1}\,dt}{1-xt},$$

from which, since $|1 - xt| \geqq 1$, we obtain

$$\left|\int_0^\infty \frac{e^{-t}\,dt}{1-xt} - s_n(x)\right| \leqq |x|^{n+1}\int_0^\infty e^{-t}t^{n+1}\,dt, \qquad \text{in } \operatorname{Re}(x) \leqq 0.$$

We may conclude that

$$\left|\int_0^\infty \frac{e^{-t}\,dt}{1-xt} - s_n(x)\right| \leqq (n+1)!\,|x|^{n+1}, \qquad \text{in } \operatorname{Re}(x) \leqq 0. \tag{9}$$

From (9) it follows at once that the condition (4), page 34, is satisfied, which concludes the proof. Actually (9) gives more information than that. Let $E_n(x)$ be the error made in computing the sum function by discarding all terms after the term $n!x^n$. Then $|E_n(x)|$ is the left member of (9), and the inequality (9) shows that $|E_n(x)|$ is smaller than the magnitude of the first term omitted. This property, although not possessed by all asymptotic series, is one of frequent occurrence.

The preceding example gives little indication of methods for obtaining asymptotic expansions. Later we shall exhibit two common methods, successive integration by parts and term-by-term integration of power series.

Extension of the concept of an asymptotic expansion to one in which the variable approaches any specific point in the finite plane is direct. For finite z_0 we say that

$$f(z) \sim \sum_{n=0}^{\infty} a_n(z-z_0)^n, \qquad \text{as } z \to z_0 \text{ in } R,$$

if and only if, for each fixed n,

$$f(z) - s_n(z) = o([z - z_0]^n), \qquad \text{as } z \to z_0 \text{ in } R,$$

in which

$$s_n(z) = \sum_{k=0}^{n} a_k(z - z_0)^k.$$

24. Asymptotic expansions about infinity. Asymptotic series are often used for large $|z|$. We say that

$$f(z) \sim \sum_{n=0}^{\infty} a_n z^{-n}, \text{ as } z \to \infty \text{ in } R, \tag{1}$$

if and only if, for each fixed n,

$$f(z) - s_n(z) = o(z^{-n}), \qquad \text{as } z \to \infty \text{ in } R, \tag{2}$$

in which

$$s_n(z) = \sum_{k=0}^{n} a_k z^{-k}. \tag{3}$$

At times, as in the subsequent example, we wish to work only along the axis of reals. We then use (1), (2), and (3) for a real variable x, with the region R replaced by a direction along the real axis.

One last extension of the term asymptotic expansion follows. It may be that $f(z)$ itself has no asymptotic expansion in the sense of the foregoing definitions. We do, however, write

$$f(z) \sim h(z) + g(z) \sum_{n=0}^{\infty} a_n z^{-n}, \qquad \text{as } z \to \infty \text{ in } R, \tag{4}$$

if and only if

$$\frac{f(z) - h(z)}{g(z)} \sim \sum_{n=0}^{\infty} a_n z^{-n}, \qquad \text{as } z \to \infty \text{ in } R, \tag{5}$$

and similarly for asymptotic expansions about a point in the finite plane.

Example: Obtain, for real x, as $x \to \infty$, an asymptotic expansion of the error function

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-t^2)\, dt. \tag{6}$$

From the fact that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, it follows at once that

$$\lim_{x\to\infty} \operatorname{erf}(x) = 1.$$

Let us write

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^\infty \exp(-t^2)\,dt - \frac{2}{\sqrt{\pi}}\int_x^\infty \exp(-t^2)\,dt$$

$$= 1 - \frac{2}{\sqrt{\pi}}\int_x^\infty \exp(-t^2)\,dt.$$

Now consider the function

$$B(x) = \int_x^\infty \exp(-t^2)\,dt$$

and integrate by parts to get

$$B(x) = -\tfrac{1}{2}\Big[t^{-1}\exp(-t^2)\Big]_x^\infty - \tfrac{1}{2}\int_x^\infty t^{-2}\exp(-t^2)\,dt$$

$$= \tfrac{1}{2}x^{-1}\exp(-x^2) - \tfrac{1}{2}\int_x^\infty t^{-2}\exp(-t^2)\,dt.$$

Iteration of the integration by parts soon yields

$$B(x) = \exp(-x^2)\left[\frac{1}{2x} - \frac{1}{2^2x^3} + \frac{1\cdot 3}{2^3x^5} - \frac{1\cdot 3\cdot 5}{2^4x^7} + \cdots + \frac{(-1)^n 1\cdot 3\cdot 5\cdots(2n-1)}{2^{n+1}x^{2n+1}}\right]$$

$$+ \frac{(-1)^{n+1}1\cdot 3\cdot 5\cdots(2n+1)}{2^{n+1}}\int_x^\infty t^{-2n-2}\exp(-t^2)\,dt,$$

or

$$B(x) = \frac{1}{2}\exp(-x^2)\sum_{k=0}^{n}\frac{(-1)^k(\frac{1}{2})_k}{x^{2k+1}} + (-1)^{n+1}\left(\frac{1}{2}\right)_{n+1}\int_x^\infty t^{-2n-2}\exp(-t^2)\,dt. \tag{7}$$

Let

$$s_n(x) = \frac{1}{2}\sum_{k=0}^{n}\frac{(-1)^k(\frac{1}{2})_k}{x^{2k+1}}.$$

Then, from (7),

$$\exp(x^2)B(x) - s_n(x) = (-1)^{n+1}(\tfrac{1}{2})_{n+1}\exp(x^2)\int_x^\infty t^{-2n-2}\exp(-t^2)\,dt.$$

The variable of integration is never less than x. We replace the factor t^{-2n-2} in the integrand by tx^{-2n-3} and thus obtain

$$\left|\exp(x^2)B(x) - s_n(x)\right| < \frac{(\tfrac{1}{2})_{n+1}\exp(x^2)}{x^{2n+3}}\int_x^\infty t\exp(-t^2)\,dt,$$

from which it follows that

$$\left|\exp(x^2)B(x) - s_n(x)\right| < \frac{(\tfrac{1}{2})_{n+1}}{2x^{2n+3}}. \tag{8}$$

Hence

$$\exp(x^2)B(x) - s_n(x) = o(x^{-2n-2}), \qquad \text{as } x \to \infty,$$

which permits us to write the asymptotic expansion

$$\exp(x^2)B(x) \sim \sum_{n=0}^{\infty}\frac{(-1)^n(\tfrac{1}{2})_n}{2x^{2n+1}}, \qquad x \to \infty.$$

But $\operatorname{erf}(x) = 1 - \dfrac{2}{\sqrt{\pi}}B(x)$. Hence

$$\operatorname{erf}(x) \sim 1 - \frac{1}{\sqrt{\pi}}\exp(-x^2)\sum_{n=0}^{\infty}\frac{(-1)^n(\tfrac{1}{2})_n}{x^{2n+1}}, \qquad x \to \infty. \tag{9}$$

Note also the useful bound in (8).

25. Algebraic properties. Asymptotic expansions behave like convergent power series in many ways. We shall treat only expansions as $z \to \infty$ in some region R. The reader can easily extend the results to theorems in which $z \to z_0$ in the finite plane.

THEOREM 14. *If, as* $z \to \infty$ in R,

$$f(z) \sim \sum_{n=0}^{\infty} a_n z^{-n} \tag{1}$$

and

$$g(z) \sim \sum_{n=0}^{\infty} b_n z^{-n}, \tag{2}$$

then

$$f(z) + g(z) \sim \sum_{n=0}^{\infty}(a_n + b_n)z^{-n} \tag{3}$$

and

$$f(z)g(z) \sim \sum_{n=0}^{\infty}\sum_{k=0}^{n} a_k b_{n-k} z^{-n}. \tag{4}$$

Proof: Let

$$S_n(z) = \sum_{k=0}^{n} a_k z^{-k}, \qquad T_n(z) = \sum_{k=0}^{n} b_k z^{-k}.$$

From (1) and (2) we know that

$$f(z) - S_n(z) = o(z^{-n}), \tag{5}$$

$$g(z) - T_n(z) = o(z^{-n}), \tag{6}$$

from which

$$f(z) + g(z) - [S_n(z) + T_n(z)] = o(z^{-n}),$$

yielding (3).

To prove the validity of (4), first put

$$Q_n(z) = \sum_{k=0}^{n} \sum_{i=0}^{k} a_i b_{k-i} z^{-k},$$

which is the "nth partial sum" of the series on the right in (4). By direct multiplication,

$$S_n(z)T_n(z) = Q_n(z) + o(z^{-n}),$$

and by (5) and (6),

$$f(z)g(z) = S_n(z)T_n(z) + o(z^{-n}).$$

Hence

$$f(z)g(z) = Q_n(z) + o(z^{-n}),$$

which shows the validity of (4).

The right member in (4) is the ordinary Cauchy product of the series (1) and (2).

26. Term-by-term integration. Suppose that for real x we have

$$f(x) \sim \sum_{n=0}^{\infty} a_n x^{-n}, \qquad x \to \infty. \tag{1}$$

Surely we are interested here in large x, so that an integral which it is natural to consider is $\int_y^\infty f(x)\,dx$. But $\int_y^\infty a_0\,dx$ and $\int_y^\infty a_1 x^{-1}\,dx$ do not exist. Therefore we restrict ourselves to the consideration of an expansion

$$g(x) \sim \sum_{n=2}^{\infty} a_n x^{-n}, \qquad x \to \infty, \tag{2}$$

and seek $\int_y^\infty g(x)\,dx$. Of course $g(x) = f(x) - a_0 - a_1x^{-1}$.

Let

$$s_n(x) = \sum_{k=2}^{n} a_k x^{-k}.$$

Then

$$g(x) - s_n(x) = o(x^{-n}), \qquad x \to \infty,$$

and

$$\begin{aligned}\left|\int_y^\infty g(x)\,dx - \int_y^\infty s_n(x)\,dx\right| &\leq \int_y^\infty |g(x) - s_n(x)|\,dx \\ &< \int_y^\infty |o(x^{-n})|\,dx \\ &= o(y^{-n+1}).\end{aligned}$$

But

$$\int_y^\infty s_n(x)\,dx = \sum_{k=2}^{n} a_k \int_y^\infty x^{-k}\,dx = \sum_{k=2}^{n} \frac{a_k y^{-k+1}}{(k-1)}.$$

Hence

$$(3) \qquad \int_y^\infty g(x)\,dx \sim \sum_{n=2}^{\infty} \frac{a_n y^{-n+1}}{n-1}, \qquad y \to \infty,$$

the desired result.

27. Uniqueness. Since $e^{-x} = o(x^k)$, as $x \to \infty$, for any real k, whole classes of functions have the same asymptotic expansion. Surely if

$$f(x) \sim \sum_{n=0}^{\infty} A_n x^{-n},$$

then also

$$f(x) + ce^{-x} \sim \sum_{n=0}^{\infty} A_n x^{-n},$$

and numerous similar examples are easily concocted.

On the other hand a given function cannot have more than one asymptotic expansion as $z \to z_0$, finite or infinite. Let us use $z \to \infty$ in a region R as a representative example.

THEOREM 15. *If*

$$(1) \qquad f(z) \sim \sum_{n=0}^{\infty} A_n z^{-n}, \qquad z \to \infty \text{ in } R,$$

and

(2) $$f(z) \sim \sum_{n=0}^{\infty} B_n z^{-n}, \qquad z \to \infty \text{ in } R,$$

then $A_n = B_n$.

Proof: From (1) and (2) we have

$$f(z) - \sum_{k=0}^{n} A_k z^{-k} = o(z^{-n}),$$

$$f(z) - \sum_{k=0}^{n} B_k z^{-k} = o(z^{-n}),$$

from which it follows that

$$\sum_{k=0}^{n} (A_k - B_k) z^{-k} = o(z^{-n}),$$

or its equivalent

$$\sum_{k=0}^{n} (A_k - B_k) z^{n-k} = o(1), \qquad z \to \infty \text{ in } R,$$

for each n. Therefore $A_k = B_k$ for each k. The expansion (1) associated with $z \to \infty$ in a particular region R is unique. The function $f(z)$ may, of course, have a different asymptotic expansion as $z \to \infty$ in some region other than R.

28. Watson's lemma. The following useful result due to Watson [1;236] gives conditions under which the term-by-term Laplace transform of a series yields an asymptotic representation for the transform of the sum of the series. For details on Laplace transforms see Churchill [1].

Since relatively complicated exponents appear in the following few pages, we shall simplify the printing by the introduction of a notation similar to the common one, $\exp u = e^u$. The symbol $\exp_x(m)$ is defined by

$$\exp_x(m) = x^m.$$

Watson's Lemma. Let $F(t)$ satisfy the following conditions:

(1) $$F(t) = \sum_{n=1}^{\infty} a_n \exp_t\left(\frac{n}{r} - 1\right), \textit{ in } |t| \leqq a + \delta, \textit{ with } a, \delta, r > 0;$$

(2) *There exist positive constants K and b such that*

$$|F(t)| < Ke^{bt}, \qquad \textit{for } t \geqq a.$$

Then

$$(3) \qquad f(s) = \int_0^\infty e^{-st}F(t)\,dt \sim \sum_{n=1}^\infty \frac{a_n\Gamma(n/r)}{s^{n/r}},$$

as $|s| \to \infty$ *in the region* $|\arg s| \leqq \frac{1}{2}\pi - \Delta$, *for arbitrarily small positive* Δ.

Note that (1) implies that $F(t)$ is either analytic at $t = 0$ or has at most a certain type of branch point there.

Proof: It is not difficult to show (Exs. 1 and 2 at the end of this chapter) that under the conditions of Watson's lemma, there exist positive constants c and β such that for all $t \geqq 0$, whether $t \leqq a$ or $t > a$,

$$(4) \qquad \left| F(t) - \sum_{k=1}^n a_k \exp_t\left(\frac{k}{r} - 1\right) \right| < c \exp_t\left(\frac{n+1}{r} - 1\right)e^{\beta t}.$$

We know also the Laplace transform of a power of t,

$$(5) \qquad \int_0^\infty e^{-st}t^m\,dt = \Gamma(m+1)s^{-m-1}, \qquad m > -1,\ \mathrm{Re}(s) > 0.$$

In order to derive (3), we need to show that for each fixed n

$$\left| f(s) - \sum_{k=1}^n a_k\Gamma\left(\frac{k}{r}\right)s^{-k/r} \right| \cdot \left| s \right|^{n/r} = o(1),$$

as $|s| \to \infty$ in $|\arg s| \leqq \frac{1}{2}\pi - \Delta$, $\Delta > 0$.

Now

$$f(s) - \sum_{k=1}^n a_k\Gamma\left(\frac{k}{r}\right)s^{-k/r} = \int_0^\infty e^{-st}\left[F(t) - \sum_{k=1}^n a_k \exp_t\left(\frac{k}{r} - 1\right)\right] dt.$$

Hence, with the aid of (4),

$$\begin{aligned}
|s|^{n/r}&\left| f(s) - \sum_{k=1}^n a_k\Gamma\left(\frac{k}{r}\right)s^{-k/r} \right| \\
&< |s|^{n/r}c\int_0^\infty |e^{-st}|\exp_t\left(\frac{n+1}{r} - 1\right)e^{\beta t}\,dt \\
&< c|s|^{n/r}\int_0^\infty e^{-\mathrm{Re}(s)t}\exp_t\left(\frac{n+1}{r} - 1\right)e^{\beta t}\,dt \\
&< c|s|^{n/r}\Gamma\left(\frac{n+1}{r}\right)[\mathrm{Re}(s) - \beta]^{-\frac{(n+1)}{r}},
\end{aligned}$$

if $\mathrm{Re}(s) > \beta$. In the region $|\arg s| \leqq \frac{1}{2}\pi - \Delta$, $\Delta > 0$, $\mathrm{Re}(s) > \beta$

as soon as we choose $|s| > \beta(\sin \Delta)^{-1}$. Therefore, as $|s| \to \infty$ in the region $|\arg s| \leqq \frac{1}{2}\pi - \Delta$,

$$|s|^{n/r}\left|f(s) - \sum_{k=1}^{n} a_k \Gamma\left(\frac{k}{r}\right)s^{-k/r}\right| = o(1),$$

as desired.

Example: Obtain an asymptotic expansion of

$$f(x) = \int_0^\infty \frac{e^{-xt}\, dt}{1 + t^2}, \qquad |x| \to \infty \text{ in } |\arg x| \leqq \tfrac{1}{2}\pi - \Delta,\ \Delta > 0.$$

Note that the result will be valid in particular for real $x \to \infty$.

We shall apply Watson's lemma with $F(t) = 1/(1 + t^2)$. Then

$$F(t) = \sum_{n=0}^{\infty} (-1)^n t^{2n} = \sum_{n=1}^{\infty} (-1)^{n+1} t^{2n-2}, \qquad |t| < 1,$$

so that we may write

$$F(t) = \sum_{n=1}^{\infty} a_n t^{n-1}, \qquad \text{in } |t| \leqq \frac{5}{6},$$

in which $a_{2n} = 0$, $a_{2n-1} = (-1)^{n+1}$, and we have chosen $r = 1$, $a = \frac{1}{2}$, $\delta = \frac{1}{3}$ in the notation of Watson's lemma.

For $t \geqq \frac{1}{2}$, $e^t > 1$ and $1/(1 + t^2) < 1$, from which

$$F(t) = \frac{1}{1 + t^2} < e^t.$$

We may therefore conclude from Watson's lemma that

$$\int_0^\infty \frac{e^{-xt}\, dt}{1 + t^2} \sim \sum_{n=1}^{\infty} a_n \Gamma(n) x^{-n},$$

or

$$\int_0^\infty \frac{e^{-xt}\, dt}{1 + t^2} \sim \sum_{n=1}^{\infty} \frac{(-1)^{n+1}\Gamma(2n - 1)}{x^{2n-1}},$$

and finally that

(6) $$\int_0^\infty \frac{e^{-xt}\, dt}{1 + t^2} \sim \sum_{n=0}^{\infty} \frac{(-1)^n (2n)!}{x^{2n+1}},$$

as $|x| \to \infty$ in $|\arg x| \leqq \frac{1}{2}\pi - \Delta$, $\Delta > 0$.

EXERCISES

1. With the assumptions of Watson's lemma, page 41, show, with the aid of the convergence of the series in (1), that for $0 \leqq t \leqq a$, there exists a positive constant c_1 such that

$$\left| F(t) - \sum_{k=1}^{n} a_k \exp_t\left(\frac{k}{r} - 1\right) \right| < c_1 \exp_t\left(\frac{n+1}{r} - 1\right).$$

2. With the assumptions of Watson's lemma, page 41, show that for $t > a$, there exist positive constants c_2 and β such that

$$\left| F(t) - \sum_{k=1}^{n} a_k \exp_t\left(\frac{k}{r} - 1\right) \right| < c_2 \exp_t\left(\frac{n+1}{r} - 1\right) e^{\beta t}.$$

3. Derive the asymptotic expansion (6) immediately preceding these exercises by applying Watson's lemma to the function

$$f'(x) = -\int_0^\infty \frac{te^{-xt}\, dt}{1 + t^2}$$

and then integrating the resultant expansion term by term.

4. Establish (6), page 43, directly, first showing that

$$f(x) - \sum_{k=0}^{n} (-1)^k (2k)! x^{-2k-1} = (-1)^{n+1} \int_0^\infty \frac{e^{-xt} t^{2n+2}\, dt}{1 + t^2},$$

and thus obtain not only (6) but also a bound on the error made in computing with the series involved.

5. Use integration by parts to establish that for real $x \to \infty$,

$$\int_x^\infty e^{-t} t^{-1}\, dt \sim e^{-x} \sum_{n=0}^{\infty} (-1)^n n! x^{-n-1}.$$

6. Let the Hermite polynomials $H_n(x)$ be defined by

$$\exp(2xt - t^2) = \sum_{n=0}^{\infty} \frac{H_n(x) t^n}{n!}$$

for all x and t, as in Chapter 11. Also let the complementary error function erfc x be defined by

$$\operatorname{erfc} x = 1 - \operatorname{erf} x = \frac{2}{\sqrt{\pi}} \int_x^\infty \exp(-\beta^2)\, d\beta.$$

Apply Watson's lemma to the function $F(t) = \exp(2xt - t^2)$; obtain

$$\exp(x - \tfrac{1}{2}s)^2 \int_{\frac{1}{2}s - x}^{\infty} \exp(-\beta^2)\, d\beta \sim \sum_{n=0}^{\infty} H_n(x) s^{-n-1}, \qquad s \to \infty,$$

and thus arrive at the result

$$\tfrac{1}{2} t^{-1} \sqrt{\pi} \exp[(\tfrac{1}{2}t^{-1} - x)^2] \operatorname{erfc}(\tfrac{1}{2}t^{-1} - x) \sim \sum_{n=0}^{\infty} H_n(x) t^n, \qquad t \to 0^+.$$

7. Use integration by parts to show that if $\operatorname{Re}(\alpha) > 0$, and if x is real,

$$\int_x^\infty e^{-t} t^{-\alpha}\, dt \sim x^{1-\alpha} e^{-x} \sum_{n=0}^{\infty} \frac{(-1)^n (\alpha)_n}{x^{n+1}}, \qquad x \to \infty,$$

of which Ex. 5 is the special case $\alpha = 1$.

CHAPTER 4

The Hypergeometric Function

29. The function $F(a, b; c; z)$. In the study of second-order linear differential equations with three regular singular points, there arises the function

$$F(a, b; c; z) = 1 + \sum_{n=1}^{\infty} \frac{(a)_n (b)_n z^n}{(c)_n n!}, \tag{1}$$

for c neither zero nor a negative integer. In (1) the notation

$$(\alpha)_n = \alpha(\alpha + 1)(\alpha + 2) \cdots (\alpha + n - 1), \qquad n \geqq 1, \tag{2}$$
$$(\alpha)_0 = 1, \qquad \alpha \neq 0,$$

of Section 18 is used. We are here concerned with various properties of the special functions under consideration; that (1) satisfies a certain differential equation is, for us, only one among many facts of interest.

Since

$$\lim_{n \to \infty} \left| \frac{(a)_{n+1}(b)_{n+1} z^{n+1}}{(c)_{n+1}(n + 1)!} \cdot \frac{(c)_n n!}{(a)_n (b)_n z^n} \right|$$
$$= \lim_{n \to \infty} \left| \frac{(a + n)(b + n)z}{(c + n)(n + 1)} \right| = |z|,$$

so long as none of a, b, c is zero or a negative integer, the series in (1) has the circle $|z| < 1$ as its circle of convergence. If either or both of a and b is zero or a negative integer, the series terminates, and convergence does not enter the discussion.

On the boundary $|z| = 1$ of the region of convergence, a sufficient condition for absolute convergence of the series is $\text{Re}(c-a-b) > 0$. To prove this, let

$$\delta = \tfrac{1}{2}\,\text{Re}(c - a - b) > 0,$$

and compare terms of the series

$$1 + \sum_{n=1}^{\infty} \left| \frac{(a)_n(b)_n z^n}{(c)_n n!} \right| \tag{3}$$

with corresponding terms of the series

$$\sum_{n=1}^{\infty} \frac{1}{n^{1+\delta}}, \tag{4}$$

known to be convergent. Since $|z| = 1$ and

$$\begin{aligned}
&\lim_{n\to\infty} \left| \frac{n^{1+\delta}(a)_n(b)_n}{(c)_n n!} \right| \\
&\quad = \lim_{n\to\infty} \left| \frac{(a)_n}{(n-1)!n^a} \cdot \frac{(b)_n}{(n-1)!n^b} \cdot \frac{(n-1)!n^c}{(c)_n} \cdot \frac{(n-1)!n^{1+\delta}}{n!n^{c-a-b}} \right| \\
&\quad = \left| \frac{1}{\Gamma(a)} \cdot \frac{1}{\Gamma(b)} \cdot \frac{\Gamma(c)}{1} \right| \lim_{n\to\infty} \left| \frac{1}{n^{c-a-b-\delta}} \right| = 0,
\end{aligned}$$

because $\text{Re}(c - a - b - \delta) = 2\delta - \delta > 0$, the series in (1) is absolutely convergent on $|z| = 1$ when $\text{Re}(c - a - b) > 0$.

A mild variation of the notation $F(a, b; c; z)$ is often used; it is

$$F\left[\begin{matrix} a, b; & \\ & z \\ c; & \end{matrix}\right], \tag{5}$$

which is sometimes more convenient for printing and which has the advantage of exhibiting the numerator parameters a and b above the denominator parameter c, thus making it easy to remember the respective roles of a, b, and c. When we come to the generalized hypergeometric functions, we shall frequently use a notation like that in (5).

The series on the right in (1) or in

$$F\left[\begin{matrix} a, b; & \\ & z \\ c; & \end{matrix}\right] = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n z^n}{(c)_n n!} \tag{6}$$

is called the *hypergeometric series.* The special case $a = c$, $b = 1$ yields the elementary geometric series $\sum_{n=0}^{\infty} z^n$; hence the term *hypergeometric.* The function in (6) or in (1) is correspondingly called the *hypergeometric function.* Although Euler obtained many properties of the function $F(a, b; c; z)$, we owe much of our knowledge of the subject to the more systematic and detailed study made by Gauss.

30. A simple integral form. If n is a non-negative integer,

$$\frac{(b)_n}{(c)_n} = \frac{\Gamma(b+n)\,\Gamma(c)}{\Gamma(c+n)\,\Gamma(b)} = \frac{\Gamma(c)}{\Gamma(b)\,\Gamma(c-b)} \cdot \frac{\Gamma(b+n)\,\Gamma(c-b)}{\Gamma(c+n)}.$$

If $\mathrm{Re}(c) > \mathrm{Re}(b) > 0$, we know from Theorem 7, page 19, and the integral definition of the Beta function, that

$$\frac{\Gamma(b+n)\,\Gamma(c-b)}{\Gamma(c+n)} = \int_0^1 t^{b+n-1}(1-t)^{c-b-1}\,dt.$$

Therefore, for $|z| < 1$,

$$F(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(b)\,\Gamma(c-b)} \sum_{n=0}^{\infty} \frac{(a)_n}{n!} \int_0^1 t^{b+n-1}(1-t)^{c-b-1} z^n\,dt$$

$$= \frac{\Gamma(c)}{\Gamma(b)\,\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1} \sum_{n=0}^{\infty} \frac{(a)_n (zt)^n\,dt}{n!}.$$

The binomial theorem states that

$$(1-y)^{-a} = \sum_{n=0}^{\infty} \frac{(-a)(-a-1)(-a-2)\cdots(-a-n+1)(-1)^n y^n}{n!},$$

which may be written

$$(1-y)^{-a} = \sum_{n=0}^{\infty} \frac{a(a+1)(a+2)\cdots(a+n-1)y^n}{n!}.$$

Therefore, in factorial function notation,

$$(1-y)^{-a} = \sum_{n=0}^{\infty} \frac{(a)_n y^n}{n!},$$

which we use with $y = zt$ to obtain the following result.

THEOREM 16. *If* $|z| < 1$ *and if* $\mathrm{Re}(c) > \mathrm{Re}(b) > 0$,

$$F(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(b)\,\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\,dt.$$

31. $F(a, b; c; 1)$ as a function of the parameters. We know already that if c is neither zero nor a negative integer and if $\mathrm{Re}(c - a - b) > 0$, the series

$$(1) \qquad F(a, b; c; 1) = 1 + \sum_{n=1}^{\infty} \frac{(a)_n(b)_n}{(c)_n n!}$$

is absolutely convergent.

Let δ be any positive number. We shall show that in the region $\mathrm{Re}(c - a - b) \geqq 2\delta > 0$, the series (1) for $F(a, b; c; 1)$ is uniformly convergent. To fix the ideas, it may be desirable to think of $\mathrm{Re}(c - a - b) \geqq 2\delta > 0$ as a region in the c-plane, with a and b chosen first. It is not necessary to look on the region in that way.

The series of positive constants

$$(2) \qquad \sum_{n=1}^{\infty} \frac{1}{n^{1+\delta}}$$

is convergent because $\delta > 0$. We show that for n sufficiently large and for all a, b, c in the region $\mathrm{Re}(c - a - b) \geqq 2\delta > 0$, with c neither zero nor a negative integer,

$$(3) \qquad \left| \frac{(a)_n(b)_n}{(c)_n n!} \right| < \frac{1}{n^{1+\delta}}.$$

Now (see page 46)

$$\lim_{n\to\infty} \left| \frac{(a)_n(b)_n n^{1+\delta}}{(c)_n n!} \right| = \left| \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \right| \lim_{n\to\infty} \left| \frac{1}{n^{c-a-b-\delta}} \right| = 0,$$

since $\mathrm{Re}(c - a - b - \delta) \geqq 2\delta - \delta = \delta > 0$. Hence (3) is true for n sufficiently large, and the Weierstrass M-test can be applied to the series in equation (1).

Theorem 17. *If c is neither zero nor a negative integer and $\mathrm{Re}(c - a - b) > 0$, $F(a, b; c; 1)$ is an analytic function of a, b, c.*

32. Evaluation of $F(a, b; c; 1)$. If $\mathrm{Re}(c - a - b) > 0$, Theorem 17 permits us to extend the integral form for $F(a, b; c; z)$, page 47, to the point $z = 1$ in the following manner. Since $\mathrm{Re}(c - a - b) > 0$, we may write

$$F(a, b; c; 1) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n n!}.$$

If we also stipulate that $\mathrm{Re}(c) > \mathrm{Re}(b) > 0$, it follows by the technique of Section 30 that

$$F(a, b; c; 1) = \frac{\Gamma(c)}{\Gamma(b)\,\Gamma(c-b)} \sum_{n=0}^{\infty} \frac{(a)_n}{n!} \int_0^1 t^{b+n-1}(1-t)^{c-b-1}\,dt$$

$$= \frac{\Gamma(c)}{\Gamma(b)\,\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-t)^{-a}\,dt.$$

Therefore, if $\operatorname{Re}(c - a - b) > 0$, if $\operatorname{Re}(c) > \operatorname{Re}(b) > 0$, and since c is neither zero nor a negative integer,

$$F(a, b; c; 1) = \frac{\Gamma(c)}{\Gamma(b)\,\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-a-b-1}\,dt$$

$$= \frac{\Gamma(c)}{\Gamma(b)\,\Gamma(c-b)} \cdot \frac{\Gamma(b)\,\Gamma(c-a-b)}{\Gamma(c-a)}$$

$$= \frac{\Gamma(c)\,\Gamma(c-a-b)}{\Gamma(c-a)\,\Gamma(c-b)}.$$

We now resort to Theorem 17 and analytic continuation to conclude that the foregoing evaluation of $F(a, b; c; 1)$ is valid without the condition $\operatorname{Re}(c) > \operatorname{Re}(b) > 0$.

THEOREM 18. *If* $\operatorname{Re}(c - a - b) > 0$ *and if c is neither zero nor a negative integer,*

$$F(a, b; c; 1) = \frac{\Gamma(c)\,\Gamma(c-a-b)}{\Gamma(c-a)\,\Gamma(c-b)}.$$

The value of $F(a, b; c; 1)$ will play a vital role in many of the results to be obtained in this and later chapters. Theorem 18 can be proved without the aid of the integral in Theorem 16. For such a proof see Whittaker and Watson [1;281–282].

EXAMPLE: Show that if $\operatorname{Re}(b) > 0$ and if n is a non-negative integer,

$$F\left[\begin{matrix} -\tfrac{1}{2}n, \ -\tfrac{1}{2}n + \tfrac{1}{2}; & \\ & 1 \\ b + \tfrac{1}{2}; & \end{matrix}\right] = \frac{2^n (b)_n}{(2b)_n}.$$

By Theorem 18 we get

$$F\left[\begin{matrix} -\tfrac{1}{2}n, \ -\tfrac{1}{2}n + \tfrac{1}{2}; & \\ & 1 \\ b + \tfrac{1}{2}; & \end{matrix}\right] = \frac{\Gamma(b + \tfrac{1}{2})\,\Gamma(b + n)}{\Gamma(b + \tfrac{1}{2}n)\,\Gamma(b + \tfrac{1}{2}n + \tfrac{1}{2})}$$

$$= \frac{(b)_n\,\Gamma(b)\,\Gamma(b + \tfrac{1}{2})}{\Gamma(b + \tfrac{1}{2}n)\,\Gamma(b + \tfrac{1}{2}n + \tfrac{1}{2})}.$$

Legendre's duplication formula for the Gamma function, page 24, yields

$$\Gamma(b)\,\Gamma(b + \tfrac{1}{2}) = 2^{1-2b}\sqrt{\pi}\,\Gamma(2b),$$

$$\Gamma(b + \tfrac{1}{2}n)\,\Gamma(b + \tfrac{1}{2}n + \tfrac{1}{2}) = 2^{1-2b-n}\sqrt{\pi}\,\Gamma(2b + n).$$

Therefore

$$F\left[\begin{matrix} -\tfrac{1}{2}n, \ -\tfrac{1}{2}n + \tfrac{1}{2}; & \\ & 1 \\ b + \tfrac{1}{2}; & \end{matrix}\right] = \frac{(b)_n 2^n \Gamma(2b)}{\Gamma(2b + n)} = \frac{2^n (b)_n}{(2b)_n},$$

as desired.

33. The contiguous function relations. Gauss defined as *contiguous* to $F(a, b; c; z)$ each of the six functions obtained by increasing or decreasing one of the parameters by unity. For simplicity in printing, we use the notations

$$F = F(a, b; c; z), \tag{1}$$

$$F(a+) = F(a + 1, b; c; z), \tag{2}$$

$$F(a-) = F(a - 1, b; c; z), \tag{3}$$

together with similar notations $F(b+)$, $F(b-)$, $F(c+)$, $F(c-)$ for the other four of the six functions contiguous to F.

Gauss proved, and we shall follow his technique, that between F and any two of its contiguous functions, there exists a linear relation with coefficients at most linear in z. The proof is one of remarkable directness; we prove that the relations exist by obtaining them. There are, of course, fifteen (six things taken two at a time) such relations.

Put

$$\delta_n = \frac{(a)_n (b)_n z^n}{(c)_n n!}$$

so that

$$F = \sum_{n=0}^{\infty} \delta_n \tag{4}$$

and

$$F(a+) = \sum_{n=0}^{\infty} \frac{(a + 1)_n\, \delta_n}{(a)_n}.$$

Since $a(a + 1)_n = (a + n)\ (a)_n$, we may write the six functions contiguous to F in the form

$$(5)\qquad F(a+) = \sum_{n=0}^{\infty} \frac{a+n}{a}\,\delta_n, \qquad F(a-) = \sum_{n=0}^{\infty} \frac{a-1}{a-1+n}\,\delta_n,$$

$$F(b+) = \sum_{n=0}^{\infty} \frac{b+n}{b}\,\delta_n, \qquad F(b-) = \sum_{n=0}^{\infty} \frac{b-1}{b-1+n}\,\delta_n,$$

$$F(c+) = \sum_{n=0}^{\infty} \frac{c}{c+n}\,\delta_n, \qquad F(c-) = \sum_{n=0}^{\infty} \frac{c-1+n}{c-1}\,\delta_n.$$

We also employ the differential operator $\theta = z\left(\frac{d}{dz}\right)$. This operator has the particularly pleasant property that $\theta z^n = nz^n$, thus making it handy to use on power series.

Since

$$(6)\qquad (\theta + a)F = \sum_{n=0}^{\infty} (a+n)\,\delta_n,$$

it can be seen with the aid of (5) that

$$(7)\qquad (\theta + a)F = aF(a+),$$

$$(8)\qquad (\theta + b)F = bF(b+),$$

$$(9)\qquad (\theta + c - 1)F = (c-1)F(c-).$$

From (7), (8), and (9), it follows at once that

$$(10)\qquad (a-b)F = aF(a+) - bF(b+),$$

and

$$(11)\qquad (a-c+1)F = aF(a+) - (c-1)F(c-),$$

two of the simplest of the contiguous function relations.

Next consider

$$\theta F = \sum_{n=1}^{\infty} \frac{n(a)_n(b)_n z^n}{(c)_n n!} = z \sum_{n=0}^{\infty} \frac{(a)_{n+1}(b)_{n+1} z^n}{(c)_{n+1} n!},$$

from which

$$(12)\qquad \theta F = z \sum_{n=0}^{\infty} \frac{(a+n)(b+n)\,\delta_n}{c+n}.$$

Now

$$\frac{(a+n)(b+n)}{c+n} = n + (a+b-c) + \frac{(c-a)(c-b)}{c+n}.$$

Hence equation (12) yields

$$\theta F = z\sum_{n=0}^{\infty} n\,\delta_n + (a+b-c)z\sum_{n=0}^{\infty}\delta_n + \frac{(c-a)(c-b)z}{c}\sum_{n=0}^{\infty}\frac{c}{c+n}\,\delta_n,$$

or

(13) $(1-z)\theta F = (a+b-c)zF + c^{-1}(c-a)(c-b)zF(c+).$

From (7) we obtain

$$(1-z)\theta F = -a(1-z)F + a(1-z)F(\text{a}+),$$

which combines with (13) to yield another of the contiguous function relations,

(14) $$[(1-z)a+(a+b-c)z]F = a(1-z)F(a+) - c^{-1}(c-a)(c-b)zF(c+).$$

The coefficient of F on the left in (14) is in a form desirable for certain later developments. Equation (14) may be replaced by

(15) $[a+(b-c)z]F = a(1-z)F(a+) - c^{-1}(c-a)(c-b)zF(c+).$

Next let us operate with θ on the series defining $F(a-)$. We thus obtain

$$\theta F(a-) = \sum_{n=1}^{\infty}\frac{(a-1)_n(b)_n z^n}{(c)_n(n-1)!} = \sum_{n=0}^{\infty}\frac{(a-1)_{n+1}(b)_{n+1}z^{n+1}}{(c)_{n+1}n!},$$

or

(16) $$\theta F(a-) = (a-1)z\sum_{n=0}^{\infty}\frac{b+n}{c+n}\,\delta_n.$$

But

$$\frac{b+n}{c+n} = 1 - \frac{c-b}{c+n},$$

so that (16) becomes

$$\theta F(a-) = (a-1)z\sum_{n=0}^{\infty}\delta_n - \frac{(a-1)(c-b)z}{c}\sum_{n=0}^{\infty}\frac{c}{c+n}\,\delta_n,$$

which, in view of (5), yields

(17) $\theta F(a-) = (a-1)zF - c^{-1}(a-1)(c-b)zF(c+).$

We return to (7) and replace a by $(a-1)$ in it to get

(18) $\theta F(a-) = (a-1)F - (a-1)F(a-).$

From (17) and (18) it follows that

(19) $(1-z)F = F(a-) - c^{-1}(c-b)zF(c+).$

Similarly, since a and b may be interchanged without affecting the hypergeometric series, we may write

$$(20) \qquad (1 - z)F = F(b-) - c^{-1}(c - a)zF(c+).$$

We now have five contiguous function relations, (19) and (20) together with

$$(10) \qquad (a - b)F = aF(a+) - bF(b+),$$

$$(11) \qquad (a - c + 1)F = aF(a+) - (c - 1)F(c-),$$

and

$$(15) \qquad [a+(b-c)z]F = a(1-z)F(a+) - c^{-1}(c-a)(c-b)zF(c+).$$

From these five relations the remaining ten may be obtained by performing suitable eliminations. See Ex. 21 at the end of this chapter.

34. The hypergeometric differential equation. The operator $\theta = z\left(\dfrac{d}{dz}\right)$, already used in the derivation of the contiguous function relations, is helpful in deriving a differential equation satisfied by

$$(1) \qquad w = F(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n z^n}{(c)_n n!}.$$

From (1) we obtain

$$\theta(\theta + c - 1)w = \sum_{n=0}^{\infty} \frac{n(n + c - 1)(a)_n(b)_n z^n}{(c)_n n!}$$

$$= \sum_{n=1}^{\infty} \frac{(a)_n(b)_n z^n}{(c)_{n-1}(n - 1)!}.$$

A shift of index yields

$$\theta(\theta + c - 1)w = \sum_{n=0}^{\infty} \frac{(a)_{n+1}(b)_{n+1} z^{n+1}}{(c)_n n!}$$

$$= z \sum_{n=0}^{\infty} \frac{(a + n)(b + n)(a)_n(b)_n z^n}{(c)_n n!}$$

$$= z(\theta + a)(\theta + b)w.$$

We have shown that $w = F(a, b; c; z)$ is a solution of the differential equation

$$(2) \qquad [\theta(\theta + c - 1) - z(\theta + a)(\theta + b)]w = 0. \qquad \theta = z\frac{d}{dz}.$$

Equation (2) is easily put in the form

$$(3) \qquad z(1-z)w'' + [c - (a+b+1)z]w' - abw = 0$$

by employing the relations $\theta w = zw'$ and $\theta(\theta - 1)w = z^2 w''$.

The second-order linear differential equation (3) is treated in many texts* and therefore we omit details here. In order to avoid tedious repetition, we shall, in this chapter only, refer to the text mentioned in the footnote as IDE.

In IDE, pages 144–148, it is shown that if c is nonintegral, two linearly independent solutions of (3) in $|z| < 1$ are

$$(4) \qquad w_1 = F(a, b; c; z)$$

and

$$(5) \qquad w_2 = z^{1-c}F(a+1-c, b+1-c; 2-c; z).$$

We shall also make free use of Kummer's 24 solutions of equation (3) as listed in IDE, pages 157–158. In any specific instance, however, previous knowledge of Kummer's 24 solutions is not necessary; the desired solution can be obtained directly with the aid of simple changes of variable performed on the differential equation (3). See Ex. 12 at the end of this chapter.

35. Logarithmic solutions of the hypergeometric equation. If c is not an integer, the hypergeometric equation

$$(1) \qquad z(1-z)w'' + [c - (a+b+1)z]w' - abw = 0$$

always has in $|z| < 1$ the two power series solutions (4) and (5) of the preceding section. If c is an integer, one solution may or may not, depending on the values of a and b, become logarithmic. In this book we are primarily interested in the functions rather than in the differential equations. We shall, whenever it is feasible, avoid discussion of logarithmic solutions.

If c is a positive integer and neither a nor b is an integer, two linearly independent solutions of (1) are as listed below. These solutions may be obtained by standard elementary techniques (see Rainville [1], Sections 92 and 94), and the details are therefore omitted here.

If $c = 1$ and neither a nor b is zero or a negative integer, two linearly independent solutions of (1) valid in $0 < |z| < 1$ are

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*See, for example, Chapter 6 of Rainville [2].

$$(2) \qquad w_1 = F(a, b; 1; z),$$

$$(3) \qquad w_3 = F(a, b; 1; z)\log z + \sum_{n=1}^{\infty} \frac{(a)_n(b)_n z^n}{(n!)^2}\{H(a, n) + H(b, n) - 2H(1, n)\},$$

in which

$$(4) \qquad H(a, n) = \sum_{k=1}^{n} \frac{1}{a + k - 1},$$

including the ordinary harmonic sum $H(1, n) = H_n$.

If c is an integer > 1 and neither a nor b is an integer, two linearly independent solutions of (1) valid in $0 < |z| < 1$ are

$$(5) \qquad w_1 = F(a, b; c; z)$$

and

$$(6) \qquad w_4 = F(a, b; c; z)\log z + \sum_{n=1}^{\infty} \frac{(a)_n(b)_n z^n}{(c)_n n!}\{H(a, n) + H(b, n) - H(c, n) - H(1, n)\} - \sum_{n=0}^{c-2} \frac{n!(1 - c)_{n+1}}{(1 - a)_{n+1}(1 - b)_{n+1} z^{n+1}}.$$

If c is an integer, $c \leqq 0$, equation (1) may be transformed by using $w = z^{1-c}y$ into a hypergeometric equation for y with new parameters $a' = a + 1 - c$, $b' = b + 1 - c$, and $c' = 2 - c$. If neither a' nor b' is an integer, the y-equation can be solved by using (5) and (6).

36. $F(a, b; c; z)$ as a function of its parameters. We have already noted that the series in

$$(1) \qquad F(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n z^n}{(c)_n n!}$$

is absolutely convergent (ratio test) for $|z| < 1$, independent of the choice of a, b, c as long as c is neither zero nor a negative integer. Recall that $(c)_n = \Gamma(c + n)/\Gamma(c)$. Consider the function

$$(2) \qquad \frac{F(a, b; c; z)}{\Gamma(c)} = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n z^n}{\Gamma(c + n)n!},$$

in which the possibility of zero denominators on the right has been removed. In any closed region in the finite a, b, and c planes,

$$\lim_{n\to\infty}\left|\frac{(a)_n(b)_n z^{\frac{1}{2}n}}{\Gamma(c+n)n!}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(a)_n}{(n-1)!n^a}\frac{(b)_n}{(n-1)!n^b}\frac{(n-1)!n^c}{\Gamma(c+n)}\frac{z^{\frac{1}{2}n}}{n^{1+c-a-b}}\right|$$

$$= \frac{1}{\Gamma(a)\Gamma(b)}\lim_{n\to\infty}\left|\frac{z^{\frac{1}{2}n}}{n^{1+c-a-b}}\right| = 0, \qquad \text{for } |z| < 1.$$

Therefore, for any fixed z in $|z| < 1$, there exists a constant K independent of a, b, c and such that

$$\left|\frac{(a)_n(b)_n z^n}{\Gamma(c+n)n!}\right| < K|z|^{\frac{1}{2}n}.$$

Since $\sum_{n=0}^{\infty} K|z|^{\frac{1}{2}n}$ is convergent, the series on the right in (2) is absolutely and uniformly convergent for all finite a, b, c as long as $|z| < 1$.

We know the location and character of the singular points of $\Gamma(c)$ and are now able to stipulate the behavior, with regard to analyticity, of the hypergeometric function with z fixed, $|z| < 1$, and a, b, c as variables.

THEOREM 19. *For $|z| < 1$ the function $F(a, b; c; z)$ is analytic in a, b, and c for all finite a, b, and c except for simple poles at c = zero and c = each negative integer.*

Reference to Theorem 19 will enable us to simplify many proofs in later work.

37. Elementary series manipulations. A common tool to be used in much of our later work is the rearrangement of terms in iterated series. Here we prove two basic lemmas of the kind needed. When convergent power series are involved, the infinite rearrangements can be justified in the elementary sense. In our study of generating functions of sets of polynomials, we sometimes deal with divergent power series. For such series the identities of this section may be considered as purely formal, but we shall find that the manipulative techniques are fully as useful as when applied to convergent series.

Lemma 10.

$$(1) \qquad \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A(k, n) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} A(k, n-k),$$

and

$$(2)\qquad \sum_{n=0}^{\infty}\sum_{k=0}^{n} B(k, n) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} B(k, n+k).$$

Proof: Consider the series

$$(3)\qquad \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A(k, n)t^{n+k}$$

in which t^{n+k} has been inserted for convenience and will be removed later by placing $t = 1$. Let us collect the powers of t in (3). We introduce new indices of summation j and m by

$$(4)\qquad k = j, \qquad n = m - j,$$

so that the exponent $(n + k)$ in (3) becomes m. The old indices n and k in (3) are restricted, as indicated in the summation symbol, by the inequalities

$$(5)\qquad n \geqq 0, \qquad k \geqq 0.$$

Because of (4) the inequalities (5) become

$$m - j \geqq 0, \qquad j \geqq 0,$$

or $0 \leqq j \leqq m$ with m, the exponent on t, restricted only in that it must be a non-negative integer. Thus we arrive at

$$(6)\qquad \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A(k, n)t^{n+k} = \sum_{m=0}^{\infty}\sum_{j=0}^{m} A(j, m-j)t^{m},$$

and the identity (1) of Lemma 10 follows by putting $t = 1$ and replacing the dummy indices j and m on the right by dummy indices k and n.

There is no need to use k and n for indices in both members of (1), but neither is there harm in it once a small degree of sophistication is acquired. We frequently employ many parameters and prefer to keep to a minimum the number of different symbols used.

In Lemma 10, equation (2) is merely (1) written in reverse; it needs no separate derivation.

Lemma 11.

$$(7)\qquad \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A(k, n) = \sum_{n=0}^{\infty}\sum_{k=0}^{[n/2]} A(k, n-2k),$$

and

$$(8)\qquad \sum_{n=0}^{\infty}\sum_{k=0}^{[n/2]} B(k, n) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} B(k, n+2k).$$

Proof: Consider the series

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} A(k, n) t^{n+2k} \tag{9}$$

and in it collect powers of t, introducing new indices by

$$k = j, \qquad n = m - 2j, \tag{10}$$

so that $n + 2k = m$. Since

$$n \geqq 0, \qquad k \geqq 0, \tag{11}$$

we conclude that

$$m - 2j \geqq 0, \qquad j \geqq 0,$$

from which $0 \leqq 2j \leqq m$ and $m \geqq 0$. Since $0 \leqq j \leqq \frac{1}{2}m$ and j is integral, the index j runs from 0 to the greatest integer in $\frac{1}{2}m$. Thus we obtain

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} A(k, n) t^{n+2k} = \sum_{m=0}^{\infty} \sum_{j=0}^{[\frac{1}{2}m]} A(j, m - 2j) t^m, \tag{12}$$

from which (7) follows by placing $t = 1$ and making the proper change of letters for the dummy indices on the right in (12). Equation (8) is (7) written in reverse order.

There is no bound to the number of such identities. The reader should now be able to obtain whatever lemmas he needs along these lines.

Note also that a combination of Lemmas 10 and 11 yields

$$\sum_{n=0}^{\infty} \sum_{k=0}^{n} C(k, n) = \sum_{n=0}^{\infty} \sum_{k=0}^{[n/2]} C(k, n - k). \tag{13}$$

38. Simple transformations. It is convenient for us to write the ordinary binomial expansion with the factorial function notation,

$$(1 - z)^{-a} = \sum_{n=0}^{\infty} \frac{(a)_n z^n}{n!} \tag{1}$$

and to recall the result of Ex. 8, page 32,

$$(\alpha)_{n-k} = \frac{(-1)^k (\alpha)_n}{(1 - \alpha - n)_k}, \qquad 0 \leqq k \leqq n. \tag{2}$$

In particular $\alpha = 1$ in (2) yields

$$(n - k)! = \frac{(-1)^k n!}{(-n)_k}, \qquad 0 \leqq k \leqq n. \tag{3}$$

Consider now the product

$$(1-z)^{-a}F\left[\begin{matrix} a, c-b; & \\ & \dfrac{-z}{1-z} \\ c; & \end{matrix}\right] = \sum_{k=0}^{\infty} \frac{(a)_k(c-b)_k(-1)^k z^k}{(c)_k k!(1-z)^{k+a}}.$$

With the aid of (1) we may write

$$(1-z)^{-a}F\left[\begin{matrix} a, c-b; & \\ & \dfrac{-z}{1-z} \\ c; & \end{matrix}\right] = \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a)_k(c-b)_k(a+k)_n(-1)^k z^{n+k}}{(c)_k k! n!}.$$

Now $(a)_k(a+k)_n = (a)_{n+k}$, so that

$$(1-z)^{-a}F\left[\begin{matrix} a, c-b; & \\ & \dfrac{-z}{1-z} \\ c; & \end{matrix}\right] = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{(c-b)_k(a)_{n+k}(-1)^k z^{n+k}}{(c)_k k! n!}$$

and we collect powers of z to obtain

$$(1-z)^{-a}F\left[\begin{matrix} a, c-b; & \\ & \dfrac{-z}{1-z} \\ c; & \end{matrix}\right] = \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{(c-b)_k(a)_n(-1)^k z^n}{(c)_k k!(n-k)!}$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{(-n)_k(c-b)_k}{(c)_k k!} \cdot \frac{(a)_n z^n}{n!},$$

by (3). The inner sum on the right is a terminating hypergeometric series. Hence

$$(1-z)^{-a}F\left[\begin{matrix} a, c-b; & \\ & \dfrac{-z}{1-z} \\ c; & \end{matrix}\right] = \sum_{n=0}^{\infty} F\left[\begin{matrix} -n, c-b; & \\ & 1 \\ c; & \end{matrix}\right]\frac{(a)_n z^n}{n!}.$$

Since $F(-n, c-b; c; 1)$ terminates, we may write

$$(1-z)^{-a}F\left[\begin{matrix} a, c-b; & \\ & \dfrac{-z}{1-z} \\ c; & \end{matrix}\right] = \sum_{n=0}^{\infty} \frac{\Gamma(c)\,\Gamma(b+n)(a)_n z^n}{\Gamma(c+n)\,\Gamma(b) n!}$$

$$= \sum_{n=0}^{\infty} \frac{(b)_n(a)_n z^n}{(c)_n n!}$$

$$= F(a, b; c; z),$$

a result valid where both $|z| < 1$ and $|z/(1-z)| < 1$ (for which see Figure 3, page 60).

THEOREM 20. *If* $|z| < 1$ *and* $|z/(1-z)| < 1$,

$$(4)\quad F\left[\begin{matrix} a, b; & \\ & z \\ c; & \end{matrix}\right] = (1-z)^{-a}\, F\left[\begin{matrix} a, c-b; & \\ & \dfrac{-z}{1-z} \\ c; & \end{matrix}\right].$$

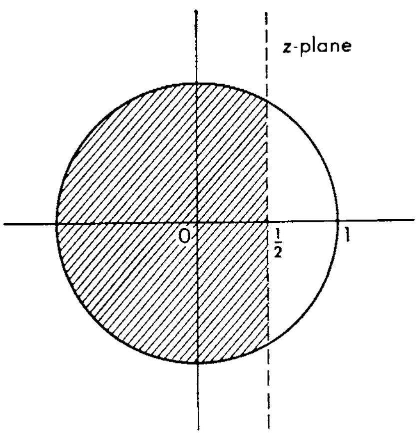

Figure 3

The roles of a and b may be interchanged in (4).

The type of series manipulations involved above in arriving at the identity (4) will be used frequently throughout the remainder of this book, and such steps will be taken hereafter without detailed explanation.

Let us use Theorem 20 on the hypergeometric function on the right in (4). Put

$$y = \frac{-z}{1-z}.$$

Then

$$F\left[\begin{matrix} a, c-b; & \\ & y \\ c; & \end{matrix}\right] = (1-y)^{-c+b} F\left[\begin{matrix} c-a, c-b; & \\ & \dfrac{-y}{1-y} \\ c; & \end{matrix}\right].$$

But $1 - y = (1-z)^{-1}$ and $-y/(1-y) = z$. Hence

$$F\left[\begin{matrix} a, c-b; & \\ & \dfrac{-z}{1-z} \\ c; & \end{matrix}\right] = (1-z)^{c-b} F\left[\begin{matrix} c-a, c-b; & \\ & z \\ c; & \end{matrix}\right],$$

which combines with Theorem 20 to yield the following result due to Euler.

THEOREM 21. *If* $|z| < 1$,

$$(5)\qquad F(a, b; c; z) = (1-z)^{c-a-b} F(c-a, c-b; c; z).$$

The identities in Theorems 20 and 21 are statements of equality among certain of the 24 Kummer solutions of the hypergeometric differential equation. In the terminology of IDE, pages 157–158, we have shown that IIIa = Va = IIIb. Alternate proofs of Theorems 20 and 21, making use of the differential equation, are left as exercises.

39. Relation between functions of z and $1 - z$. The hypergeometric differential equation

$$z(1-z)w'' + [c - (a+b+1)z]w' - abw = 0 \tag{1}$$

has, in $|1 - z| < 1$, the solution

$$w = F(a, b; a + b + 1 - c; 1 - z), \tag{2}$$

as indicated in IDE, page 157, formula IVa. The solution (2) can be obtained independently by placing $z = 1 - y$ in the differential equation (1) and observing that the transformed equation is also hypergeometric with parameters $a' = a$, $b' = b$, $c' = a + b + 1 - c$, and argument $y = 1 - z$.

We already know that in $|z| < 1$, the equation (1) has the linearly independent solutions

$$w_1 = F(a, b; c; z), \tag{3}$$

$$w_2 = z^{1-c}F(a + 1 - c, b + 1 - c; 2 - c; z). \tag{4}$$

Then there must exist constants A and B such that

$$\begin{aligned} F(a, b; a + b + 1 - c; 1 - z) &= AF(a, b; c; z) \\ &\quad + Bz^{1-c}F(a + 1 - c, b + 1 - c; 2 - c; z) \end{aligned} \tag{5}$$

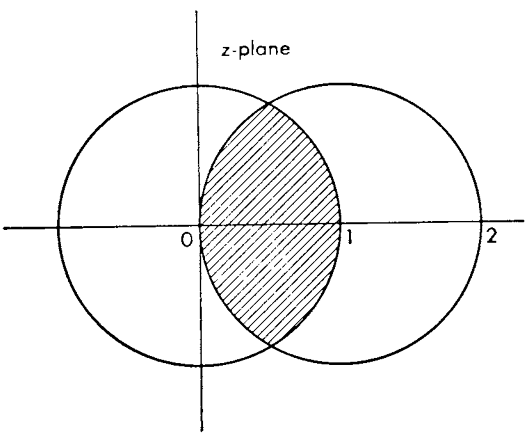

Figure 4

is an identity in the region (Figure 4) where both $|z| < 1$ and $|1 - z| < 1$. If we insist that $\text{Re}(1 - c) > 0$ and let $z \to 0$ from within the pertinent region, (5) yields

$$F(a, b; a + b + 1 - c; 1) = A \cdot 1 + B \cdot 0,$$

from which, by Theorem 18, page 49,

$$A = \frac{\Gamma(a+b+1-c)\,\Gamma(1-c)}{\Gamma(a+1-c)\,\Gamma(b+1-c)}. \tag{6}$$

Again from (5) if we let $z \to 1$ from inside the region and insist that $\mathrm{Re}(c-a-b) > 0$, we obtain

$$1 = AF(a, b; c; 1) + BF(a+1-c, b+1-c; 2-c; 1),$$

or, by Theorem 18,

$$\frac{\Gamma(c)\,\Gamma(c-a-b)A}{\Gamma(c-a)\,\Gamma(c-b)} + \frac{\Gamma(2-c)\,\Gamma(c-a-b)B}{\Gamma(1-a)\,\Gamma(1-b)} = 1. \tag{7}$$

Employing (6) in (7) leads to

$$\frac{\Gamma(2-c)\,\Gamma(c-a-b)B}{\Gamma(1-a)\,\Gamma(1-b)} = 1 - \frac{\Gamma(a+b+1-c)\,\Gamma(1-c)\,\Gamma(c)\,\Gamma(c-a-b)}{\Gamma(a+1-c)\,\Gamma(b+1-c)\,\Gamma(c-a)\,\Gamma(c-b)}. \tag{8}$$

In Ex. 15, page 32, we showed that the right member of (8) is equal to

$$\frac{\Gamma(2-c)\,\Gamma(c-1)\,\Gamma(c-a-b)\,\Gamma(a+b+1-c)}{\Gamma(a)\,\Gamma(1-a)\,\Gamma(b)\,\Gamma(1-b)}. \tag{9}$$

From (8) and (9) we obtain

$$B = \frac{\Gamma(a+b+1-c)\,\Gamma(c-1)}{\Gamma(a)\,\Gamma(b)}, \tag{10}$$

which completes the proof of the following statement.

THEOREM 22. *If* $|z| < 1$ *and* $|1-z| < 1$, *if* $\mathrm{Re}(c) < 1$ *and* $\mathrm{Re}\,(c-a-b) > 0$, *and if none of* $a, b, c, c-a, c-b, c-a-b$ *is an integer,*

$$\begin{aligned} (11)\quad & F(a, b; a+b+1-c; 1-z) \\ &= \frac{\Gamma(a+b+1-c)\,\Gamma(1-c)}{\Gamma(a+1-c)\,\Gamma(b+1-c)} \cdot F(a, b; c; z) \\ &+ \frac{\Gamma(a+b+1-c)\,\Gamma(c-1)}{\Gamma(a)\,\Gamma(b)} \cdot z^{1-c}F(a+1-c, b+1-c; 2-c; z). \end{aligned}$$

The restrictions on a, b, c can be relaxed somewhat, if desired. The expression of $F(a, b; c; z)$ as a linear combination of functions of $(1 - z)$ is left as an exercise; see Ex. 15 at the end of this chapter.

Theorems 20, 21, and 22 exhibit only three of the numerous relations among the 24 Kummer solutions. For other such relations see volume one of the Bateman Manuscript Project work, Erdélyi [1; 106–109].

40. A quadratic transformation. In any detailed study of the hypergeometric differential equation

$$z(1 - z)w'' + [c - (a + b + 1)z]w' - abw = 0, \tag{1}$$

the derivation of the 24 Kummer solutions is a natural result of the study of transformations of equation (1) into itself under linear fractional transformations on the independent variable. It is reasonable to attempt to use a quadratic transformation on the independent variable for the same purpose. Such a study shows that the parameters a, b, c need to be related for the new equation to be of hypergeometric character. Since differential equations are not our primary interest, we bypass the fairly simple determination of all such quadratic transformations and corresponding relations among a, b, and c. Here we move directly to the particular change of variables which leads to the relation we need for our later work.

In equation (1) put $c = 2b$ to get

$$z(1 - z)w'' + [2b - (a + b + 1)z]w' - abw = 0, \tag{2}$$

of which one solution is $w = F(a, b; 2b; z)$. Next let

$$z = \frac{4x}{(1 + x)^2}, \tag{3}$$

and obtain, after the usual labor involved in changing independent variables, the equation

$$x(1 - x)(1 + x)^2\frac{d^2w}{dx^2} + 2(1 + x)(b - 2ax + bx^2 - x^2)\frac{dw}{dx} - 4ab(1 - x)w = 0, \tag{4}$$

of which one solution is therefore

$$w = F\left[\begin{matrix} a, b; \\ 2b; \end{matrix} \quad \frac{4x}{(1 + x)^2}\right]. \tag{5}$$

In (4) put $w = (1 + x)^{2a}y$ to obtain the equation

$$x(1-x^2)y''+2[b-(2a-b+1)x^2]y'-2ax(1+2a-2b)y=0, \tag{6}$$

of which one solution is

$$(7) \qquad y = (1 + x)^{-2a} F\left[\begin{matrix} a, b; & \dfrac{4x}{(1+x)^2} \\ 2b; & \end{matrix}\right].$$

The differential equation (6) is invariant under a change from x to $(-x)$. Hence we introduce a new independent variable $v = x^2$. The equation in y and v is found to be

$$(8) \qquad v(1-v)\frac{d^2y}{dv^2} + \left[b + \frac{1}{2} - \left(2a - b + \frac{3}{2}\right)v\right]\frac{dy}{dv} - a\left(a - b + \frac{1}{2}\right)y = 0,$$

which has, in $|v| < 1$, the general solution

$$(9) \quad y = AF\left[\begin{matrix} a, a - b + \frac{1}{2}; & v \\ b + \frac{1}{2}; & \end{matrix}\right] + Bv^{\frac{1}{2}-b}F\left[\begin{matrix} a - b + \frac{1}{2}, a + 1 - 2b; & v \\ \frac{3}{2} - b; & \end{matrix}\right].$$

We now have the following situation. The differential equation (6) has a solution (7) valid in

$$\left|\frac{4x}{(1+x)^2}\right| < 1$$

as long as $2b$ is neither zero nor a negative integer. At the same time, equation (6) has the general solution* (9) with $v = x^2$, this solution valid in $|x| < 1$.

Therefore, if both $|x| < 1$ and $\left|\dfrac{4x}{(1+x)^2}\right| < 1$ and if $2b$ is neither zero nor a negative integer, there exist constants A and B such that

$$(10) \quad (1 + x)^{-2a} F\left[\begin{matrix} a, b; & \dfrac{4x}{(1+x)^2} \\ 2b; & \end{matrix}\right] = AF\left[\begin{matrix} a, a - b + \frac{1}{2}; & x^2 \\ b + \frac{1}{2}; & \end{matrix}\right]$$
$$+ Bx^{1-2b}F\left[\begin{matrix} a - b + \frac{1}{2}, a + 1 - 2b; & x^2 \\ \frac{3}{2} - b; & \end{matrix}\right].$$

In (10) the left member and the first term on the right are analytic at $x = 0$, but the last term is not analytic at $x = 0$ because of the

*If $2b$ is a positive integer, the second term on the right in (9) may or may not need to be replaced by a logarithmic solution. If such a logarithmic solution is involved in (9), reasoning parallel to that following equation (10) shows again that $B = 0$.

factor x^{1-2b}. Hence $B = 0$ and A is easily determined by using $x = 0$ in the resultant identity

$$(11)\quad (1+x)^{-2a}F\left[\begin{matrix} a, b; & \dfrac{4x}{(1+x)^2} \\ 2b; & \end{matrix}\right] = AF\left[\begin{matrix} a, a-b+\frac{1}{2}; & x^2 \\ b+\frac{1}{2}; & \end{matrix}\right].$$

Thus $A = 1$, and we are led to the following result due to Gauss.

THEOREM 23. *If $2b$ is neither zero nor a negative integer, and if both $|x| < 1$ and $|4x(1+x)^{-2}| < 1$,*

$$(12)\quad (1+x)^{-2a}F\left[\begin{matrix} a, b; & \dfrac{4x}{(1+x)^2} \\ 2b; & \end{matrix}\right] = F\left[\begin{matrix} a, a-b+\frac{1}{2}; & x^2 \\ b+\frac{1}{2}; & \end{matrix}\right].$$

41. Other quadratic transformations. For variety of technique we shall now prove the following theorem without recourse to the differential equation.

THEOREM 24. *If $2b$ is neither zero nor a negative integer and if $|y| < \frac{1}{2}$ and $|y/(1-y)| < 1$,*

$$(1)\quad (1-y)^{-a}F\left[\begin{matrix} \frac{1}{2}a, \frac{1}{2}a+\frac{1}{2}; & \dfrac{y^2}{(1-y)^2} \\ b+\frac{1}{2}; & \end{matrix}\right] = F\left[\begin{matrix} a, b; & 2y \\ 2b; & \end{matrix}\right].$$

Proof: Let ψ denote the left member of (1). Then

$$\psi = \sum_{k=0}^{\infty} \frac{(\frac{1}{2}a)_k(\frac{1}{2}a+\frac{1}{2})_k y^{2k}}{(b+\frac{1}{2})_k k!(1-y)^{a+2k}} = \sum_{k=0}^{\infty} \frac{(a)_{2k}y^{2k}}{2^{2k}(b+\frac{1}{2})_k k!(1-y)^{a+2k}}$$

with the aid of Lemma 5, page 22. Also

$$(1-y)^{-a-2k} = \sum_{n=0}^{\infty} \frac{(a+2k)_n y^n}{n!}$$

and $(a)_{2k}\,(a+2k)_n = (a)_{n+2k}$. Hence

$$\psi = \sum_{n,k=0}^{\infty} \frac{(a)_{n+2k}y^{n+2k}}{2^{2k}(b+\frac{1}{2})_k k!n!}.$$

Using Lemma 11, page 57, we may collect powers of y and obtain

$$\psi = \sum_{n=0}^{\infty}\sum_{k=0}^{[n/2]} \frac{(a)_n y^n}{2^{2k}(b+\frac{1}{2})_k k!(n-2k)!}.$$

We know that $(n-2k)! = n!/(-n)_{2k}$ and that

$$(-n)_{2k} = 2^{2k}(-\tfrac{1}{2}n)_k(-\tfrac{1}{2}n + \tfrac{1}{2})_k.$$

Therefore we have

$$\psi = \sum_{n=0}^{\infty}\sum_{k=0}^{[n/2]}\frac{(-\frac{1}{2}n)_k(-\frac{1}{2}n+\frac{1}{2})_k}{(b+\frac{1}{2})_k k!}\cdot\frac{(a)_n y^n}{n!}$$

$$= \sum_{n=0}^{\infty} F\left[\begin{matrix}-\frac{1}{2}n, -\frac{1}{2}n+\frac{1}{2}; & \\ & 1\\ b+\frac{1}{2}; & \end{matrix}\right]\frac{(a)_n y^n}{n!}.$$

In the example on page 49 we found that the terminating hypergeometric function above has the value $2^n(b)_n/(2b)_n$. Hence

$$\psi = \sum_{n=0}^{\infty}\frac{2^n(b)_n(a)_n y^n}{(2b)_n n!} = F\left[\begin{matrix}a, b; & \\ & 2y\\ 2b; & \end{matrix}\right],$$

which completes the proof of Theorem 24.

In Theorem 24 put $y = 2x/(1+x)^2$. Then

$$1 - y = \frac{1+x^2}{(1+x)^2}, \qquad \frac{y}{1-y} = \frac{2x}{1+x^2}$$

and we may write

$$(1+x^2)^{-a}(1+x)^{2a}F\left[\begin{matrix}\frac{1}{2}a, \frac{1}{2}a+\frac{1}{2}; & \\ & \dfrac{4x^2}{(1+x^2)^2}\\ b+\frac{1}{2}; & \end{matrix}\right] = F\left[\begin{matrix}a, b; & \\ & \dfrac{4x}{(1+x)^2}\\ 2b; & \end{matrix}\right].$$

In view of Theorem 23 we may now conclude that

$$(1+x^2)^{-a}F\left[\begin{matrix}\frac{1}{2}a, \frac{1}{2}a+\frac{1}{2}; & \\ & \dfrac{4x^2}{(1+x^2)^2}\\ b+\frac{1}{2}; & \end{matrix}\right] = F\left[\begin{matrix}a, a-b+\frac{1}{2}; & \\ & x^2\\ b+\frac{1}{2}; & \end{matrix}\right].$$

Now put $x^2 = z$ and replace b by $(\frac{1}{2}+a-b)$ to obtain

$$(2)\qquad (1+z)^{-a}F\left[\begin{matrix}\frac{1}{2}a, \frac{1}{2}a+\frac{1}{2}; & \\ & \dfrac{4z}{(1+z)^2}\\ 1+a-b; & \end{matrix}\right] = F\left[\begin{matrix}a, b; & \\ & z\\ 1+a-b; & \end{matrix}\right].$$

By Theorem 20, page 60, with appropriate substitutions for the a, b, c and z of the theorem,

$$F\left[\begin{matrix}\frac{1}{2}a, \frac{1}{2}a+\frac{1}{2}; & \\ & \dfrac{4z}{(1+z)^2}\\ 1+a-b; & \end{matrix}\right] = \left(\frac{1-z}{1+z}\right)^{-a}F\left[\begin{matrix}\frac{1}{2}a, \frac{1}{2}+\frac{1}{2}a-b; & \\ & \dfrac{-4z}{(1-z)^2}\\ 1+a-b; & \end{matrix}\right].$$

We may therefore rewrite (2) in the form

$$(3)\quad (1-z)^{-a}F\left[\begin{matrix}\tfrac{1}{2}a,\ \tfrac{1}{2}+\tfrac{1}{2}a-b; \\ 1+a-b;\end{matrix}\ \frac{-4z}{(1-z)^2}\right] = F\left[\begin{matrix}a,\ b; \\ 1+a-b;\end{matrix}\ z\right],$$

which will be useful in Section 42.

Let us next return to the differential equation to establish one more relation involving a quadratic transformation.

THEOREM 25. *If $a + b + \frac{1}{2}$ is neither zero nor a negative integer, and if $|x| < 1$ and $|4x(1-x)| < 1$,*

$$(4)\qquad F\left[\begin{matrix}a,\ b; \\ a+b+\tfrac{1}{2};\end{matrix}\ 4x(1-x)\right] = F\left[\begin{matrix}2a,\ 2b; \\ a+b+\tfrac{1}{2};\end{matrix}\ x\right].$$

The function

$$(5)\qquad y = F\left[\begin{matrix}a,\ b; \\ a+b+\tfrac{1}{2};\end{matrix}\ z\right]$$

is a solution of the differential equation

$$(6)\quad z(1-z)\frac{d^2y}{dz^2} + [a+b+\tfrac{1}{2}-(a+b+1)z]\frac{dy}{dz} - aby = 0.$$

In (6) put $z = 4x(1-x)$, and with some labor thus obtain

$$(7)\qquad x(1-x)y'' + [a+b+\tfrac{1}{2}-(2a+2b+1)x]y' - 4aby = 0.$$

Equation (7) is hypergeometric in character and has the general solution

$$y = AF\left[\begin{matrix}2a,\ 2b; \\ a+b+\tfrac{1}{2};\end{matrix}\ x\right] + Bx^{\frac{1}{2}-a-b}F\left[\begin{matrix}\tfrac{1}{2}+a-b,\ \tfrac{1}{2}+b-a; \\ \tfrac{3}{2}-a-b;\end{matrix}\ x\right],$$

as well as the solution

$$y = F\left[\begin{matrix}a,\ b; \\ a+b+\tfrac{1}{2};\end{matrix}\ 4x(1-x)\right]$$

from (5) above. By the usual argument it is easy to conclude the validity of (4).

42. A theorem due to Kummer. Let us return to equation (3) of the preceding section and let $z \to -1$. The result is

$$2^{-a}F\begin{bmatrix} \tfrac{1}{2}a, \tfrac{1}{2} + \tfrac{1}{2}a - b; & \\ & 1 \\ 1 + a - b; & \end{bmatrix} = F\begin{bmatrix} a, b; & \\ & -1 \\ 1 + a - b; & \end{bmatrix}.$$

We can sum the series on the left and thus obtain

$$(1)\qquad F\begin{bmatrix} a, b; & \\ & -1 \\ 1 + a - b; & \end{bmatrix} = \frac{\Gamma(1 + a - b)\Gamma(\tfrac{1}{2})}{2^{a}\Gamma(1 + \tfrac{1}{2}a - b)\Gamma(\tfrac{1}{2} + \tfrac{1}{2}a)}.$$

Legendre's duplication formula, page 24, yields

$$\Gamma(\tfrac{1}{2})\Gamma(1 + a) = 2^{a}\Gamma(\tfrac{1}{2} + \tfrac{1}{2}a)\Gamma(1 + \tfrac{1}{2}a),$$

which may be used on the right in (1).

Theorem 26. *If* $(1 + a - b)$ *is neither zero nor a negative integer, and* $\mathrm{Re}(b) < 1$ *for convergence,*

$$(2)\qquad F\begin{bmatrix} a, b; & \\ & -1 \\ 1 + a - b; & \end{bmatrix} = \frac{\Gamma(1 + a - b)\Gamma(1 + \tfrac{1}{2}a)}{\Gamma(1 + \tfrac{1}{2}a - b)\Gamma(1 + a)}.$$

43. Additional properties. Further results applying to special hypergeometric functions appear in later chapters, where we shall find that the polynomials of Legendre, Jacobi, Gegenbauer, and others are terminating hypergeometric series.

We now obtain one more identity as an example of those resulting from combinations of the theorems proved earlier in this chapter. In the identity of Theorem 25, page 67, replace a by $(\tfrac{1}{2}c - \tfrac{1}{2}a)$ and b by $(\tfrac{1}{2}c + \tfrac{1}{2}a - \tfrac{1}{2})$ to get

$$F\begin{bmatrix} \tfrac{1}{2}c - \tfrac{1}{2}a, \tfrac{1}{2}c + \tfrac{1}{2}a - \tfrac{1}{2}; & \\ & 4x(1 - x) \\ c; & \end{bmatrix} = F\begin{bmatrix} c - a, c + a - 1; & \\ & x \\ c; & \end{bmatrix}.$$

Theorem 21, page 60, yields

$$F\begin{bmatrix} c - a, c + a - 1; & \\ & x \\ c; & \end{bmatrix} = (1 - x)^{1-c}F\begin{bmatrix} a, 1 - a; & \\ & x \\ c; & \end{bmatrix},$$

which leads to the desired result.

THEOREM 27. *If c is neither zero nor a negative integer, and if both $|x| < 1$ and $|4x(1-x)| < 1$,*

$$F\left[\begin{matrix} a, 1-a; & \\ & x \\ c; & \end{matrix}\right] = (1-x)^{c-1} F\left[\begin{matrix} \frac{1}{2}c - \frac{1}{2}a, \frac{1}{2}c + \frac{1}{2}a - \frac{1}{2}; & \\ & 4x(1-x) \\ c; & \end{matrix}\right].$$

EXERCISES

1. Show that

$$\frac{d}{dx} F\left[\begin{matrix} a, b; & \\ & x \\ c; & \end{matrix}\right] = \frac{ab}{c} F\left[\begin{matrix} a+1, b+1; & \\ & x \\ c+1; & \end{matrix}\right].$$

2. Show that

$$F\left[\begin{matrix} 2a, 2b; & \\ & \frac{1}{2} \\ a+b+\frac{1}{2}; & \end{matrix}\right] = \frac{\Gamma(a+b+\frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(a+\frac{1}{2})\Gamma(b+\frac{1}{2})}.$$

3. Show that

$$F\left[\begin{matrix} a, 1-a; & \\ & \frac{1}{2} \\ c; & \end{matrix}\right] = \frac{2^{1-c}\Gamma(c)\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2}c + \frac{1}{2}a)\Gamma(\frac{1}{2}c - \frac{1}{2}a + \frac{1}{2})}.$$

4. Obtain the result

$$F\left[\begin{matrix} -n, b; & \\ & 1 \\ c; & \end{matrix}\right] = \frac{(c-b)_n}{(c)_n}.$$

5. Obtain the result

$$F\left[\begin{matrix} -n, a+n; & \\ & 1 \\ c; & \end{matrix}\right] = \frac{(-1)^n(1+a-c)_n}{(c)_n}.$$

6. Show that

$$F\left[\begin{matrix} -n, 1-b-n; & \\ & 1 \\ a; & \end{matrix}\right] = \frac{(a+b-1)_{2n}}{(a)_n(a+b-1)_n}.$$

7. Prove that if $g_n = F(-n, \alpha; 1+\alpha-n; 1)$ and α is not an integer, then $g_n = 0$ for $n \geqq 1$, $g_0 = 1$.

8. Show that

$$\frac{d^n}{dx^n}[x^{a-1+n}F(a, b; c; x)] = (a)_n x^{a-1} F(a+n, b; c; x).$$

9. Use equation (2), page 66, with $z = -x$, $b = -n$, in which n is a non-negative integer, to conclude that

$$F\left[\begin{matrix} -n, a; \\ 1+a+n; \end{matrix} \quad -x\right] = (1-x)^{-a}F\left[\begin{matrix} \frac{1}{2}a, \frac{1}{2}a+\frac{1}{2}; \\ 1+a+n; \end{matrix} \quad \frac{-4x}{(1-x)^2}\right].$$

10. In Theorem 23, page 65, put $b = \gamma$, $a = \gamma + \frac{1}{2}$, $4x(1+x)^{-2} = z$ and thus prove that

$$F\left[\begin{matrix} \gamma, \gamma+\frac{1}{2}; \\ 2\gamma; \end{matrix} \quad z\right] = (1-z)^{-\frac{1}{2}}\left[\frac{2}{1+\sqrt{1-z}}\right]^{2\gamma-1}$$

and further that

$$F\left[\begin{matrix} \gamma, \gamma-\frac{1}{2}; \\ 2\gamma; \end{matrix} \quad z\right] = \left(\frac{2}{1+\sqrt{1-z}}\right)^{2\gamma-1}.$$

11. Use Theorem 27, page 69, to show that

$$(1-x)^{1-c}F\left[\begin{matrix} a, 1-a; \\ c; \end{matrix} \quad x\right] = (1-2x)^{a-c}F\left[\begin{matrix} \frac{1}{2}c-\frac{1}{2}a, \frac{1}{2}c-\frac{1}{2}a+\frac{1}{2}; \\ c; \end{matrix} \quad \frac{4x(x-1)}{(1-2x)^2}\right].$$

12. In the differential equation (3), page 54, for

$$w = F(a, b; c; z)$$

introduce a new dependent variable u by $w = (1-z)^{-a}u$, thus obtaining

$$z(1-z)^2u'' + (1-z)[c + (a-b-1)z]u' + a(c-b)u = 0.$$

Next change the independent variable to x by putting $x = -z/(1-z)$. Show that the equation for u in terms of x is

$$x(1-x)\frac{d^2u}{dx^2} + [c - (a+c-b+1)x]\frac{du}{dx} - a(c-b)u = 0,$$

and thus derive the solution

$$w = (1-z)^{-a}F\left[\begin{matrix} a, c-b; \\ c; \end{matrix} \quad \frac{-z}{1-z}\right].$$

13. Use the result of Ex. 12 and the method of Section 40 to prove Theorem 20, page 60.

14. Prove Theorem 21, page 60, by the method suggested by Exs. 12 and 13.

15. Use the method of Section 39 to prove that if both $|z| < 1$ and $|1-z| < 1$, and if a, b, c are suitably restricted,

$$F\left[\begin{matrix} a, b; \\ c; \end{matrix} \quad z\right] = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}F\left[\begin{matrix} a, b; \\ a+b+1-c; \end{matrix} \quad 1-z\right]$$

$$+ \frac{\Gamma(c)\Gamma(a+b-c)(1-z)^{c-a-b}}{\Gamma(a)\Gamma(b)}F\left[\begin{matrix} c-a, c-b; \\ c-a-b+1; \end{matrix} \quad 1-z\right].$$

16. In a common notation for the Laplace transform

$$L\{F(t)\} = \int_0^\infty e^{-st}F(t)\,dt = f(s); \qquad L^{-1}\{f(s)\} = F(t).$$

Show that

$$L^{-1}\left\{\frac{1}{s}F\left[\begin{matrix} a, b; \\ \\ s+1; \end{matrix} \quad z\right]\right\} = F\left[\begin{matrix} a, b; \\ \\ 1; \end{matrix} \quad z(1-e^{-t})\right].$$

17. With the notation of Ex. 16 show that

$$L\{t^n \sin at\} = \frac{a\Gamma(n+2)}{s^{n+2}}F\left[\begin{matrix} 1+\frac{1}{2}n, \frac{3}{2}+\frac{1}{2}n; \\ \\ \frac{3}{2}; \end{matrix} \quad -\frac{a^2}{s^2}\right].$$

18. Obtain the results

$$\begin{aligned} \operatorname{Log}(1+x) &= xF(1, 1; 2; -x), \\ \operatorname{Arcsin} x &= xF(\tfrac{1}{2}, \tfrac{1}{2}; \tfrac{3}{2}; x^2), \\ \operatorname{Arctan} x &= xF(\tfrac{1}{2}, 1; \tfrac{3}{2}; -x^2). \end{aligned}$$

19. The complete elliptic integral of the first kind is

$$K = \int_0^{\frac{1}{2}\pi} \frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}}.$$

Show that $K = \frac{1}{2}\pi F(\frac{1}{2}, \frac{1}{2}; 1; k^2)$.

20. The complete elliptic integral of the second kind is

$$E = \int_0^{\frac{1}{2}\pi} \sqrt{1-k^2\sin^2\theta}\,d\theta.$$

Show that $E = \frac{1}{2}\pi F(\frac{1}{2}, -\frac{1}{2}; 1; k^2)$.

21. From the contiguous function relations

(1) $(a-b)F = aF(a+) - bF(b+)$,
(2) $(a-c+1)F = aF(a+) - (c-1)F(c-)$,
(3) $[a+(b-c)z]F = a(1-z)F(a+) - c^{-1}(c-a)(c-b)zF(c+)$,
(4) $(1-z)F = F(a-) - c^{-1}(c-b)zF(c+)$,
(5) $(1-z)F = F(b-) - c^{-1}(c-a)zF(c+)$, derived in Section 33,

obtain the remaining ten such relations:

(6) $[2a-c+(b-a)z]F = a(1-z)F(a+) - (c-a)F(a-)$,
(7) $(a+b-c)F = a(1-z)F(a+) - (c-b)F(b-)$,
(8) $(c-a-b)F = (c-a)F(a-) - b(1-z)F(b+)$,
(9) $(b-a)(1-z)F = (c-a)F(a-) - (c-b)F(b-)$,
(10) $[1-a+(c-b-1)z]F = (c-a)F(a-) - (c-1)(1-z)F(c-)$,
(11) $[2b-c+(a-b)z]F = b(1-z)F(b+) - (c-b)F(b-)$,
(12) $[b+(a-c)z]F = b(1-z)F(b+) - c^{-1}(c-a)(c-b)zF(c+)$,
(13) $(b-c+1)F = bF(b+) - (c-1)F(c-)$,
(14) $[1-b+(c-a-1)z]F = (c-b)F(b-) - (c-1)(1-z)F(c-)$,
(15) $[c-1+(a+b+1-2c)z]F = (c-1)(1-z)F(c-) - c^{-1}(c-a)(c-b)zF(c+)$.

22. The notation used in Ex. 21 and in Section 33 is often extended as in the examples

$$F(a-, b+) = F(a-1, b+1; c; z),$$
$$F(b+, c+) = F(a, b+1; c+1; z).$$

Use the relations (4) and (5) of Ex. 21 to obtain

$$F(a-) - F(b-) + c^{-1}(b-a)zF(c+) = 0$$

and from it, by changing b to $(b+1)$ arrive at

$$F = F(a-, b+) + c^{-1}(b+1-a)zF(b+, c+),$$

a relation we wish to use in Chapter 16.

23. In equation (9) of Ex. 21 shift b to $(b+1)$ to obtain the relation

$$(c-1-b)F = (c-a)F(a-, b+) + (a-1-b)(1-z)F(b+),$$

or

$$(c-1-b)F(a, b; c; z)$$
$$= (c-a)F(a-1, b+1; c; z) + (a-1-b)(1-z)F(a, b+1; c; z),$$

another relation we wish to use in Chapter 16.

1

BASIC HYPERGEOMETRIC SERIES

1.1 Introduction

Our main objective in this chapter is to present the definitions and notations for hypergeometric and basic hypergeometric series, and to derive the elementary formulas that form the basis for most of the summation, transformation and expansion formulas, basic integrals, and applications to orthogonal polynomials and to other fields that follow in the subsequent chapters. We begin by defining Gauss' ${}_2F_1$ hypergeometric series, the ${}_rF_s$ (generalized) hypergeometric series, and pointing out some of their most important special cases. Next we define Heine's ${}_2\phi_1$ basic hypergeometric series which contains an additional parameter q, called the base, and then give the definition and notations for ${}_r\phi_s$ basic hypergeometric series. Basic hypergeometric series are called q-analogues (basic analogues or q-extensions) of hypergeometric series because an ${}_rF_s$ series can be obtained as the $q \to 1$ limit case of an ${}_r\phi_s$ series.

Since the binomial theorem is at the foundation of most of the summation formulas for hypergeometric series, we then derive a q-analogue of it, called the q-binomial theorem, and use it to derive Heine's q-analogues of Euler's transformation formulas, Jacobi's triple product identity, and summation formulas that are q-analogues of those for hypergeometric series due to Chu and Vandermonde, Gauss, Kummer, Pfaff and Saalschütz, and to Karlsson and Minton. We also introduce q-analogues of the exponential, gamma and beta functions, as well as the concept of a q-integral that allows us to give a q-analogue of Euler's integral representation of a hypergeometric function. Many additional formulas and q-analogues are given in the exercises at the end of the chapter.

1.2 Hypergeometric and basic hypergeometric series

In 1812, Gauss presented to the Royal Society of Sciences at Göttingen his famous paper (Gauss [1813]) in which he considered the infinite series

$$1+\frac{ab}{1\cdot c}z+\frac{a(a+1)b(b+1)}{1\cdot 2\cdot c(c+1)}z^2+\frac{a(a+1)(a+2)b(b+1)(b+2)}{1\cdot 2\cdot 3\cdot c(c+1)(c+2)}z^3+\cdots \quad (1.2.1)$$

as a function of a, b, c, z, where it is assumed that $c \neq 0, -1, -2, \ldots$, so that no zero factors appear in the denominators of the terms of the series. He showed that the series converges absolutely for $|z| < 1$, and for $|z| = 1$ when Re $(c - a - b) > 0$, gave its (contiguous) recurrence relations, and derived his famous formula (see (1.2.11) below) for the sum of this series when $z = 1$ and Re $(c - a - b) > 0$.

Although Gauss used the notation $F(a,b,c,z)$ for his series, it is now customary to use $F(a,b;c;z)$ or either of the notations

$$ {}_2F_1(a,\ b;\ c;\ z), \qquad {}_2F_1\left[\begin{matrix} a,b \\ c \end{matrix};z\right] $$

for this series (and for its sum when it converges), because these notations separate the numerator parameters a,b from the denominator parameter c and the variable z. In view of Gauss' paper, his series is frequently called *Gauss' series.* However, since the special case $a=1, b=c$ yields the geometric series

$$ 1+z+z^2+z^3+\cdots, $$

Gauss' series is also called the *(ordinary) hypergeometric series* or the *Gauss hypergeometric series.*

Some important functions which can be expressed by means of Gauss' series are

$$ \begin{aligned} (1+z)^a &= F(-a,b;b;-z), \\ \log(1+z) &= zF(1,1;2;-z), \\ \sin^{-1} z &= zF(1/2,1/2;3/2;z^2), \\ \tan^{-1} z &= zF(1/2,1;3/2;-z^2), \\ e^z &= \lim_{a\to\infty} F(a,b;b;z/a), \end{aligned} \tag{1.2.2} $$

where $|z|<1$ in the first four formulas. Also expressible by means of Gauss' series are the classical orthogonal polynomials, such as the *Tchebichef polynomials of the first and second kinds*

$$ T_n(x) = F(-n,n;1/2;(1-x)/2), \tag{1.2.3} $$

$$ U_n(x) = (n+1)F(-n,n+2;3/2;(1-x)/2), \tag{1.2.4} $$

the *Legendre polynomials*

$$ P_n(x) = F(-n,n+1;1;(1-x)/2), \tag{1.2.5} $$

the *Gegenbauer (ultraspherical) polynomials*

$$ C_n^\lambda(x) = \frac{(2\lambda)_n}{n!}\, F(-n,n+2\lambda;\lambda+1/2;(1-x)/2), \tag{1.2.6} $$

and the more general *Jacobi polynomials*

$$ P_n^{(\alpha,\beta)}(x) = \frac{(\alpha+1)_n}{n!}\, F(-n,n+\alpha+\beta+1;\ \alpha+1;(1-x)/2), \tag{1.2.7} $$

where $n=0,1,\ldots,$ and $(a)_n$ denotes the *shifted factorial* defined by

$$ (a)_0=1,\ (a)_n = a(a+1)\cdots(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)}, \quad n=1,2,\ldots. \tag{1.2.8} $$

Before Gauss, Chu [1303] (see Needham [1959, p. 138], Takács [1973] and Askey [1975, p. 59]) and Vandermonde [1772] had proved the summation formula

$$ F(-n,b;c;1) = \frac{(c-b)_n}{(c)_n}, \quad n=0,1,\ldots, \tag{1.2.9} $$

which is now called *Vandermonde's formula* or the *Chu–Vandermonde formula*, and Euler [1748] had derived several results for hypergeometric series, including his transformation formula

$$F(a,b;c;z) = (1-z)^{c-a-b}\, F(c-a,c-b;c;z), \quad |z|<1. \tag{1.2.10}$$

Formula (1.2.9) is the terminating case $a=-n$ of the summation formula

$$F(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \quad \mathrm{Re}(c-a-b)>0, \tag{1.2.11}$$

which Gauss proved in his paper.

Thirty-three years after Gauss' paper, Heine [1846, 1847, 1878] introduced the series

$$1+\frac{(1-q^a)(1-q^b)}{(1-q)(1-q^c)}z+\frac{(1-q^a)(1-q^{a+1})(1-q^b)(1-q^{b+1})}{(1-q)(1-q^2)(1-q^c)(1-q^{c+1})}z^2+\cdots, \tag{1.2.12}$$

where it is assumed that $q\neq 1$, $c\neq 0,-1,-2,\ldots$ and the principal value of each power of q is taken. This series converges absolutely for $|z|<1$ when $|q|<1$ and it tends (at least termwise) to Gauss' series as $q\to 1$, because

$$\lim_{q\to 1}\frac{1-q^a}{1-q} = a. \tag{1.2.13}$$

The series in (1.2.12) is usually called *Heine's series* or, in view of the base q, the *basic hypergeometric series* or *q-hypergeometric series.*

Analogous to Gauss' notation, Heine used the notation $\phi(a,b,c,q,z)$ for his series. However, since one would like to also be able to consider the case when q to the power a, b, or c is replaced by zero, it is now customary to define the *basic hypergeometric series* by

$$\begin{aligned}\phi(a,b;c;q,z) &\equiv {}_2\phi_1(a,b;c;q,z) \equiv {}_2\phi_1\left[\begin{matrix}a,b\\c\end{matrix};q,z\right]\\ &= \sum_{n=0}^{\infty}\frac{(a;q)_n(b;q)_n}{(q;q)_n(c;q)_n}z^n,\end{aligned} \tag{1.2.14}$$

where

$$(a;q)_n = \begin{cases}1, & n=0,\\ (1-a)(1-aq)\cdots(1-aq^{n-1}), & n=1,2,\ldots,\end{cases} \tag{1.2.15}$$

is the *q-shifted factorial* and it is assumed that $c\neq q^{-m}$ for $m=0,1,\ldots$. Some other notations that have been used in the literature for the product $(a;q)_n$ are $(a)_{q,n}$, $[a]_n$, and even $(a)_n$ when (1.2.8) is not used and the base is not displayed.

Another generalization of Gauss' series is the *(generalized) hypergeometric series* with r numerator parameters $a_1,\ldots,a_r$ and s denominator parameters $b_1,\ldots,b_s$ defined by

$$\begin{aligned}&{}_rF_s(a_1,a_2,\ldots,a_r;\ b_1,\ldots,b_s;z) \equiv {}_rF_s\left[\begin{matrix}a_1,a_2,\ldots,a_r\\ b_1,\ldots,b_s\end{matrix};z\right]\\ &= \sum_{n=0}^{\infty}\frac{(a_1)_n(a_2)_n\cdots(a_r)_n}{n!(b_1)_n\cdots(b_s)_n}z^n.\end{aligned} \tag{1.2.16}$$

Some well-known special cases are the *exponential function*

$$e^z = {}_0F_0(—;—;z), \tag{1.2.17}$$

the *trigonometric functions*

$$\begin{aligned} \sin z &= z\,{}_0F_1(—;3/2;-z^2/4), \\ \cos z &= {}_0F_1(—;1/2;-z^2/4), \end{aligned} \tag{1.2.18}$$

the *Bessel function*

$$J_\alpha(z) = (z/2)^\alpha\,{}_0F_1(—;\alpha+1;-z^2/4)/\Gamma(\alpha+1), \tag{1.2.19}$$

where a dash is used to indicate the absence of either numerator (when $r=0$) or denominator (when $s=0$) parameters. Some other well-known special cases are the *Hermite polynomials*

$$H_n(x) = (2x)^n\,{}_2F_0(-n/2,(1-n)/2;—;-x^{-2}), \tag{1.2.20}$$

and the *Laguerre polynomials*

$$L_n^\alpha(x) = \frac{(\alpha+1)_n}{n!}\,{}_1F_1(-n;\alpha+1;x). \tag{1.2.21}$$

Generalizing Heine's series, we shall define an ${}_r\phi_s$ *basic hypergeometric series* by

$$\begin{aligned} &{}_r\phi_s(a_1,a_2,\ldots,a_r;b_1,\ldots,b_s;q,z) \equiv {}_r\phi_s\left[\begin{matrix} a_1,a_2,\ldots,a_r \\ b_1,\ldots,b_s \end{matrix};q,z\right] \\ &= \sum_{n=0}^{\infty} \frac{(a_1;q)_n(a_2;q)_n\cdots(a_r;q)_n}{(q;q)_n(b_1;q)_n\cdots(b_s;q)_n}\left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r} z^n \end{aligned} \tag{1.2.22}$$

with $\binom{n}{2} = n(n-1)/2$, where $q \neq 0$ when $r > s+1$.

In (1.2.16) and (1.2.22) it is assumed that the parameters $b_1,\ldots,b_s$ are such that the denominator factors in the terms of the series are never zero. Since

$$(-m)_n = \left(q^{-m};q\right)_n = 0, \quad n = m+1, m+2, \ldots, \tag{1.2.23}$$

an ${}_rF_s$ series terminates if one of its numerator parameters is zero or a negative integer, and an ${}_r\phi_s$ series terminates if one of its numerator parameters is of the form q^{-m} with $m = 0,1,2,\ldots$, and $q \neq 0$. Basic analogues of the classical orthogonal polynomials will be considered in Chapter 7 as well as in the exercises at the ends of the chapters.

Unless stated otherwise, when dealing with nonterminating basic hypergeometric series we shall assume that $|q| < 1$ and that the parameters and variables are such that the series converges absolutely. Note that if $|q| > 1$, then we can perform an inversion with respect to the base by setting $p = q^{-1}$ and using the identity

$$(a;q)_n = (a^{-1};p)_n(-a)^n p^{-\binom{n}{2}} \tag{1.2.24}$$

to convert the series (1.2.22) to a similar series in base p with $|p| < 1$ (see Ex. 1.4(i)). The inverted series will have a finite radius of convergence if the original series does.

Observe that if we denote the terms of the series (1.2.16) and (1.2.22) which contain z^n by u_n and v_n, respectively, then

$$\frac{u_{n+1}}{u_n} = \frac{(a_1+n)(a_2+n)\cdots(a_r+n)}{(1+n)(b_1+n)\cdots(b_s+n)}\, z \tag{1.2.25}$$

is a rational function of n, and

$$\frac{v_{n+1}}{v_n} = \frac{(1-a_1q^n)(1-a_2q^n)\cdots(1-a_rq^n)}{(1-q^{n+1})(1-b_1q^n)\cdots(1-b_sq^n)}\left(-q^n\right)^{1+s-r} z \tag{1.2.26}$$

is a rational function of q^n. Conversely, if $\sum_{n=0}^{\infty} u_n$ and $\sum_{n=0}^{\infty} v_n$ are power series with $u_0 = v_0 = 1$ such that u_{n+1}/u_n is a rational function of n and v_{n+1}/v_n is a rational function of q^n, then these series are of the forms (1.2.16) and (1.2.22), respectively.

By the ratio test, the ${}_rF_s$ series converges absolutely for all z if $r \le s$, and for $|z| < 1$ if $r = s+1$. By an extension of the ratio test (Bromwich [1959, p. 241]), it converges absolutely for $|z| = 1$ if $r = s+1$ and Re $[b_1 + \cdots + b_s - (a_1 + \cdots + a_r)] > 0$. If $r > s+1$ and $z \ne 0$ or $r = s+1$ and $|z| > 1$, then this series diverges, unless it terminates.

If $0 < |q| < 1$, the ${}_r\phi_s$ series converges absolutely for all z if $r \le s$ and for $|z| < 1$ if $r = s+1$. This series also converges absolutely if $|q| > 1$ and $|z| < |b_1b_2\cdots b_sq|/|a_1a_2\cdots a_r|$. It diverges for $z \ne 0$ if $0 < |q| < 1$ and $r > s+1$, and if $|q| > 1$ and $|z| > |b_1b_2\cdots b_sq|/|a_1a_2\cdots a_r|$, unless it terminates. As is customary, the ${}_rF_s$ and ${}_r\phi_s$ notations are also used for the sums of these series inside the circle of convergence and for their analytic continuations (called *hypergeometric functions* and *basic hypergeometric functions*, respectively) outside the circle of convergence.

Observe that the series (1.2.22) has the property that if we replace z by z/a_r and let $a_r \to \infty$, then the resulting series is again of the form (1.2.22) with r replaced by $r-1$. Because this is not the case for the ${}_r\phi_s$ series defined without the factors $\left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r}$ in the books of Bailey [1935] and Slater [1966] and we wish to be able to handle such limit cases, we have chosen to use the series defined in (1.2.22). There is no loss in generality since the Bailey and Slater series can be obtained from the $r = s+1$ case of (1.2.22) by choosing s sufficiently large and setting some of the parameters equal to zero.

An ${}_{r+1}F_r$ series is called *k-balanced* if $b_1 + b_2 + \cdots + b_r = k + a_1 + a_2 + \cdots + a_{r+1}$ and $z = 1$; a 1-balanced series is called *balanced* (or *Saalschützian*). Analogously, an ${}_{r+1}\phi_r$ series is called *k-balanced* if $b_1b_2\cdots b_r = q^k a_1a_2\cdots a_{r+1}$ and $z = q$, and a 1-balanced series is called *balanced* (or *Saalschützian*). We will first encounter balanced series in §1.7, where we derive a summation formula for such a series.

For negative subscripts, the *shifted factorial* and the *q-shifted factorials* are defined by

$$(a)_{-n} = \frac{1}{(a-1)(a-2)\cdots(a-n)} = \frac{1}{(a-n)_n} = \frac{(-1)^n}{(1-a)_n}, \tag{1.2.27}$$

$$(a;q)_{-n} = \frac{1}{(1-aq^{-1})(1-aq^{-2})\cdots(1-aq^{-n})} = \frac{1}{(aq^{-n};q)_n} = \frac{(-q/a)^n q^{\binom{n}{2}}}{(q/a;q)_n}, \tag{1.2.28}$$

where $n = 0, 1, \ldots$. We also define

$$(a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k) \tag{1.2.29}$$

for $|q| < 1$. Since the infinite product in (1.2.29) diverges when $a \neq 0$ and $|q| \geq 1$, whenever $(a;q)_\infty$ appears in a formula, we shall assume that $|q| < 1$. The following easily verified identities will be frequently used in this book:

$$(a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty}, \tag{1.2.30}$$

$$(a^{-1}q^{1-n};q)_n = (a;q)_n(-a^{-1})^n q^{-\binom{n}{2}}, \tag{1.2.31}$$

$$(a;q)_{n-k} = \frac{(a;q)_n}{(a^{-1}q^{1-n};q)_k}\,(-qa^{-1})^k q^{\binom{k}{2}-nk}, \tag{1.2.32}$$

$$(a;q)_{n+k} = (a;q)_n(aq^n;q)_k, \tag{1.2.33}$$

$$(aq^n;q)_k = \frac{(a;q)_k(aq^k;q)_n}{(a;q)_n}, \tag{1.2.34}$$

$$(aq^k;q)_{n-k} = \frac{(a;q)_n}{(a;q)_k}, \tag{1.2.35}$$

$$(aq^{2k};q)_{n-k} = \frac{(a;q)_n(aq^n;q)_k}{(a;q)_{2k}}, \tag{1.2.36}$$

$$(q^{-n};q)_k = \frac{(q;q)_n}{(q;q)_{n-k}}(-1)^k q^{\binom{k}{2}-nk}, \tag{1.2.37}$$

$$(aq^{-n};q)_k = \frac{(a;q)_k(qa^{-1};q)_n}{(a^{-1}q^{1-k};q)_n}q^{-nk}, \tag{1.2.38}$$

$$(a;q)_{2n} = (a;q^2)_n(aq;q^2)_n, \tag{1.2.39}$$

$$(a^2;q^2)_n = (a;q)_n(-a;q)_n, \tag{1.2.40}$$

where n and k are integers. A more complete list of useful identities is given in Appendix I at the end of the book.

Since products of q-shifted factorials occur so often, to simplify them we shall frequently use the more compact notations

$$(a_1, a_2, \ldots, a_m;q)_n = (a_1;q)_n(a_2;q)_n\cdots(a_m;q)_n, \tag{1.2.41}$$

$$(a_1, a_2, \ldots, a_m;q)_\infty = (a_1;q)_\infty(a_2;q)_\infty\cdots(a_m;q)_\infty. \tag{1.2.42}$$

The ratio $(1-q^a)/(1-q)$ considered in (1.2.13) is called a *q-number* (or *basic number*) and it is denoted by

$$[a]_q = \frac{1-q^a}{1-q}, \quad q \neq 1. \tag{1.2.43}$$

It is also called a q-analogue, q-deformation, q-extension, or a q-generalization of the complex number a. In terms of q-numbers the *q-number factorial* $[n]_q!$ is defined for a nonnegative integer n by

$$[n]_q! = \prod_{k=1}^{n} [k]_q, \tag{1.2.44}$$

and the corresponding *q-number shifted factorial* is defined by

$$[a]_{q;n} = \prod_{k=0}^{n-1} [a+k]_q. \tag{1.2.45}$$

Clearly,

$$\lim_{q\to 1} [n]_q! = n!, \quad \lim_{q\to 1} [a]_q = a, \tag{1.2.46}$$

and

$$[a]_{q;n} = (1-q)^{-n}(q^a;q)_n, \quad \lim_{q\to 1} [a]_{q;n} = (a)_n. \tag{1.2.47}$$

Corresponding to (1.2.41) we can use the compact notation

$$[a_1, a_2, \ldots, a_m]_{q;n} = [a_1]_{q;n}[a_2]_{q;n} \cdots [a_m]_{q;n}. \tag{1.2.48}$$

Since

$$\begin{aligned} &\sum_{n=0}^{\infty} \frac{[a_1, a_2, \ldots, a_r]_{q;n}}{[n]_q![b_1, \ldots, b_s]_{q;n}} \left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r} z^n \\ &= {}_r\phi_s\left(q^{a_1}, q^{a_2}, \ldots, q^{a_r}; q^{b_1}, \ldots, q^{b_s}; q, z(1-q)^{1+s-r}\right), \end{aligned} \tag{1.2.49}$$

anyone working with q-numbers and the *q-number hypergeometric series* on the left-hand side of (1.2.49) can use the formulas for ${}_r\phi_s$ series in this book that have no zero parameters by replacing the parameters by q^{n} powers and applying (1.2.49).

As in Frenkel and Turaev [1995] one can define a *trigonometric number* $[a;\sigma]$ by

$$[a;\sigma] = \frac{\sin(\pi\sigma a)}{\sin(\pi\sigma)} \tag{1.2.50}$$

for noninteger values of σ and view $[a;\sigma]$ as a *trigonometric deformation* of a since $\lim_{\sigma\to 0}[a;\sigma] = a$. The corresponding ${}_rt_s$ *trigonometric hypergeometric series* can be defined by

$$\begin{aligned} &{}_rt_s(a_1, a_2, \ldots, a_r; b_1, \ldots, b_s; \sigma, z) \\ &= \sum_{n=0}^{\infty} \frac{[a_1, a_2, \ldots, a_r; \sigma]_n}{[n;\sigma]![b_1, \ldots, b_s; \sigma]_n} \left[(-1)^n e^{\pi i \sigma \binom{n}{2}}\right]^{1+s-r} z^n, \end{aligned} \tag{1.2.51}$$

where

$$[n;\sigma]! = \prod_{k=1}^{n}[k;\sigma], \quad [a;\sigma]_n = \prod_{k=0}^{n-1}[a+k;\sigma], \tag{1.2.52}$$

and

$$[a_1,a_2,\ldots,a_m;\sigma]_n = [a_1;\sigma]_n[a_2;\sigma]_n\cdots[a_m;\sigma]_n. \tag{1.2.53}$$

From

$$[a;\sigma] = \frac{e^{\pi i\sigma a}-e^{-\pi i\sigma a}}{e^{\pi i\sigma}-e^{-\pi i\sigma}} = \frac{q^{a/2}-q^{-a/2}}{q^{1/2}-q^{-1/2}} = \frac{1-q^a}{1-q}q^{(1-a)/2}, \tag{1.2.54}$$

where $q=e^{2\pi i\sigma}$, it follows that

$$[a;\sigma]_n = \frac{(q^a;q)_n}{(1-q)^n}q^{n(1-a)/2-n(n-1)/4}, \tag{1.2.55}$$

and hence

$$\begin{aligned} &{}_rt_s(a_1,a_2,\ldots,a_r;b_1,\ldots,b_s;\sigma,z)\\ &= {}_r\phi_s(q^{a_1},q^{a_2},\ldots,q^{a_r};q^{b_1},\ldots,q^{b_s};q,cz) \end{aligned} \tag{1.2.56}$$

with

$$c = (1-q)^{1+s-r}q^{r/2-s/2+(b_1+\cdots+b_s)/2-(a_1+\cdots+a_r)/2}, \tag{1.2.57}$$

which shows that the ${}_rt_s$ series is equivalent to the ${}_r\phi_s$ series in (1.2.49).

Elliptic numbers $[a;\sigma,\tau]$, which are a one-parameter generalization (deformation) of trigonometric numbers, are considered in §1.6, and the corresponding elliptic (and theta) hypergeometric series and their summation and transformation formulas are considered in Chapter 11.

We close this section with two identities involving ordinary binomial coefficients, which are particularly useful in handling some powers of q that arise in the derivations of many formulas containing q-series:

$$\binom{n+k}{2} = \binom{n}{2}+\binom{k}{2}+kn, \tag{1.2.58}$$

$$\binom{n-k}{2} = \binom{n}{2}+\binom{k}{2}+k-kn. \tag{1.2.59}$$

1.3 The q-binomial theorem

One of the most important summation formulas for hypergeometric series is given by the *binomial theorem:*

$${}_2F_1(a,c;c;z) = {}_1F_0(a;—;z) = \sum_{n=0}^{\infty}\frac{(a)_n}{n!}z^n = (1-z)^{-a}, \tag{1.3.1}$$

where $|z|<1$. We shall show that this formula has the following q-analogue

$${}_1\phi_0(a;—;q,z) = \sum_{n=0}^{\infty}\frac{(a;q)_n}{(q;q)_n}z^n = \frac{(az;q)_\infty}{(z;q)_\infty}, \quad |z|<1,\ |q|<1, \tag{1.3.2}$$

which was derived by Cauchy [1843], Heine [1847] and by other mathematicians. See Askey [1980a], which also cites the books by Rothe [1811] and Schweins [1820], and the remark on p. 491 of Andrews, Askey, and Roy [1999] concerning the terminating form of the q-binomial theorem in Rothe [1811].

Heine's proof of (1.3.2), which can also be found in the books Heine [1878], Bailey [1935, p. 66] and Slater [1966, p. 92], is better understood if one first follows Askey's [1980a] approach of evaluating the sum of the binomial series in (1.3.1), and then carries out the analogous steps for the series in (1.3.2).

Let us set

$$f_a(z) = \sum_{n=0}^{\infty} \frac{(a)_n}{n!} z^n. \tag{1.3.3}$$

Since this series is uniformly convergent in $|z| \leq \epsilon$ when $0 < \epsilon < 1$, we may differentiate it termwise to get

$$\begin{aligned} f_a'(z) &= \sum_{n=1}^{\infty} \frac{n(a)_n}{n!} z^{n-1} \\ &= \sum_{n=0}^{\infty} \frac{(a)_{n+1}}{n!} z^n = a f_{a+1}(z). \end{aligned} \tag{1.3.4}$$

Also

$$\begin{aligned} f_a(z) - f_{a+1}(z) &= \sum_{n=1}^{\infty} \frac{(a)_n - (a+1)_n}{n!} z^n \\ &= \sum_{n=1}^{\infty} \frac{(a+1)_{n-1}}{n!} [a - (a+n)]\, z^n = -\sum_{n=1}^{\infty} \frac{n(a+1)_{n-1}}{n!} z^n \\ &= -\sum_{n=0}^{\infty} \frac{(a+1)_n}{n!} z^{n+1} = -z f_{a+1}(z). \end{aligned} \tag{1.3.5}$$

Eliminating $f_{a+1}(z)$ from (1.3.4) and (1.3.5), we obtain the first order differential equation

$$f_a'(z) = \frac{a}{1-z} f_a(z), \tag{1.3.6}$$

subject to the initial condition $f_a(0) = 1$, which follows from the definition (1.3.3) of $f_a(z)$. Solving (1.3.6) under this condition immediately gives that $f_a(z) = (1-z)^{-a}$ for $|z| < 1$.

Analogously, let us now set

$$h_a(z) = \sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} z^n, \quad |z| < 1,\ |q| < 1. \tag{1.3.7}$$

Clearly, $h_{q^a}(z) \to f_a(z)$ as $q \to 1$. Since $h_{aq}(z)$ is a q-analogue of $f_{a+1}(z)$, we first compute the difference

$$h_a(z) - h_{aq}(z) = \sum_{n=1}^{\infty} \frac{(a;q)_n - (aq;q)_n}{(q;q)_n} z^n$$

$$
\begin{aligned}
&= \sum_{n=1}^{\infty} \frac{(aq;q)_{n-1}}{(q;q)_n} \left[1 - a - (1 - aq^n)\right] z^n \\
&= -a \sum_{n=1}^{\infty} \frac{(1-q^n)(aq;q)_{n-1}}{(q;q)_n} z^n \\
&= -a \sum_{n=1}^{\infty} \frac{(aq;q)_{n-1}}{(q;q)_{n-1}} z^n = -azh_{aq}(z),
\end{aligned} \tag{1.3.8}
$$

giving an analogue of (1.3.5). Observing that

$$
f'(z) = \lim_{q\to 1} \frac{f(z) - f(qz)}{(1-q)z} \tag{1.3.9}
$$

for a differentiable function f, we next compute the difference

$$
\begin{aligned}
h_a(z) - h_a(qz) &= \sum_{n=1}^{\infty} \frac{(a;q)_n}{(q;q)_n} (z^n - q^n z^n) \\
&= \sum_{n=1}^{\infty} \frac{(a;q)_n}{(q;q)_{n-1}} z^n = \sum_{n=0}^{\infty} \frac{(a;q)_{n+1}}{(q;q)_n} z^{n+1} \\
&= (1-a) z h_{aq}(z).
\end{aligned} \tag{1.3.10}
$$

Eliminating $h_{aq}(z)$ from (1.3.8) and (1.3.10) gives

$$
h_a(z) = \frac{1-az}{1-z} h_a(qz). \tag{1.3.11}
$$

Iterating this relation $n-1$ times and then letting $n \to \infty$ we obtain

$$
\begin{aligned}
h_a(z) &= \frac{(az;q)_n}{(z;q)_n} h_a(q^n z) \\
&= \frac{(az;q)_\infty}{(z;q)_\infty} h_a(0) = \frac{(az;q)_\infty}{(z;q)_\infty},
\end{aligned} \tag{1.3.12}
$$

since $q^n \to 0$ as $n \to \infty$ and $h_a(0) = 1$ by (1.3.7), which completes the proof of (1.3.2).

One consequence of (1.3.2) is the product formula

$$
{}_1\phi_0(a;—;q,z)\ {}_1\phi_0(b;—;q,az) = {}_1\phi_0(ab;—;q,z), \tag{1.3.13}
$$

which is a q-analogue of $(1-z)^{-a}(1-z)^{-b} = (1-z)^{-a-b}$.

In the special case $a = q^{-n}, n = 0, 1, 2, \ldots$, (1.3.2) gives

$$
{}_1\phi_0(q^{-n};—;q,z) = (zq^{-n};q)_n = (-z)^n q^{-n(n+1)/2} (q/z;q)_n, \tag{1.3.14}
$$

where, by analytic continuation, z can be any complex number. From now on, unless stated othewise, whenever $q^{-j}, q^{-k}, q^{-m}, q^{-n}$ appear as numerator parameters in basic series it will be assumed that j, k, m, n, respectively, are nonnegative integers.

If we set $a = 0$ in (1.3.2), we get

$$
{}_1\phi_0(0;—;q,z) = \sum_{n=0}^{\infty} \frac{z^n}{(q;q)_n} = \frac{1}{(z;q)_\infty}, \ |z| < 1, \tag{1.3.15}
$$

which is a q-analogue of the exponential function e^z. Another q-analogue of e^z can be obtained from (1.3.2) by replacing z by $-z/a$ and then letting $a \to \infty$ to get

$$ {}_0\phi_0(—;—;q,-z) = \sum_{n=0}^{\infty} \frac{q^{n(n-1)/2}}{(q;q)_n} z^n = (-z;q)_\infty. \tag{1.3.16}$$

Observe that if we denote the *q-exponential* functions in (1.3.15) and (1.3.16) by $e_q(z)$ and $E_q(z)$, respectively, then $e_q(z)E_q(-z) = 1$, $e_{q^{-1}}(z) = E_q(-qz)$ by (1.2.24), and

$$ \lim_{q\to 1^-} e_q(z(1-q)) = \lim_{q\to 1^-} E_q(z(1-q)) = e^z. \tag{1.3.17}$$

In deriving q-analogues of various formulas we shall sometimes use the observation that

$$ \frac{(q^a z;q)_\infty}{(z;q)_\infty} = {}_1\phi_0(q^a;—;q,z) \to {}_1F_0(a;—;z) = (1-z)^{-a} \text{ as } q \to 1^-. \tag{1.3.18}$$

Thus

$$ \lim_{q\to 1^-} \frac{(q^a z;q)_\infty}{(z;q)_\infty} = (1-z)^{-a}, \quad |z|<1, \quad a \text{ real}. \tag{1.3.19}$$

By analytic continuation this holds for z in the complex plane cut along the positive real axis from 1 to ∞, with $(1-z)^{-a}$ positive when z is real and less than 1.

Let Δ and ∇ be the *forward* and *backward q-difference operators*, respectively, defined by

$$ \Delta f(z) = f(qz) - f(z), \quad \nabla f(z) = f(q^{-1}z) - f(z), \tag{1.3.20}$$

where we take $0 < q < 1$, without any loss of generality. Then the unique analytic solutions of

$$ \frac{\Delta f(z)}{\Delta z} = f(z), \quad f(0)=1 \quad \text{and} \quad \frac{\nabla g(z)}{\nabla z} = g(z), \quad g(0)=1, \tag{1.3.21}$$

are

$$ f(z) = e_q(z(1-q)) \quad \text{and} \quad g(z) = E_q(z(1-q)). \tag{1.3.22}$$

The *symmetric q-difference operator* δ_q is defined by

$$ \delta_q f(z) = f(zq^{1/2}) - f(zq^{-1/2}). \tag{1.3.23}$$

If we seek an analytic solution of the initial-value problem

$$ \frac{\delta_q f(z)}{\delta_q z} = f(z), \quad f(0)=1, \tag{1.3.24}$$

in the form $\sum_{n=0}^{\infty} a_n z^n$, then we find that

$$ a_{n+1} = \frac{1-q}{1-q^{n+1}} q^{n/2} a_n, \quad a_0 = 1, \tag{1.3.25}$$

$n = 0, 1, 2, \ldots$. Hence, $a_n = (1-q)^n q^{(n^2-n)/4}/(q;q)_n$, and we have a third q-exponential function

$$\exp_q(z) = \sum_{n=0}^{\infty} \frac{(1-q)^n q^{(n^2-n)/4}}{(q;q)_n} z^n = \sum_{n=0}^{\infty} \frac{1}{[n;\sigma]!} z^n \tag{1.3.26}$$

with $q = e^{2\pi i\sigma}$. This q-exponential function has the properties

$$\exp_{q^{-1}}(z) = \exp_q(z), \qquad \lim_{q\to 1} \exp_q(z) = e^z, \tag{1.3.27}$$

and it is an entire function of z of order zero with an infinite product representation in terms of its zeros. See Nelson and Gartley [1994], and Atakishiyev and Suslov [1992a]. The multi-sheet Riemann surface associated with the q-*logarithm* inverse function $z = \ln_q(w)$ of $w = \exp_q(z)$ is considered in Nelson and Gartley [1996].

Ismail and Zhang [1994] found an extension of $\exp_q(z)$ in the form

$$f(z) = \sum_{m=0}^{\infty} \frac{q^{m^2/4}}{(q;q)_m} \left(aq^{\frac{1-m}{2}+z}, aq^{\frac{1-m}{2}-z}; q\right)_m b^m, \tag{1.3.28}$$

which has the property

$$\frac{\delta f(z)}{\delta x(z)} = f(z), \quad \delta f(z) = f(z+1/2) - f(z-1/2), \tag{1.3.29}$$

where

$$x(z) = C(q^z + q^{-z}) \tag{1.3.30}$$

with $C = -abq^{1/4}/(1-q)$ is the so-called *q-quadratic lattice*, and a and b are arbitrary complex parameters such that $|ab| < 1$. In the particular case $q^z = e^{-i\theta}$, $0 \le \theta \le \pi$, $x = \cos\theta$, the q-exponential function in (1.3.28) becomes the function

$$\mathcal{E}_q(x;a,b) = \sum_{m=0}^{\infty} \frac{q^{m^2/4}}{(q;q)_m} \left(q^{\frac{1-m}{2}} ae^{i\theta}, q^{\frac{1-m}{2}} ae^{-i\theta}; q\right)_m b^m. \tag{1.3.31}$$

Ismail and Zhang showed that

$$\lim_{q\to 1} \mathcal{E}_q(x;a,b(1-q)) = \exp[(1+a^2-2ax)b], \tag{1.3.32}$$

and that $\mathcal{E}_q(x;a,b)$ is an entire function of x when $|ab| < 1$. From (1.3.32) they observed that $\mathcal{E}_q(x;-i,-it/2)$ is a q-analogue of e^{xt}. It is now standard to use the notation in Suslov [2003] for the slightly modified q-exponential function

$$\mathcal{E}_q(x;\alpha) = \frac{(\alpha^2;q^2)_\infty}{(q\alpha^2;q^2)_\infty} \sum_{m=0}^{\infty} \frac{q^{m^2/4}}{(q;q)_m} (-i\alpha)^m \left(-iq^{\frac{1-m}{2}} e^{i\theta}, -iq^{\frac{1-m}{2}} e^{-i\theta}; q\right)_m, \tag{1.3.33}$$

which, because of the normalizing factor that he introduced, has the nice property that $\mathcal{E}_q(0;\alpha) = 1$ (see Suslov [2003, p. 17]).

1.4 Heine's transformation formulas for ${}_2\phi_1$ series

Heine [1847, 1878] showed that

$$ {}_2\phi_1(a,b;c;q,z) = \frac{(b,az;q)_\infty}{(c,z;q)_\infty}\, {}_2\phi_1(c/b,z;az;q,b), \tag{1.4.1}$$

where $|z|<1$ and $|b|<1$. To prove this transformation formula, first observe from the q-binomial theorem (1.3.2) that

$$\frac{(cq^n;q)_\infty}{(bq^n;q)_\infty} = \sum_{m=0}^\infty \frac{(c/b;q)_m}{(q;q)_m}(bq^n)^m.$$

Hence, for $|z|<1$ and $|b|<1$,

$$\begin{aligned}
{}_2\phi_1(a,b;c;q,z) &= \frac{(b;q)_\infty}{(c;q)_\infty}\sum_{n=0}^\infty \frac{(a;q)_n(cq^n;q)_\infty}{(q;q)_n(bq^n;q)_\infty} z^n \\
&= \frac{(b;q)_\infty}{(c;q)_\infty}\sum_{n=0}^\infty \frac{(a;q)_n}{(q;q)_n} z^n \sum_{m=0}^\infty \frac{(c/b;q)_m}{(q;q)_m}(bq^n)^m \\
&= \frac{(b;q)_\infty}{(c;q)_\infty}\sum_{m=0}^\infty \frac{(c/b;q)_m}{(q;q)_m} b^m \sum_{n=0}^\infty \frac{(a;q)_n}{(q;q)_n}(zq^m)^n \\
&= \frac{(b;q)_\infty}{(c;q)_\infty}\sum_{m=0}^\infty \frac{(c/b;q)_m}{(q;q)_m} b^m \frac{(azq^m;q)_\infty}{(zq^m;q)_\infty} \\
&= \frac{(b,az;q)_\infty}{(c,z;q)_\infty}\, {}_2\phi_1(c/b,z;az;q,b)
\end{aligned}$$

by (1.3.2), which gives (1.4.1).

Heine also showed that Euler's transformation formula

$$ {}_2F_1(a,b;c;z) = (1-z)^{c-a-b}\, {}_2F_1(c-a,c-b;c;z) \tag{1.4.2}$$

has a q-analogue of the form

$$ {}_2\phi_1(a,b;c;q,z) = \frac{(abz/c;q)_\infty}{(z;q)_\infty}\, {}_2\phi_1(c/a,c/b;c;q,abz/c). \tag{1.4.3}$$

A short way to prove this formula is just to iterate (1.4.1) as follows

$$ {}_2\phi_1(a,b;c;q,z) = \frac{(b,az;q)_\infty}{(c,z;q)_\infty}\, {}_2\phi_1(c/b,z;az;q,b) \tag{1.4.4}$$

$$ = \frac{(c/b,bz;q)_\infty}{(c,z;q)_\infty}\, {}_2\phi_1(abz/c,b;bz;q,c/b) \tag{1.4.5}$$

$$ = \frac{(abz/c;q)_\infty}{(z;q)_\infty}\, {}_2\phi_1(c/a,c/b;c;q,abz/c). \tag{1.4.6}$$

1.5 Heine's q-analogue of Gauss' summation formula

In order to derive Heine's [1847] q-analogue of Gauss' summation formula (1.2.11) it suffices to set $z = c/ab$ in (1.4.1), assume that $|b| < 1$, $|c/ab| < 1$, and observe that the series on the right side of

$$ {}_2\phi_1(a, b; c; q, c/ab) = \frac{(b, c/b; q)_\infty}{(c, c/ab; q)_\infty} \, {}_1\phi_0(c/ab; —; q, b) $$

can be summed by (1.3.2) to give

$$ {}_2\phi_1(a, b; c; q, c/ab) = \frac{(c/a, c/b; q)_\infty}{(c, c/ab; q)_\infty}. \tag{1.5.1} $$

By analytic continuation, we may drop the assumption that $|b| < 1$ and require only that $|c/ab| < 1$ for (1.5.1) to be valid.

For the terminating case when $a = q^{-n}$, (1.5.1) reduces to

$$ {}_2\phi_1(q^{-n}, b; c; q, cq^n/b) = \frac{(c/b; q)_n}{(c; q)_n}. \tag{1.5.2} $$

By inversion or by changing the order of summation it follows from (1.5.2) that

$$ {}_2\phi_1(q^{-n}, b; c; q, q) = \frac{(c/b; q)_n}{(c; q)_n} b^n. \tag{1.5.3} $$

Both (1.5.2) and (1.5.3) are q-analogues of Vandermonde's formula (1.2.9). These formulas can be used to derive other important formulas such as, for example, Jackson's [1910a] transformation formula

$$ \begin{aligned} {}_2\phi_1(a, b; c; q, z) &= \frac{(az; q)_\infty}{(z; q)_\infty} \sum_{k=0}^{\infty} \frac{(a, c/b; q)_k}{(q, c, az; q)_k} (-bz)^k q^{\binom{k}{2}} \\ &= \frac{(az; q)_\infty}{(z; q)_\infty} \, {}_2\phi_2(a, c/b; c, az; q, bz). \end{aligned} \tag{1.5.4} $$

This formula is a q-analogue of the Pfaff–Kummer transformation formula

$$ {}_2F_1(a, b; c; z) = (1 - z)^{-a} \, {}_2F_1(a, c - b; c; z/(z - 1)). \tag{1.5.5} $$

To prove (1.5.4), we use (1.5.2) to write

$$ \frac{(b; q)_k}{(c; q)_k} = \sum_{n=0}^{k} \frac{(q^{-k}, c/b; q)_n}{(q, c; q)_n} \left(bq^k\right)^n $$

and hence

$$\begin{aligned}
&{}_2\phi_1(a,b;c;q,z)\\
&=\sum_{k=0}^{\infty}\frac{(a;q)_k}{(q;q)_k}z^k\sum_{n=0}^{k}\frac{(q^{-k},c/b;q)_n}{(q,c;q)_n}\left(bq^k\right)^n\\
&=\sum_{n=0}^{\infty}\sum_{k=n}^{\infty}\frac{(a;q)_k(c/b;q)_n}{(q;q)_{k-n}(q,c;q)_n}z^k(-b)^nq^{\binom{n}{2}}\\
&=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(a;q)_{k+n}(c/b;q)_n}{(q;q)_k(q,c;q)_n}(-bz)^nz^kq^{\binom{n}{2}}\\
&=\sum_{n=0}^{\infty}\frac{(a,c/b;q)_n}{(q,c;q)_n}(-bz)^nq^{\binom{n}{2}}\sum_{k=0}^{\infty}\frac{(aq^n;q)_k}{(q;q)_k}z^k\\
&=\frac{(az;q)_\infty}{(z;q)_\infty}\sum_{n=0}^{\infty}\frac{(a,c/b;q)_n}{(q,c,az;q)_n}(-bz)^nq^{\binom{n}{2}},
\end{aligned}$$

by (1.3.2). Also see Andrews [1973]. If $a=q^{-n}$, then the series on the right side of (1.5.4) can be reversed (by replacing k by $n-k$) to yield Sears' [1951c] transformation formula

$$\begin{aligned}
&{}_2\phi_1(q^{-n},b;c;q,z)\\
&=\frac{(c/b;q)_n}{(c;q)_n}\left(\frac{bz}{q}\right)^n{}_3\phi_2(q^{-n},q/z,c^{-1}q^{1-n};bc^{-1}q^{1-n},0;q,q).
\end{aligned}\tag{1.5.6}$$

1.6 Jacobi's triple product identity, theta functions, and elliptic numbers

Jacobi's [1829] well-known *triple product identity* (see Andrews [1971])

$$(zq^{\frac{1}{2}},q^{\frac{1}{2}}/z,q;q)_\infty=\sum_{n=-\infty}^{\infty}(-1)^nq^{n^2/2}z^n,\quad z\neq 0,\tag{1.6.1}$$

can be easily derived by using Heine's summation formula (1.5.1).

First, set $c=bzq^{\frac{1}{2}}$ in (1.5.1) and then let $b\to 0$ and $a\to\infty$ to obtain

$$\sum_{n=0}^{\infty}\frac{(-1)^nq^{n^2/2}}{(q;q)_n}z^n=(zq^{\frac{1}{2}};q)_\infty.\tag{1.6.2}$$

Similarly, setting $c=zq$ in (1.5.1) and letting $a\to\infty$ and $b\to\infty$ we get

$$\sum_{n=0}^{\infty}\frac{q^{n^2}z^n}{(q,zq;q)_n}=\frac{1}{(zq;q)_\infty}.\tag{1.6.3}$$

Now use (1.6.2) to find that

$$\begin{aligned}
&(zq^{\frac{1}{2}},q^{\frac{1}{2}}/z;q)_\infty\\
&=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^{m+n}q^{(m^2+n^2)/2}}{(q;q)_m(q;q)_n}z^{m-n}
\end{aligned}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2/2}}{(q;q)_n} z^n \sum_{k=0}^{\infty} \frac{q^{k^2}}{(q,q^{n+1};q)_k} q^{nk}$$
$$+ \sum_{n=1}^{\infty} \frac{(-1)^n q^{n^2/2}}{(q;q)_n} z^{-n} \sum_{k=0}^{\infty} \frac{q^{k^2}}{(q,q^{n+1};q)_k} q^{nk}. \tag{1.6.4}$$

Formula (1.6.1) then follows from (1.6.3) by observing that

$$\frac{1}{(q;q)_n} \sum_{k=0}^{\infty} \frac{q^{k^2}}{(q,q^{n+1};q)_k} q^{nk} = \frac{1}{(q;q)_n (q^{n+1};q)_\infty} = \frac{1}{(q;q)_\infty}.$$

An important application of (1.6.1) is that it can be used to express the *theta functions* (Whittaker and Watson [1965, Chapter 21])

$$\vartheta_1(x,q) = 2 \sum_{n=0}^{\infty} (-1)^n q^{(n+1/2)^2} \sin(2n+1)x, \tag{1.6.5}$$

$$\vartheta_2(x,q) = 2 \sum_{n=0}^{\infty} q^{(n+1/2)^2} \cos(2n+1)x, \tag{1.6.6}$$

$$\vartheta_3(x,q) = 1 + 2 \sum_{n=1}^{\infty} q^{n^2} \cos 2nx, \tag{1.6.7}$$

$$\vartheta_4(x,q) = 1 + 2 \sum_{n=1}^{\infty} (-1)^n q^{n^2} \cos 2nx \tag{1.6.8}$$

in terms of infinite products. Just replace q by q^2 in (1.6.1) and then set z equal to $qe^{2ix}, -qe^{2ix}, -e^{2ix}, e^{2ix}$, respectively, to obtain

$$\vartheta_1(x,q) = 2q^{1/4} \sin x \prod_{n=1}^{\infty} (1-q^{2n})(1-2q^{2n}\cos 2x + q^{4n}), \tag{1.6.9}$$

$$\vartheta_2(x,q) = 2q^{1/4} \cos x \prod_{n=1}^{\infty} (1-q^{2n})(1+2q^{2n}\cos 2x + q^{4n}), \tag{1.6.10}$$

$$\vartheta_3(x,q) = \prod_{n=1}^{\infty} (1-q^{2n})(1+2q^{2n-1}\cos 2x + q^{4n-2}), \tag{1.6.11}$$

and

$$\vartheta_4(x,q) = \prod_{n=1}^{\infty} (1-q^{2n})(1-2q^{2n-1}\cos 2x + q^{4n-2}). \tag{1.6.12}$$

It is common to write $\vartheta_k(x)$ for $\vartheta_k(x,q)$, $k = 1, \ldots, 4$.

Since, from (1.6.9) and (1.6.10),

$$\lim_{q\to 0} 2^{-1} q^{-\frac{1}{4}} \vartheta_1(x,q) = \sin x, \quad \lim_{q\to 0} 2^{-1} q^{-\frac{1}{4}} \vartheta_2(x,q) = \cos x, \tag{1.6.13}$$

one can think of the theta functions $\vartheta_1(x,q)$ and $\vartheta_2(x,q)$ as one-parameter deformations (generalizations) of the trigonometric functions $\sin x$ and $\cos x$, respectively. This led Frenkel and Turaev [1995] to define an *elliptic number* $[a;\sigma,\tau]$ by

$$[a;\sigma,\tau]=\frac{\vartheta_1(\pi\sigma a,e^{\pi i\tau})}{\vartheta_1(\pi\sigma,e^{\pi i\tau})}, \tag{1.6.14}$$

where a is a complex number and the *modular parameters* σ and τ are fixed complex numbers such that Im $(\tau)>0$ and $\sigma\neq m+n\tau$ for integer values of m and n, so that the denominator $\vartheta_1(\pi\sigma,e^{\pi i\tau})$ in (1.6.14) is never zero. Then, from (1.6.9) it is clear that $[a;\sigma,\tau]$ is well-defined, $[-a;\sigma,\tau]=-[a;\sigma,\tau]$, $[1;\sigma,\tau]=1$, and

$$\lim_{\mathrm{I}\mu\ \tau\to\infty}[a;\sigma,\tau]=\frac{\sin(\pi\sigma a)}{\sin(\pi\sigma)}=[a;\sigma]. \tag{1.6.15}$$

Hence, the elliptic number $[a;\sigma,\tau]$ is a one-parameter deformation of the trigonometric number $[a;\sigma]$ and a two-parameter deformation of the number a. Notice that $[a;\sigma,\tau]$ is called an "elliptic number" even though it is not an elliptic (doubly periodic and meromorphic) function of a. However, $[a;\sigma,\tau]$ is a quotient of ϑ_1 functions and, as is well-known (see Whittaker and Watson [1965, §21.5]), any (doubly periodic meromorphic) elliptic function can be written as a constant multiple of a quotient of products of ϑ_1 functions. The corresponding elliptic hypergeometric series are considered in Chapter 11.

1.7 A q-analogue of Saalschütz's summation formula

Pfaff [1797] discovered the summation formula

$${}_3F_2(a,b,-n;c,1+a+b-c-n;1)=\frac{(c-a)_n(c-b)_n}{(c)_n(c-a-b)_n},\quad n=0,1,\ldots, \tag{1.7.1}$$

which sums a terminating balanced ${}_3F_2(1)$ series with argument 1. It was rediscovered by Saalschütz [1890] and is usually called *Saalschütz formula* or the *Pfaff–Saalschütz formula*; see Askey [1975]. To derive a q-analogue of (1.7.1), observe that since, by (1.3.2),

$$\frac{(abz/c;q)_\infty}{(z;q)_\infty}=\sum_{k=0}^{\infty}\frac{(ab/c;q)_k}{(q;q)_k}z^k$$

the right side of (1.4.3) equals

$$\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\frac{(ab/c;q)_k(c/a,c/b;q)_m}{(q;q)_k(q,c;q)_m}\left(\frac{ab}{c}\right)^m z^{k+m},$$

and hence, equating the coefficients of z^n on both sides of (1.4.3) we get

$$\sum_{j=0}^{n}\frac{(q^{-n},c/a,c/b;q)_j}{(q,c,cq^{1-n}/ab;q)_j}q^j=\frac{(a,b;q)_n}{(c,ab/c;q)_n}.$$

Replacing a,b by $c/a,c/b$, respectively, this gives the following sum of a terminating balanced ${}_3\phi_2$ series

$${}_3\phi_2(a,b,q^{-n};c,abc^{-1}q^{1-n};q,q)=\frac{(c/a,c/b;q)_n}{(c,c/ab;q)_n},\quad n=0,1,\ldots, \tag{1.7.2}$$

which was first derived by Jackson [1910a]. It is easy to see that (1.7.1) follows from (1.7.2) by replacing a, b, c in (1.7.2) by q^a, q^b, q^c, respectively, and letting $q \to 1$. Note that letting $a \to \infty$ in (1.7.2) gives (1.5.2), while letting $a \to 0$ gives (1.5.3).

1.8 The Bailey–Daum summation formula

Bailey [1941] and Daum [1942] independently discovered the summation formula

$$ {}_2\phi_1(a, b; aq/b; q, -q/b) = \frac{(-q;q)_\infty (aq, aq^2/b^2; q^2)_\infty}{(aq/b, -q/b; q)_\infty}, \tag{1.8.1} $$

which is a q-analogue of Kummer's formula

$$ {}_2F_1(a, b; 1+a-b; -1) = \frac{\Gamma(1+a-b)\Gamma(1+\frac{1}{2}a)}{\Gamma(1+a)\Gamma(1+\frac{1}{2}a-b)}. \tag{1.8.2} $$

Formula (1.8.1) can be easily obtained from (1.4.1) by using the identity (1.2.40) and a limiting form of (1.2.39), namely, $(a;q)_\infty = (a, aq; q^2)_\infty$, to see that

$$ \begin{aligned} &{}_2\phi_1(a, b; aq/b; q, -q/b) \\ &= \frac{(a, -q; q)_\infty}{(aq/b, -q/b; q)_\infty} \, {}_2\phi_1(q/b, -q/b; -q; q, a) \\ &= \frac{(a, -q; q)_\infty}{(aq/b, -q/b; q)_\infty} \sum_{n=0}^\infty \frac{(q^2/b^2; q^2)_n}{(q^2; q^2)_n} a^n \\ &= \frac{(a, -q; q)_\infty}{(aq/b, -q/b; q)_\infty} \frac{(aq^2/b^2; q^2)_\infty}{(a; q^2)_\infty} \qquad \text{by (1.3.2)} \\ &= \frac{(-q;q)_\infty (aq, aq^2/b^2; q^2)_\infty}{(aq/b, -q/b; q)_\infty}. \end{aligned} $$

1.9 q-analogues of the Karlsson–Minton summation formulas

Minton [1970] showed that if a is a negative integer and $m_1, m_2, \ldots, m_r$ are nonnegative integers such that $-a \geq m_1 + \cdots + m_r$, then

$$ \begin{aligned} &{}_{r+2}F_{r+1}\left[\begin{matrix} a, b, b_1+m_1, \ldots, b_r+m_r \\ b+1, b_1, \ldots, b_r \end{matrix}; 1\right] \\ &= \frac{\Gamma(b+1)\Gamma(1-a)}{\Gamma(1+b-a)} \frac{(b_1-b)_{m_1} \cdots (b_r-b)_{m_r}}{(b_1)_{m_1} \cdots (b_r)_{m_r}} \end{aligned} \tag{1.9.1} $$

where, as usual, it is assumed that none of the factors in the denominators of the terms of the series is zero. Karlsson [1971] showed that (1.9.1) also holds when a is not a negative integer provided that the series converges, i.e., if $\mathrm{Re}(-a) > m_1 + \cdots + m_r - 1$, and he deduced from (1.9.1) that

$$ {}_{r+1}F_r\left[\begin{matrix} a, b_1+m_1, \ldots, b_r+m_r \\ b_1, \ldots, b_r \end{matrix}; 1\right] = 0, \quad \mathrm{Re}\,(-a) > m_1 + \cdots + m_r, \tag{1.9.2} $$

$$ {}_{r+1}F_r\left[\begin{matrix} -(m_1+\cdots+m_r), b_1+m_1, \ldots, b_r+m_r \\ b_1, \ldots, b_r \end{matrix}; 1\right] $$
$$ = (-1)^{m_1+\cdots+m_r} \frac{(m_1+\cdots+m_r)!}{(b_1)_{m_1}\cdots(b_r)_{m_r}}. \tag{1.9.3} $$

These formulas are particularly useful for evaluating sums that appear as solutions to some problems in theoretical physics such as the Racah coefficients. They were also used by Gasper [1981b] to prove the orthogonality on $(0, 2\pi)$ of certain functions that arose in Greiner's [1980] work on spherical harmonics on the Heisenberg group. Here we shall present Gasper's [1981a] derivation of q-analogues of the above formulas. Some of the formulas derived below will be used in Chapter 7 to prove the orthogonality relation for the continuous q-ultraspherical polynomials.

Observe that if m and n are nonnegative integers with $m \geq n$, then

$$ {}_2\phi_1(q^{-n}, q^{-m}; b_r; q, q) = \frac{(b_r q^m; q)_n}{(b_r; q)_n} q^{-mn} $$

by (1.5.3), and hence

$$ {}_{r+1}\phi_r\left[\begin{matrix} a_1, \ldots, a_r, b_r q^m \\ b_1, \ldots, b_{r-1}, b_r \end{matrix}; q, z\right] $$
$$ = \sum_{n=0}^{\infty} \frac{(a_1, \ldots, a_r; q)_n}{(q, b_1, \ldots, b_{r-1}; q)_n} z^n \sum_{k=0}^{n} \frac{(q^{-n}, q^{-m}; q)_k}{(q, b_r; q)_k} q^{mn+k} $$
$$ = \sum_{n=0}^{\infty}\sum_{k=0}^{m} \frac{(a_1, \ldots, a_r; q)_n (q^{-m}; q)_k}{(b_1, \ldots, b_{r-1}; q)_n (q; q)_{n-k} (q, b_r; q)_k} z^n (-1)^k q^{mn+k-nk+\binom{k}{2}} $$
$$ = \sum_{k=0}^{m} \frac{(q^{-m}, a_1, \ldots, a_r; q)_k}{(q, b_1, \ldots, b_r; q)_k} (-zq^m)^k q^{-\binom{k}{2}} $$
$$ \times\ {}_r\phi_{r-1}\left[\begin{matrix} a_1 q^k, \ldots, a_r q^k \\ b_1 q^k, \ldots, b_{r-1} q^k \end{matrix}; q, zq^{m-k}\right], \quad |z| < 1. \tag{1.9.4} $$

This expansion formula is a q-analogue of a formula in Fox [1927, (1.11)] and independently derived by Minton [1970, (4)].

When $r = 2$, formulas (1.9.4), (1.5.1) and (1.5.3) yield

$$ {}_3\phi_2\left[\begin{matrix} a, b, b_1 q^m \\ bq, b_1 \end{matrix}; q, a^{-1}q^{1-m}\right] = \frac{(q, bq/a; q)_\infty}{(bq, q/a; q)_\infty} {}_2\phi_1(q^{-m}, b; b_1; q, q) $$
$$ = \frac{(q, bq/a; q)_\infty (b_1/b; q)_m}{(bq, q/a; q)_\infty (b_1; q)_m} b^m, \tag{1.9.5} $$

provided that $|a^{-1}q^{1-m}| < 1$. By induction it follows from (1.9.4) and (1.9.5) that if $m_1, \ldots, m_r$ are nonnegative integers and $|a^{-1}q^{1-(m_1+\cdots+m_r)}| < 1$, then

$$ {}_{r+2}\phi_{r+1}\left[\begin{matrix} a, b, b_1 q^{m_1}, \ldots, b_r q^{m_r} \\ bq, b_1, \ldots, b_r \end{matrix}; q, a^{-1}q^{1-(m_1+\cdots+m_r)}\right] $$
$$ = \frac{(q, bq/a; q)_\infty}{(bq, q/a; q)_\infty} \frac{(b_1/b; q)_{m_1} \cdots (b_r/b; q)_{m_r}}{(b_1; q)_{m_1} \cdots (b_r; q)_{m_r}} b^{m_1+\cdots+m_r} \tag{1.9.6} $$

which is a q-analogue of (1.9.1). Formula (1.9.1) can be derived from (1.9.6) by replacing $a, b, b_1, \ldots, b_r$ by $q^a, q^b, q^{b_1}, \ldots, q^{b_r}$, respectively, and letting $q \to 1$.

Setting $b_r = b, m_r = 1$ and then replacing r by $r+1$ in (1.9.6) gives

$$ {}_{r+1}\phi_r \left[\begin{matrix} a, b_1 q^{m_1}, \ldots, b_r q^{m_r} \\ b_1, \ldots, b_r \end{matrix} ; q, a^{-1} q^{-(m_1 + \cdots + m_r)} \right] = 0, \ |a^{-1} q^{-(m_1 + \cdots + m_r)}| < 1, \tag{1.9.7} $$

while letting $b \to \infty$ in the case $a = q^{-(m_1 + \cdots + m_r)}$ of (1.9.6) gives

$$ \begin{aligned} &{}_{r+1}\phi_r \left[\begin{matrix} q^{-(m_1 + \cdots + m_r)}, b_1 q^{m_1}, \ldots, b_r q^{m_r} \\ b_1, \ldots, b_r \end{matrix} ; q, 1 \right] \\ &= \frac{(-1)^{m_1 + \cdots + m_r} (q;q)_{m_1 + \cdots + m_r}}{(b_1;q)_{m_1} \cdots (b_r;q)_{m_r}} q^{-(m_1 + \cdots + m_r)(m_1 + \cdots + m_r + 1)/2}, \end{aligned} \tag{1.9.8} $$

which are q-analogues of (1.9.2) and (1.9.3). Another q-analogue of (1.9.3) can be derived by letting $b \to 0$ in (1.9.6) to obtain

$$ \begin{aligned} &{}_{r+1}\phi_r \left[\begin{matrix} a, b_1 q^{m_1}, \ldots, b_r q^{m_r} \\ b_1, \ldots, b_r \end{matrix} ; q, a^{-1} q^{1-(m_1 + \cdots + m_r)} \right] \\ &= \frac{(-1)^{m_1 + \cdots + m_r} (q;q)_\infty b_1^{m_1} \cdots b_r^{m_r}}{(q/a;q)_\infty (b_1;q)_{m_1} \cdots (b_r;q)_{m_r}} q^{\binom{m_1}{2} + \cdots + \binom{m_r}{2}}, \end{aligned} \tag{1.9.9} $$

when $|a^{-1} q^{1-(m_1 + \cdots + m_r)}| < 1$.

In addition, if $a = q^{-n}$ and n is a nonnegative integer then we can reverse the order of summation of the series in (1.9.6), (1.9.7) and (1.9.9) to obtain

$$ \begin{aligned} &{}_{r+2}\phi_{r+1} \left[\begin{matrix} q^{-n}, b, b_1 q^{m_1}, \ldots, b_r q^{m_r} \\ bq, b_1, \ldots, b_r \end{matrix} ; q, q \right] \\ &= \frac{b^n (q;q)_n (b_1/b;q)_{m_1} \cdots (b_r/b;q)_{m_r}}{(bq;q)_n (b_1;q)_{m_1} \cdots (b_r;q)_{m_r}}, \quad n \geq m_1 + \cdots + m_r, \end{aligned} \tag{1.9.10} $$

$$ {}_{r+1}\phi_r \left[\begin{matrix} q^{-n}, b_1 q^{m_1}, \ldots, b_r q^{m_r} \\ b_1, \ldots, b_r \end{matrix} ; q, q \right] = 0, \quad n > m_1 + \cdots + m_r, \tag{1.9.11} $$

and the following generalization of (1.9.8)

$$ {}_{r+1}\phi_r \left[\begin{matrix} q^{-n}, b_1 q^{m_1}, \ldots, b_r q^{m_r} \\ b_1, \ldots, b_r \end{matrix} ; q, 1 \right] = \frac{(-1)^n (q;q)_n q^{-n(n+1)/2}}{(b_1;q)_{m_1} \cdots (b_r;q)_{m_r}}, \tag{1.9.12} $$

where $n \geq m_1 + \cdots + m_r$, which also follows by letting $b \to \infty$ in (1.9.10). Note that the $b \to 0$ limit case of (1.9.10) is (1.9.11) when $n > m_1 + \cdots + m_r$, and it is the $a = q^{-(m_1 + \cdots + m_r)}$ special case of (1.9.9) when $n = m_1 + \cdots + m_r$.

1.10 The q-gamma and q-beta functions

The *q-gamma* function

$$ \Gamma_q(x) = \frac{(q;q)_\infty}{(q^x;q)_\infty} (1-q)^{1-x}, \ 0 < q < 1, \tag{1.10.1} $$

was introduced by Thomae [1869] and later by Jackson [1904e]. Heine [1847] gave an equivalent definition, but without the factor $(1-q)^{1-x}$. When $x = n+1$ with n a nonnegative integer, this definition reduces to

$$\Gamma_q(n+1) = 1(1+q)(1+q+q^2)\cdots(1+q+\cdots+q^{n-1}), \tag{1.10.2}$$

which clearly approaches $n!$ as $q \to 1^-$. Hence $\Gamma_q(n+1)$ tends to $\Gamma(n+1) = n!$ as $q \to 1^-$. The definition of $\Gamma_q(x)$ can be extended to $|q| < 1$ by using the principal values of q^x and $(1-q)^{1-x}$ in (1.10.1).

To show that

$$\lim_{q\to 1^-} \Gamma_q(x) = \Gamma(x) \tag{1.10.3}$$

we shall give a simple, formal proof due to Gosper; see Andrews [1986]. From (1.10.1),

$$\begin{aligned}\Gamma_q(x+1) &= \frac{(q;q)_\infty}{(q^{x+1};q)_\infty}(1-q)^{-x} \\ &= \prod_{n=1}^{\infty} \frac{(1-q^n)(1-q^{n+1})^x}{(1-q^{n+x})(1-q^n)^x}.\end{aligned}$$

Hence

$$\begin{aligned}\lim_{q\to 1^-} \Gamma_q(x+1) &= \prod_{n=1}^{\infty} \frac{n}{n+x}\left(\frac{n+1}{n}\right)^x \\ &= x\left[x^{-1}\prod_{n=1}^{\infty}\left(1+\frac{x}{n}\right)^{-1}\left(1+\frac{1}{n}\right)^x\right] \\ &= x\Gamma(x) = \Gamma(x+1)\end{aligned}$$

by Euler's product formula (see Whittaker and Watson [1965, §12.11]) and the well-known functional equation for the gamma function

$$\Gamma(x+1) = x\,\Gamma(x), \quad \Gamma(1) = 1. \tag{1.10.4}$$

For a rigorous justification of the above steps see Koornwinder [1990]. From (1.10.1) it is easily seen that, analogous to (1.10.4), $\Gamma_q(x)$ satisfies the functional equation

$$f(x+1) = \frac{1-q^x}{1-q} f(x), \quad f(1) = 1. \tag{1.10.5}$$

Askey [1978] derived analogues of many of the well-known properties of the gamma function, including its log-convexity (see the exercises at the end of this chapter), which show that (1.10.1) is a natural q-analogue of $\Gamma(x)$.

It is obvious from (1.10.1) that $\Gamma_q(x)$ has poles at $x = 0, -1, -2, \ldots$. The residue at $x = -n$ is

$$\begin{aligned}\lim_{x\to -n}(x+n)\Gamma_q(x) &= \frac{(1-q)^{n+1}}{(1-q^{-n})(1-q^{1-n})\cdots(1-q^{-1})}\lim_{x\to -n}\frac{x+n}{1-q^{x+n}} \\ &= \frac{(1-q)^{n+1}}{(q^{-n};q)_n \log q^{-1}}.\end{aligned} \tag{1.10.6}$$

The q-gamma function has no zeros, so its reciprocal is an entire function with zeros at $x = 0, -1, -2, \ldots$. Since

$$\frac{1}{\Gamma_q(x)} = (1-q)^{x-1} \prod_{n=0}^{\infty} \frac{1-q^{n+x}}{1-q^{n+1}}, \tag{1.10.7}$$

the function $1/\Gamma_q(x)$ has zeros at $x = -n \pm 2\pi i k/\log q$, where k and n are nonnegative integers.

A q-analogue of Legendre's duplication formula

$$\Gamma(2x)\Gamma\left(\frac{1}{2}\right) = 2^{2x-1}\Gamma(x)\Gamma\left(x+\frac{1}{2}\right) \tag{1.10.8}$$

can be easily derived by observing that

$$\begin{aligned}\frac{\Gamma_{q^2}(x)\Gamma_{q^2}\left(x+\frac{1}{2}\right)}{\Gamma_{q^2}\left(\frac{1}{2}\right)} &= \frac{(q,q^2;q^2)_\infty}{(q^{2x},q^{2x+1};q^2)_\infty}(1-q^2)^{1-2x}\\ &= \frac{(q;q)_\infty}{(q^{2x};q)_\infty}(1-q^2)^{1-2x} = (1+q)^{1-2x}\Gamma_q(2x)\end{aligned}$$

and hence

$$\Gamma_q(2x)\Gamma_{q^2}\left(\frac{1}{2}\right) = (1+q)^{2x-1}\Gamma_{q^2}(x)\Gamma_{q^2}\left(x+\frac{1}{2}\right). \tag{1.10.9}$$

Similarly, it can be shown that the Gauss multiplication formula

$$\Gamma(nx)(2\pi)^{(n-1)/2} = n^{nx-\frac{1}{2}}\Gamma(x)\Gamma\left(x+\frac{1}{n}\right)\cdots\Gamma\left(x+\frac{n-1}{n}\right) \tag{1.10.10}$$

has a q-analogue of the form

$$\begin{aligned}&\Gamma_q(nx)\Gamma_r\left(\frac{1}{n}\right)\Gamma_r\left(\frac{2}{n}\right)\cdots\Gamma_r\left(\frac{n-1}{n}\right)\\ &= (1+q+\cdots+q^{n-1})^{nx-1}\Gamma_r(x)\Gamma_r\left(x+\frac{1}{n}\right)\cdots\Gamma_r\left(x+\frac{n-1}{n}\right)\end{aligned} \tag{1.10.11}$$

with $r = q^n$; see Jackson [1904e, 1905d]. The q-gamma function for $q > 1$ is considered in Exercise 1.23. For other interesting properties of the q-gamma function see Askey [1978] and Moak [1980a,b] and Ismail, Lorch and Muldoon [1986].

Since the *beta function* is defined by

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \tag{1.10.12}$$

it is natural to define the *q-beta function* by

$$B_q(x,y) = \frac{\Gamma_q(x)\Gamma_q(y)}{\Gamma_q(x+y)}, \tag{1.10.13}$$

which tends to $B(x,y)$ as $q \to 1^-$. By (1.10.1) and (1.3.2),

$$\begin{aligned} B_q(x,y) &= (1-q)\frac{(q,q^{x+y};q)_\infty}{(q^x,q^y;q)_\infty} \\ &= (1-q)\frac{(q;q)_\infty}{(q^y;q)_\infty}\sum_{n=0}^{\infty}\frac{(q^y;q)_n}{(q;q)_n}\,q^{nx} \\ &= (1-q)\sum_{n=0}^{\infty}\frac{(q^{n+1};q)_\infty}{(q^{n+y};q)_\infty}q^{nx}, \quad \operatorname{Re} x,\ \operatorname{Re} y>0. \qquad (1.10.14) \end{aligned}$$

This series expansion will be used in the next section to derive a q-integral representation for $B_q(x,y)$.

1.11 The q-integral

Thomae [1869, 1870] and Jackson [1910c, 1951] introduced the *q-integral*

$$\int_0^1 f(t)\ d_qt = (1-q)\sum_{n=0}^{\infty} f(q^n)q^n \qquad (1.11.1)$$

and Jackson gave the more general definition

$$\int_a^b f(t)\ d_qt = \int_0^b f(t)\ d_qt - \int_0^a f(t)\ d_qt, \qquad (1.11.2)$$

where

$$\int_0^a f(t)\ d_qt = a(1-q)\sum_{n=0}^{\infty} f(aq^n)q^n. \qquad (1.11.3)$$

Jackson also defined an integral on $(0,\infty)$ by

$$\int_0^\infty f(t)\ d_qt = (1-q)\sum_{n=-\infty}^{\infty} f(q^n)q^n. \qquad (1.11.4)$$

The *bilateral q-integral* is defined by

$$\int_{-\infty}^\infty f(t)\ d_qt = (1-q)\sum_{n=-\infty}^{\infty}\left[f(q^n)+f(-q^n)\right]q^n. \qquad (1.11.5)$$

If f is continuous on $[0,a]$, then it is easily seen that

$$\lim_{q\to 1}\int_0^a f(t)\ d_qt = \int_0^a f(t)\ dt \qquad (1.11.6)$$

and that a similar limit holds for (1.11.4) and (1.11.5) when f is suitably restricted. By (1.11.1), it follows from (1.10.14) that

$$B_q(x,y) = \int_0^1 t^{x-1}\frac{(tq;q)_\infty}{(tq^y;q)_\infty}\ d_qt,\ \operatorname{Re} x>0, \quad y\neq 0,-1,-2,\ldots, \qquad (1.11.7)$$

which clearly approaches the beta function integral

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\ dt, \quad \operatorname{Re} x,\ \operatorname{Re} y>0, \qquad (1.11.8)$$

as $q \to 1^-$. Thomae [1869] rewrote Heine's formula (1.4.1) in the q-integral form

$$ {}_2\phi_1(q^a, q^b; q^c; q, z) = \frac{\Gamma_q(c)}{\Gamma_q(b)\Gamma_q(c-b)} \int_0^1 t^{b-1} \frac{(tzq^a, tq; q)_\infty}{(tz, tq^{c-b}; q)_\infty}\, d_qt, \qquad (1.11.9) $$

which is a q-analogue of Euler's integral representation

$$ {}_2F_1(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\, dt, \qquad (1.11.10) $$

where $|\arg(1-z)| < \pi$ and $\mathrm{Re}\, c >\ \mathrm{Re}\, b > 0$.

The q-integral notation is, as we shall see later, quite useful in simplifying and manipulating various formulas involving sums of series.

Exercises

1.1 Verify the identities (1.2.30)–(1.2.40), and show that

(i) $$(aq^{-n}; q)_n = (q/a; q)_n \left(-\frac{a}{q}\right)^n q^{-\binom{n}{2}},$$

(ii) $$(aq^{-k-n}; q)_n = \frac{(q/a; q)_{n+k}}{(q/a; q)_k} \left(-\frac{a}{q}\right)^n q^{-nk-\binom{n}{2}},$$

(iii) $$\frac{(qa^{\frac12}, -qa^{\frac12}; q)_n}{(a^{\frac12}, -a^{\frac12}; q)_n} = \frac{1-aq^{2n}}{1-a},$$

(iv) $$(a; q)_{2n} = (a^{\frac12}, -a^{\frac12}, (aq)^{\frac12}, -(aq)^{\frac12}; q)_n,$$

(v) $$(a; q)_n (q/a; q)_{-n} = (-a)^n q^{\binom{n}{2}},$$

(vi) $$(q, -q, -q^2; q^2)_\infty = 1.$$

1.2 The *q-binomial coefficient* is defined by

$$ \begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q; q)_n}{(q; q)_k (q; q)_{n-k}} $$

for $k = 0, 1, \ldots, n$, and by

$$ \begin{bmatrix} \alpha \\ \beta \end{bmatrix}_q = \frac{(q^{\beta+1}, q^{\alpha-\beta+1}; q)_\infty}{(q, q^{\alpha+1}; q)_\infty} = \frac{\Gamma_q(\alpha+1)}{\Gamma_q(\beta+1)\Gamma_q(\alpha-\beta+1)} $$

for complex α and β when $|q| < 1$. Verify that

(i) $$\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{bmatrix} n \\ n-k \end{bmatrix}_q,$$

(ii) $$\begin{bmatrix} \alpha \\ k \end{bmatrix}_q = \frac{(q^{-\alpha}; q)_k}{(q; q)_k} (-q^\alpha)^k q^{-\binom{k}{2}},$$

(iii) $$\begin{bmatrix} k+\alpha \\ k \end{bmatrix}_q = \frac{(q^{\alpha+1}; q)_k}{(q; q)_k},$$

(iv) $$\begin{bmatrix} -\alpha \\ k \end{bmatrix}_q = \begin{bmatrix} \alpha+k-1 \\ k \end{bmatrix}_q (-q^{-\alpha})^k q^{-\binom{k}{2}},$$

(v) $$\begin{bmatrix} \alpha+1 \\ k \end{bmatrix}_q = \begin{bmatrix} \alpha \\ k \end{bmatrix}_q q^k + \begin{bmatrix} \alpha \\ k-1 \end{bmatrix}_q = \begin{bmatrix} \alpha \\ k \end{bmatrix}_q + \begin{bmatrix} \alpha \\ k-1 \end{bmatrix}_q q^{\alpha+1-k},$$

(vi) $$(z;q)_n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q (-z)^k q^{\binom{k}{2}},$$

when k and n are nonnegative integers.

1.3 (i) Show that the *binomial theorem*

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$$

where $n = 0, 1, \ldots,$ has a q-analogue of the form

$$\begin{aligned}(ab;q)_n &= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q b^k (a;q)_k (b;q)_{n-k} \\ &= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q a^{n-k} (a;q)_k (b;q)_{n-k}.\end{aligned}$$

(ii) Extend the above formula to the *q-multinomial theorem*

$$\begin{aligned}&(a_1 a_2 \cdots a_{m+1};q)_n \\ &= \sum_{\substack{0 \le k_1, \ldots, 0 \le k_m \\ k_1 + \cdots + k_m \le n}} \begin{bmatrix} n \\ k_1, \ldots, k_m \end{bmatrix}_q a_2^{k_1} a_3^{k_1+k_2} \cdots a_{m+1}^{k_1+k_2+\cdots+k_m} \\ &\quad \times (a_1;q)_{k_1} (a_2;q)_{k_2} \cdots (a_m;q)_{k_m} (a_{m+1};q)_{n-(k_1+\cdots+k_m)},\end{aligned}$$

where $m = 1, 2, \ldots,\ n = 0, 1, \ldots,$ and

$$\begin{bmatrix} n \\ k_1, \ldots, k_m \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_{k_1} \cdots (q;q)_{k_m} (q;q)_{n-(k_1+\cdots+k_m)}}$$

is the *q-multinomial coefficient.*

1.4 (i) Prove the inversion formula

$$\begin{aligned}&{}_r\phi_s \left[\begin{matrix} a_1, \ldots, a_r \\ b_1, \ldots, b_s \end{matrix} ; q, z \right] \\ &= \sum_{n=0}^{\infty} \frac{(a_1^{-1}, \ldots, a_r^{-1}; q^{-1})_n}{(q^{-1}, b_1^{-1}, \ldots, b_s^{-1}; q^{-1})_n} \left(\frac{a_1 \cdots a_r z}{b_1 \cdots b_s q} \right)^n.\end{aligned}$$

(ii) By reversing the order of summation, show that

$$\begin{aligned}&{}_{r+1}\phi_s \left[\begin{matrix} a_1, \ldots, a_r, q^{-n} \\ b_1, \ldots, b_s \end{matrix} ; q, z \right] \\ &= \frac{(a_1, \ldots, a_r; q)_n}{(b_1 \ldots, b_s; q)_n} \left(\frac{z}{q} \right)^n \left((-1)^n q^{\binom{n}{2}} \right)^{s-r-1} \\ &\quad \times \sum_{k=0}^{n} \frac{(q^{1-n}/b_1, \ldots, q^{1-n}/b_s, q^{-n}; q)_k}{(q, q^{1-n}/a_1, \ldots, q^{1-n}/a_r; q)_k} \left(\frac{b_1 \cdots b_s}{a_1 \cdots a_r} \frac{q^{n+1}}{z} \right)^k\end{aligned}$$

when $n = 0, 1, \ldots$.

(iii) Show that

$$
\begin{aligned}
&{}_{r+1}\phi_r\left[\begin{matrix} a_1, \ldots, a_{r+1} \\ b_1, \ldots, b_r \end{matrix}; q, q^z\right] \\
&= \frac{(a_1, \ldots, a_{r+1}; q)_\infty}{(1-q)(q, b_1, \ldots, b_r; q)_\infty} \int_0^1 t^{z-1} \frac{(qt, b_1 t, \ldots, b_r t; q)_\infty}{(a_1 t, \ldots, a_{r+1} t; q)_\infty} d_q t,
\end{aligned}
$$

when $0 < q < 1$, Re $z > 0$, and the series on the left side does not terminate.

1.5 Show that

$$
\frac{(c, bq^n; q)_m}{(b; q)_m} = \frac{(b/c; q)_n}{(b; q)_n} \sum_{k=0}^{n} \frac{(q^{-n}, c; q)_k q^k}{(q, cq^{1-n}/b; q)_k} (cq^k; q)_m.
$$

1.6 Prove the summation formulas

(i) ${}_2\phi_1(q^{-n}, q^{1-n}; qb^2; q^2, q^2) = \dfrac{(b^2; q^2)_n}{(b^2; q)_n} q^{-\binom{n}{2}}$,

(ii) ${}_1\phi_1(a; c; q, c/a) = \dfrac{(c/a; q)_\infty}{(c; q)_\infty}$,

(iii) ${}_2\phi_0(a, q^{-n}; -; q, q^n/a) = a^{-n}$,

(iv) $\displaystyle\sum_{n=0}^{\infty} \frac{q^{n^2-n}}{(q; q)_n^2} = \frac{2}{(q; q)_\infty}$,

(v) ${}_1\phi_0(a; -; p, z) = \dfrac{(zp^{-1}; p^{-1})_\infty}{(azp^{-1}; p^{-1})_\infty}$, $\quad |p| > 1$, $|azp^{-1}| < 1$,

(vi) ${}_2\phi_1(a, b; c; p, p) = \dfrac{(a/c, b/c; p^{-1})_\infty}{(1/c, ab/c; p^{-1})_\infty}$, $|p| > 1$.

1.7 Show that, for $|z| < 1$,

$$
{}_2\phi_1(a^2, aq; a; q, z) = (1 + az)\frac{(a^2 qz; q)_\infty}{(z; q)_\infty}.
$$

1.8 Show that, when $|a| < 1$ and $|bq/a^2| < 1$,

$$
\begin{aligned}
&{}_2\phi_1(a^2, a^2/b; b; q^2, bq/a^2) \\
&= \frac{(a^2, q; q^2)_\infty}{2(b, bq/a^2; q^2)_\infty} \left[\frac{(b/a; q)_\infty}{(a; q)_\infty} + \frac{(-b/a; q)_\infty}{(-a; q)_\infty}\right].
\end{aligned}
$$

(Andrews and Askey [1977])

1.9 Let $\phi(a, b, c)$ denote the series ${}_2\phi_1(a, b; c; q, z)$. Verify Heine's [1847] *q-contiguous relations*:

(i) $\phi(a, b, cq^{-1}) - \phi(a, b, c) = cz\dfrac{(1-a)(1-b)}{(q-c)(1-c)}\phi(aq, bq, cq)$,

(ii) $\phi(aq, b, c) - \phi(a, b, c) = az\dfrac{1-b}{1-c}\phi(aq, bq, cq)$,

(iii) $\phi(aq,b,cq)-\phi(a,b,c)=az\dfrac{(1-b)(1-c/a)}{(1-c)(1-cq)}\phi(aq,bq,cq^2)$,

(iv) $\phi(aq,bq^{-1},c)-\phi(a,b,c)=az\dfrac{(1-b/aq)}{1-c}\phi(aq,b,cq)$.

1.10 Denoting ${}_2\phi_1(a,b;c;q,z)$, ${}_2\phi_1(aq^{\pm1},b;c,q,z)$, ${}_2\phi_1(a,bq^{\pm1};c;q,z)$ and ${}_2\phi_1(a,b;cq^{\pm1};q,z)$ by $\phi,\phi(aq^{\pm1}),\phi(bq^{\pm1})$ and $\phi(cq^{\pm1})$, respectively, show that

(i) $b(1-a)\phi(aq)-a(1-b)\phi(bq)=(b-a)\phi$,
(ii) $a(1-b/c)\phi(bq^{-1})-b(1-a/c)\phi(aq^{-1})=(a-b)(1-abz/cq)\phi$,
(iii) $q(1-a/c)\phi(aq^{-1})+(1-a)(1-abz/c)\phi(aq)$
$=[1+q-a-aq/c+a^2z(1-b/a)/c]\phi$,
(iv) $(1-c)(q-c)(abz-c)\phi(cq^{-1})+(c-a)(c-b)z\phi(cq)$
$=(c-1)[c(q-c)+(ca+cb-ab-abq)z]\phi$.
(Heine [1847])

1.11 Let $g(\theta;\lambda,\mu,\nu)=(\lambda e^{i\theta},\mu\nu;q)_\infty\, {}_2\phi_1(\mu e^{-i\theta},\nu e^{-i\theta};\mu\nu;q,\lambda e^{i\theta})$. Prove that $g(\theta;\lambda,\mu,\nu)$ is symmetric in λ,μ,ν and is even in θ.

1.12 Let $\mathcal{D}_q$ be the *q-derivative operator* defined for fixed q by

$$\mathcal{D}_q f(z)=\frac{f(z)-f(qz)}{(1-q)z},$$

and let $\mathcal{D}_q^n u=\mathcal{D}_q(D_q^{n-1}u)$ for $n=1,2,\ldots$. Show that

(i) $\lim_{q\to1}\mathcal{D}_q f(z)=\dfrac{d}{dz}f(z)$ if f is differentiable at z,

(ii) $\mathcal{D}_q^n\, {}_2\phi_1(a,b;c;q,z)=\dfrac{(a,b;q)_n}{(c;q)_n(1-q)^n}\, {}_2\phi_1(aq^n,bq^n;cq^n;q,z)$,

(iii) $\mathcal{D}_q^n\left\{\dfrac{(z;q)_\infty}{(abz/c;q)_\infty}\, {}_2\phi_1(a,b;c;q,z)\right\}$
$=\dfrac{(c/a,c/b;q)_n}{(c;q)_n(1-q)^n}\left(\dfrac{ab}{c}\right)^n\dfrac{(zq^n;q)_\infty}{(abz/c;q)_\infty}\, {}_2\phi_1(a,b;cq^n;q,zq^n)$.

(iv) Prove the q-Leibniz formula

$$\mathcal{D}_q^n[f(z)g(z)]=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q \mathcal{D}_q^{n-k}f(zq^k)\mathcal{D}_q^k g(z).$$

1.13 Show that $u(z)={}_2\phi_1(a,b;c;q,z)$ satisfies (for $|z|<1$ and in the formal power series sense) the second order q-differential equation

$$z(c-abqz)\mathcal{D}_q^2u+\left[\frac{1-c}{1-q}+\frac{(1-a)(1-b)-(1-abq)}{1-q}z\right]\mathcal{D}_q u$$
$$-\frac{(1-a)(1-b)}{(1-q)^2}u=0,$$

where $\mathcal{D}_q$ is defined as in Ex. 1.12. By replacing a,b,c, respectively, by q^a,q^b,q^c and then letting $q\to1^-$ show that the above equation tends to the second order differential equation

$$z(1-z)v''+[c-(a+b+1)z]v'-abv=0$$

for the hypergeometric function $v(z) = {}_2F_1(a,b;c;z)$, where $|z|<1$. (Heine [1847])

1.14 Let $|x|<1$ and let $e_q(x)$ and $E_q(x)$ be as defined in §1.3. Define

$$\sin_q(x) = \frac{e_q(ix)-e_q(-ix)}{2i} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(q;q)_{2n+1}},$$

$$\cos_q(x) = \frac{e_q(ix)+e_q(-ix)}{2} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(q;q)_{2n}}.$$

Also define

$$\mathrm{Sin}_q(x) = \frac{E_q(ix)-E_q(-ix)}{2i}, \quad \mathrm{Cos}_q(x) = \frac{E_q(ix)+E_q(-ix)}{2}.$$

Show that

(i) $$e_q(ix) = \cos_q(x) + i\sin_q(x),$$

(ii) $$E_q(ix) = \mathrm{Cos}_q(x) + i\mathrm{Sin}_q(x),$$

(iii) $$\sin_q(x)\mathrm{Sin}_q(x) + \cos_q(x)\mathrm{Cos}_q(x) = 1,$$

(iv) $$\sin_q(x)\mathrm{Cos}_q(x) - \mathrm{Sin}_q(x)\cos_q(x) = 0.$$

For these identities and other identities involving q-analogues of $\sin x$ and $\cos x$, see Jackson [1904a] and Hahn [1949c].

1.15 Prove the transformation formulas

(i) $${}_2\phi_1\left[\begin{matrix} q^{-n}, b \\ c \end{matrix}; q, z\right] = \frac{(bzq^{-n}/c;q)_\infty}{(bz/c;q)_\infty}\, {}_3\phi_2\left[\begin{matrix} q^{-n}, c/b, 0 \\ c, cq/bz \end{matrix}; q, q\right],$$

(ii) $${}_2\phi_1\left[\begin{matrix} q^{-n}, b \\ c \end{matrix}; q, z\right] = \frac{(c/b;q)_n}{(c;q)_n} b^n\, {}_3\phi_1\left[\begin{matrix} q^{-n}, b, q/z \\ bq^{1-n}/c \end{matrix}; q, z/c\right],$$

(iii) $${}_2\phi_1\left[\begin{matrix} q^{-n}, b \\ c \end{matrix}; q, z\right] = \frac{(c/b;q)_n}{(c;q)_n}\, {}_3\phi_2\left[\begin{matrix} q^{-n}, b, bzq^{-n}/c \\ bq^{1-n}/c,\ 0 \end{matrix}; q, q\right].$$

(See Jackson [1905a, 1927])

1.16 Show that

$$\sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} q^{n(n+1)/2} = (-q;q)_\infty (aq;q^2)_\infty.$$

1.17 Show that

$$\sum_{k=0}^{n} \frac{(a,b;q)_k}{(q;q)_k} (-ab)^{n-k} q^{(n-k)(n+k-1)/2}$$

$$= (a;q)_{n+1} \sum_{k=0}^{n} \frac{(-b)^k q^{\binom{k}{2}}}{(q;q)_k (q;q)_{n-k} (1-aq^{n-k})}. \qquad \text{(Carlitz [1974])}$$

1.18 Show that

(i) $(c;q)_\infty\ {}_1\phi_1(a;c;q,z) = (z;q)_\infty\ {}_1\phi_1(az/c;z;q,c)$,
and deduce that ${}_1\phi_1(-bq;0;q,-q) = (-bq^2;q^2)_\infty/(q;q^2)_\infty$,

(ii) $(z;q)_\infty\ {}_2\phi_1(a,0;c;q,z) = (az;q)_\infty\ {}_1\phi_2(a;c,az;q,cz)$,

(iii) $$\sum_{n=0}^{\infty} \frac{(a;q)_n}{(q,a^2;q)_n} q^{\binom{n}{2}} (at/z)^n\ {}_2\phi_1(q^{-n},a;q^{1-n}/a;q,qz^2/a)$$
$$= (-zt;q)_\infty\ {}_2\phi_1(a,a/z^2;a^2;q,-zt), \quad |zt|<1.$$

1.19 Using (1.5.4) show that

(i) $${}_2\phi_2\left[\begin{matrix} a, q/a \\ -q, b \end{matrix}; q, -b\right] = \frac{(ab, bq/a;q^2)_\infty}{(b;q)_\infty},$$

(ii) $${}_2\phi_2\left[\begin{matrix} a^2, b^2 \\ abq^{\frac12}, -abq^{\frac12} \end{matrix}; q, -q\right] = \frac{(a^2q, b^2q;q^2)_\infty}{(q, a^2b^2q;q^2)_\infty}.$$

(Andrews [1973])

1.20 Prove that if Re $x > 0$ and $0 < q < 1$, then

(i) $$\Gamma_q(x) = (q;q)_\infty (1-q)^{1-x} \sum_{n=0}^{\infty} \frac{q^{nx}}{(q;q)_n},$$

(ii) $$\frac{1}{\Gamma_q(x)} = \frac{(1-q)^{x-1}}{(q;q)_\infty} \sum_{n=0}^{\infty} \frac{(-1)^n q^{nx}}{(q;q)_n} q^{\binom{n}{2}}.$$

1.21 For $0 < q < 1$ and $x > 0$, show that
$$\frac{d^2}{dx^2} \log \Gamma_q(x) = (\log q)^2 \sum_{n=0}^{\infty} \frac{q^{n+x}}{(1-q^{n+x})^2},$$
which proves that $\log \Gamma_q(x)$ is convex for $x > 0$ when $0 < q < 1$.

1.22 Conversely, prove that if $f(x)$ is a positive function defined on $(0,\infty)$which satisfies
$$f(x+1) = \frac{1-q^x}{1-q} f(x) \text{ for some } q, 0<q<1,$$
$$f(1) = 1,$$
and $\log f(x)$ is convex for $x > 0$, then $f(x) = \Gamma_q(x)$. This is Askey's [1978] q-analogue of the Bohr-Mollerup [1922] theorem for $\Gamma(x)$. For two extensions to the $q > 1$ case (with $\Gamma_q(x)$ defined as in the next exercise), see Moak [1980b].

1.23 For $q > 1$ the *q-gamma function* is defined by
$$\Gamma_q(x) = \frac{(q^{-1};q^{-1})_\infty}{(q^{-x};q^{-1})_\infty} (q-1)^{1-x} q^{x(x-1)/2}.$$

Show that this function also satisfies the functional equation (1.10.5) and that $\Gamma_q(x) \to \Gamma(x)$ as $q \to 1^+$. Show that for $q > 1$ the residue of $\Gamma_q(x)$ at $x = -n$ is

$$\frac{(q-1)^{n+1} q^{\binom{n+1}{2}}}{(q;q)_n \log q}.$$

1.24 Jackson [1905a,b,e] gave the following q-analogues of Bessel functions:

$$J_\nu^{(1)}(x;q) = \frac{(q^{\nu+1};q)_\infty}{(q;q)_\infty}(x/2)^\nu \,{}_2\phi_1(0,0;q^{\nu+1};q,-x^2/4),$$
$$J_\nu^{(2)}(x;q) = \frac{(q^{\nu+1};q)_\infty}{(q;q)_\infty}(x/2)^\nu \,{}_0\phi_1\left(—;q^{\nu+1};q,-\frac{x^2 q^{\nu+1}}{4}\right),$$
$$J_\nu^{(3)}(x;q) = \frac{(q^{\nu+1};q)_\infty}{(q;q)_\infty}(x/2)^\nu \,{}_1\phi_1(0;q^{\nu+1};q,qx^2/4),$$

where $0 < q < 1$. The above notations for the *q-Bessel functions* are due to Ismail [1981, 1982, 2003c].
Show that

$$J_\nu^{(2)}(x;q) = \left(-x^2/4;q\right)_\infty J_\nu^{(1)}(x;q), \quad |x| < 2, \quad \text{(Hahn [1949c])}$$

and

$$\lim_{q\to 1} J_\nu^{(k)}(x(1-q);q) = J_\nu(x), \qquad k = 1,2,3.$$

1.25 For the q-Bessel functions defined as in Exercise 1.24 prove that

(i) $$q^\nu J_{\nu+1}^{(k)}(x;q) = \frac{2(1-q^\nu)}{x} J_\nu^{(k)}(x;q) - J_{\nu-1}^{(k)}(x;q), \ k = 1,2;$$

(ii) $$J_\nu^{(1)}(xq^{\frac12};q) = q^{\nu/2}\left(J_\nu^{(1)}(x;q) + \frac{x}{2}J_{\nu+1}^{(1)}(x;q)\right);$$

(iii) $$J_\nu^{(1)}(xq^{\frac12};q) = q^{-\nu/2}\left(J_\nu^{(1)}(x;q) - \frac{x}{2}J_{\nu-1}^{(1)}(x;q)\right).$$

(iv) $$q^{\nu+1} J_{\nu+1}^{(3)}(xq^{1/2};q) = \frac{2(1-q^\nu)}{x} J_\nu^{(3)}(x;q) - J_{\nu-1}^{(3)}(x;q).$$

1.26 (i) Following Ismail [1982], let

$$f_\nu(x) = J_\nu^{(1)}(x;q)J_{-\nu}^{(1)}(xq^{\frac12};q) - J_{-\nu}^{(1)}(x;q)J_\nu^{(1)}(xq^{\frac12};q).$$

Show that

$$f_\nu(xq^{\frac12}) = \left(1+\frac{x^2}{4}\right) f_\nu(x)$$

and deduce that, for non-integral ν,

$$f_\nu(x) = q^{-\nu/2}(q^\nu, q^{1-\nu};q)_\infty/(q,q,-x^2/4;q)_\infty.$$

(ii) Show that

$$g_\nu(qx) + \left(x^2/4 - q^\nu - q^{-\nu}\right)g_\nu(x) + g_\nu(xq^{-1}) = 0$$

with $g_\nu(x) = J_\nu^{(3)}(xq^{\nu/2}; q^2)$ and deduce that

$$g_\nu(x)g_{-\nu}(xq^{-1}) - g_{-\nu}(x)g_\nu(xq^{-1}) = \frac{(q^{2\nu}, q^{1-2\nu}; q^2)_\infty}{(q^2, q^2; q^2)_\infty} q^{\nu(\nu-1)}.$$

(Ismail [2003c])

1.27 Show that

$$\text{(i)} \qquad \sum_{n=-\infty}^{\infty} t^n J_n^{(2)}(x;q) = \left(-x^2/4;q\right)_\infty e_q(xt/2)e_q(-x/2t),$$

$$\text{(ii)} \qquad \sum_{n=-\infty}^{\infty} t^n J_n^{(3)}(x;q) = e_q(xt/2)E_q(-qx/2t).$$

Both of these are q-analogues of the generating function

$$\sum_{n=-\infty}^{\infty} t^n J_n(x) = e^{x(t-t^{-1})/2}.$$

1.28 The *continuous q-Hermite polynomials* are defined in Askey and Ismail [1983] by

$$H_n(x|q) = \sum_{k=0}^{n} \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}} e^{i(n-2k)\theta},$$

where $x = \cos\theta$; see Szegő [1926], Carlitz [1955, 1957a, 1958, 1960] and Rogers [1894, 1917]. Derive the generating function

$$\sum_{n=0}^{\infty} \frac{H_n(x|q)}{(q;q)_n} t^n = \frac{1}{(te^{i\theta}, te^{-i\theta}; q)_\infty}, \quad |t| < 1. \quad \text{(Rogers [1894])}$$

1.29 The *continuous q-ultraspherical polynomials* are defined in Askey and Ismail [1983] by

$$C_n(x;\beta|q) = \sum_{k=0}^{n} \frac{(\beta;q)_k(\beta;q)_{n-k}}{(q;q)_k(q;q)_{n-k}} e^{i(n-2k)\theta},$$

where $x = \cos\theta$. Show that

$$\begin{aligned} C_n(x;\beta|q) &= \frac{(\beta;q)_n}{(q;q)_n} e^{in\theta} {}_2\phi_1\left[\begin{matrix} q^{-n}, \beta \\ \beta^{-1}q^{1-n} \end{matrix}; q, q\beta^{-1}e^{-2i\theta}\right] \\ &= \frac{(\beta^2;q)_n}{(q;q)_n} e^{-in\theta}\beta^{-n} {}_3\phi_2\left[\begin{matrix} q^{-n}, \beta, \beta e^{2i\theta} \\ \beta^2, 0 \end{matrix}; q, q\right], \\ &\lim_{q\to 1} C_n(x;q^\lambda|q) = C_n^\lambda(x), \end{aligned}$$

and

$$\sum_{n=0}^{\infty} C_n(x;\beta|q)t^n = \frac{(\beta te^{i\theta}, \beta te^{-i\theta}; q)_\infty}{(te^{i\theta}, te^{-i\theta}; q)_\infty}, \quad |t| < 1. \quad \text{(Rogers [1895])}$$

1.30 Show that if $m_1, \ldots, m_r$ are nonnegative integers, then

(i)
$$ {}_{r+1}\phi_{r+1}\left[\begin{matrix} b, & b_1q^{m_1}, \ldots, b_rq^{m_r} \\ & bq, b_1, \ldots, b_r \end{matrix} ; q, q^{1-(m_1+\cdots+m_r)}\right] $$
$$ = \frac{(q;q)_\infty (b_1/b;q)_{m_1}\cdots(b_r/b;q)_{m_r}}{(bq;q)_\infty (b_1;q)_{m_1}\cdots(b_r;q)_{m_r}} b^{m_1+\cdots+m_r}, $$

(ii)
$$ {}_r\phi_r\left[\begin{matrix} b_1q^{m_1}, \ldots, b_rq^{m_r} \\ b_1, \ldots, b_r \end{matrix} ; q, q^{-(m_1+\cdots+m_r)}\right] = 0, $$

(iii)
$$ {}_r\phi_r\left[\begin{matrix} b_1q^{m_1}, \ldots, b_rq^{m_r} \\ b_1, \ldots, b_r \end{matrix} ; q, q^{1-(m_1+\cdots+m_r)}\right] $$
$$ = \frac{(-1)^{m_1+\cdots+m_r}(q;q)_\infty b_1^{m_1}\cdots b_r^{m_r}}{(b_1;q)_{m_1}\cdots(b_r;q)_{m_r}} q^{\binom{m_1}{2}+\cdots+\binom{m_r}{2}}. $$

(Gasper [1981a])

1.31 Let Δ_b denote the *q-difference operator* defined for a fixed q by
$$ \Delta_b f(z) = bf(qz) - f(z). $$
Then Δ_1 is the Δ operator defined in (1.3.20). Show that
$$ \Delta_b x^n = (bq^n - 1)x^n $$
and, if
$$ v_n(z) = \frac{(a_1, \ldots, a_r; q)_n}{(q, b_1, \ldots, b_s; q)_n} (-1)^{(1+s-r)n} q^{(1+s-r)n(n-1)/2} z^n, $$
then
$$ (\Delta\Delta_{b_1/q}\Delta_{b_2/q}\cdots\Delta_{b_s/q})v_n(z) $$
$$ = z(\Delta_{a_1}\Delta_{a_2}\cdots\Delta_{a_r})v_{n-1}(zq^{1+s-r}), \quad n = 1, 2, \ldots . $$
Use this to show that the basic hypergeometric series
$$ v(z) = {}_r\phi_s(a_1, \ldots, a_r; b_1, \ldots, b_s; q, z) $$
satisfies (in the sense of formal power series) the q-difference equation
$$ (\Delta\Delta_{b_1/q}\Delta_{b_2/q}\cdots\Delta_{b_s/q})v(z) = z(\Delta_{a_1}\cdots\Delta_{a_r})v(zq^{1+s-r}). $$
This is a q-analogue of the formal differential equation for generalized hypergeometric series given, e.g. in Henrici [1974, Theorem (1.5)] and Slater [1966, (2.1.2.1)]. Also see Jackson [1910d, (15)].

1.32 The *little q-Jacobi polynomials* are defined by
$$ p_n(x; a, b; q) = {}_2\phi_1(q^{-n}, abq^{n+1}; aq; q, qx). $$
Show that these polynomials satisfy the orthogonality relation
$$ \sum_{j=0}^\infty \frac{(bq;q)_j}{(q;q)_j}(aq)^j p_n(q^j; a, b; q) p_m(q^j; a, b; q) $$
$$ = \begin{cases} 0, & \text{if } m \neq n, \\ \dfrac{(q, bq; q)_n (1 - abq)(aq)^n}{(aq, abq; q)_n (1 - abq^{2n+1})} \dfrac{(abq^2; q)_\infty}{(aq; q)_\infty}, & \text{if } m = n. \end{cases} $$

(Andrews and Askey [1977])

1.33 Show for the above little q-Jacobi polynomials that the formula

$$p_n(x;c,d;q)=\sum_{k=0}^{n} a_{k,n}p_k(x;a,b;q)$$

holds with

$$a_{k,n}=(-1)^k q^{\binom{k+1}{2}}\frac{(q^{-n},aq,cdq^{n+1};q)_k}{(q,cq,abq^{k+1};q)_k}\,{}_3\phi_2\left[\begin{matrix}q^{k-n},cdq^{n+k+1},aq^{k+1}\\ cq^{k+1},abq^{2k+2}\end{matrix};q,q\right].$$

(Andrews and Askey [1977])

1.34 (i) If $m,m_1,m_2,\ldots,m_r$ are arbitrary nonnegative integers and $|a^{-1}q^{m+1-(m_1+\cdots+m_r)}|<1$, show that

$$\begin{aligned}&{}_{r+2}\phi_{r+1}\left[\begin{matrix}a,b,b_1q^{m_1},\ldots,b_rq^{m_r}\\ bq^{1+m},b_1,\ldots,b_r\end{matrix};q,a^{-1}q^{m+1-(m_1+\cdots+m_r)}\right]\\ &=\frac{(q,bq/a;q)_\infty(bq;q)_m(b_1/b;q)_{m_1}\cdots(b_r/b;q)_{m_r}}{(bq,q/a;q)_\infty(q;q)_m(b_1;q)_{m_1}\cdots(b_r;q)_{m_r}}b^{m_1+\cdots+m_r-m}\\ &\times{}_{r+2}\phi_{r+1}\left[\begin{matrix}q^{-m},b,bq/b_1,\ldots,bq/b_r\\ bq/a,bq^{1-m_1}/b_1,\ldots,bq^{1-m_r}/b_r\end{matrix};q,q\right];\end{aligned}$$

(ii) if $m_1,m_2,\ldots,m_r$ are nonnegative integers and $|a^{-1}q^{1-(m_1+\cdots+m_r)}|<1$, $|cq|<1$, show that

$$\begin{aligned}&{}_{r+2}\phi_{r+1}\left[\begin{matrix}a,b,b_1q^{m_1},\ldots,b_rq^{m_r}\\ bcq,b_1,\ldots,b_r\end{matrix};q,a^{-1}q^{1-(m_1+\cdots+m_r)}\right]\\ &=\frac{(bq/a,cq;q)_\infty}{(bcq,q/a;q)_\infty}\frac{(b_1/b;q)_{m_1}\cdots(b_r/b;q)_{m_r}}{(b_1;q)_{m_1}\cdots(b_r;q)_{m_r}}b^{m_1+\cdots+m_r}\\ &\times{}_{r+2}\phi_{r+1}\left[\begin{matrix}c^{-1},b,bq/b_1,\ldots,bq/b_r\\ bq/a,bq^{1-m_1}/b_1,\ldots,bq^{1-m_r}/b_r\end{matrix};q,cq\right].\end{aligned}$$

(Gasper [1981a])

1.35 Use Ex. 1.2(v) to prove that if x and y are indeterminates such that $xy=qyx$, q commutes with x and y, and the associative law holds, then

$$(x+y)^n=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q y^kx^{n-k}=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_{q^{-1}}x^ky^{n-k}.$$

(See Cigler [1979], Feinsilver [1982], Koornwinder [1989], Potter[1950], Schützenberger [1953], and Yang [1991]).

1.36 Verify that if x and y are indeterminates satisfying the conditions in Ex. 1.35, then

(i) $\quad e_q(y)e_q(x)=e_q(x+y),\quad e_q(x)e_q(y)=e_q(x+y-yx);$

(ii) $\quad E_q(x)E_q(y)=E_q(x+y),\quad E_q(y)E_q(x)=E_q(x+y+yx).$

(Fairlie and Wu [1997]; Koornwinder [1997], where q-exponentials with q-Heisenberg relations and other relations are also considered.)

1.37 Show that

$$\mathcal{E}_q(z;\alpha) = \frac{(\alpha^2;q^2)_\infty}{(q\alpha^2;q^2)_\infty} \left\{ {}_2\phi_1 \left[\begin{matrix} -qe^{2i\theta}, -qe^{-2i\theta} \\ q \end{matrix} ; q^2, \alpha^2 \right] + \frac{2q^{1/4}}{1-q} \alpha \cos\theta \, {}_2\phi_1 \left[\begin{matrix} -q^2e^{2i\theta}, -q^2e^{-2i\theta} \\ q^3 \end{matrix} ; q^2, \alpha^2 \right] \right\}$$

with $z = \cos\theta$.

1.38 Extend Jacobi's triple product identity to the transformation formula

$$1 + \sum_{n=1}^{\infty} (-1)^n q^{\binom{n}{2}} (a^n + b^n) = (q,a,b;q)_\infty \sum_{n=0}^{\infty} \frac{(ab/q;q)_{2n} q^n}{(q,a,b,ab;q)_n}.$$

Deduce that

$$1 + 2\sum_{n=1}^{\infty} a^n q^{2n^2} = (q;q)_\infty (aq;q^2)_\infty \sum_{n=0}^{\infty} \frac{(-a;q)_{2n} q^n}{(q,-aq;q)_n (aq;q^2)_n}.$$

(Warnaar [2003a])

Notes

§§1.1 and 1.2 For additional material on hypergeometric series and orthogonal polynomials see, e.g., the books by Erdélyi [1953], Rainville [1960], Szegő [1975], Whittaker and Watson [1965], Agarwal [1963], Carlson [1977], T.S. Chihara [1978], Henrici [1974], Luke [1969], Miller [1968], Nikiforov and Uvarov [1988], Vilenkin [1968], and Watson [1952]. Some techniques for using symbolic computer algebraic systems such as Mathematica, Maple, and Macsyma to derive formulas containing hypergeometric and basic hypergeometric series are discussed in Gasper [1990]. Also see Andrews [1984d, 1986, 1987b], Andrews, Crippa and Simon [1997], Andrews and Knopfmacher [2001], Andrews, Knopfmacher, Paule and Zimmermann [2001], Andrews, Paule and Riese [2001a,b], Askey [1989f, 1990], Askey, Koepf and Koornwinder [1999], Böing and Koepf [1999], Garoufalidis [2003], Garoufalidis, Le and Zeilberger [2003], Garvan [1999], Garvan and Gonnet [1992], Gosper [2001], Gosper and Suslov [2000], Koepf [1998], Koornwinder [1991b, 1993a, 1998], Krattenthaler [1995b], Paule and Riese [1997], Petkovsek, Wilf and Zeilberger [1996], Riese [2003], Sills [2003c], Wilf and Zeilberger [1990], and Zeilberger [1990b].

§§1.3–1.5 The q-binomial theorem was also derived in Jacobi [1846], along with the q-Vandermonde formula. Bijective proofs of the q-binomial theorem, Heine's ${}_2\phi_1$ transformation and q-analogue of Gauss' summation formula, the q-Saalschütz formula, and of other formulas are presented in Joichi and Stanton [1987]. Rahman and Suslov [1996a] used the method of first order linear difference equations to prove the q-binomial and q-Gauss formulas. Bender [1971] used partitions to derive an extension of the q-Vandermonde sum in the form of a generalized q-binomial Vandermonde convolution. The even and odd parts of the infinite series on the right side of (1.3.33) appeared in Atakishiyev and Suslov [1992a], but without any explicit reference to the q-exponential function. Also see Suslov [1998–2003] and the q-convolutions in Carnovale [2002], Carnovale and Koornwinder [2000], and Rogov [2000].

§1.6 Other proofs of Jacobi's triple product identity and/or applications of it are presented in Adiga *et al.* [1985], Alladi and Berkovich [2003], Andrews [1965], Cheema [1964], Ewell [1981], Gustafson [1989], Joichi and Stanton [1989], Kac [1978, 1985], Lepowsky and Milne [1978], Lewis [1984], Macdonald [1972], Menon [1965], Milne [1985a], Sudler [1966], Sylvester [1882], and Wright [1965]. Concerning theta functions, see Adiga *et al.* [1985], Askey [1989c], Bellman [1961], and Jensen's use of theta functions in Pólya [1927] to derive necessary and sufficient conditions for the Riemann hypothesis to hold.

§1.7 Some applications of the q-Saalschütz formula are contained in Carlitz [1969b] and Wright [1968].

§1.9 Formulas (1.9.3) and (1.9.8) were rediscovered by Gustafson [1987a, Theorems 3.15 and 3.18] while working on multivariable orthogonal polynomials.

§1.11 Also see Jackson [1917, 1951] and, for fractional q-integrals and q-derivatives, Al-Salam [1966] and Agarwal [1969b]. Toeplitz [1963, pp. 53–55] pointed out that around 1650 Fermat used a q-integral type Riemann sum to evaluate the integral of x^k on the interval $[0, b]$. Al-Salam and Ismail [1994] evaluated a q-beta integral on the unit circle and found corresponding systems of biorthogonal rational functions.

Ex. 1.2 The q-binomial coefficient $\begin{bmatrix} n \\ k \end{bmatrix}_q$, which is also called the Gaussian binomial coefficient, counts the number of k dimensional subspaces of an n dimensional vector space over a field $GF(q)$, q a prime power (Goldman and Rota [1970]), and it is the generating function, in powers of q, for partitions into at most k parts not exceeding $n-k$ (Sylvester [1882]). It arises in such diverse fields as analysis, computer programming, geometry, number theory, physics, and statistics. See, e.g., Aigner [1979], Andrews [1971a, 1976], M. Baker and Coon [1970], Baxter and Pearce [1983], Berman and Fryer [1972], Dowling [1973], Dunkl [1981], Garvan and Stanton [1990], Handa and Mohanty [1980], Ihrig and Ismail [1981], Jimbo [1985, 1986], van Kampen [1961], Kendall and Stuart [1979, §31.25], Knuth [1971, 1973], Pólya [1970], Pólya and Alexanderson [1970], Szegő [1975, §2.7], and Zaslavsky [1987]. Sylvester [1878] used the invariant theory that he and Cayley developed to prove that the coefficients of the Gaussian polynomial $\begin{bmatrix} n \\ k \end{bmatrix}_q = \Sigma a_j q^j$ are unimodal. A constructive proof was recently given by O'Hara [1990]. Also see Bressoud [1992] and Zeilberger [1989a,b, 1990b]. The unimodality of the sequence $\left(\begin{bmatrix} n \\ k \end{bmatrix}_q : k = 0, 1, \ldots, n \right)$ is explicitly displayed in Aigner [1979, Proposition 3.13], and Macdonald [1995, Example 4 on p. 137].

Ex. 1.3 Cigler [1979] derived an operator form of the q-binomial theorem. MacMahon [1916, Arts. 105–107] showed that if a multiset is permuted, then the generating function for inversions is the q-multinomial coefficient. Also see Carlitz [1963a], Kadell [1985a], and Knuth [1973, p. 33, Ex. 16]. Gasper derived the q-multinomial theorem in part (ii) several years ago by using the q-binomial theorem and mathematical induction. Andrews observed in a 1988 letter that it can also be derived by using the expansion formula for the q-Lauricella function Φ_D stated in Andrews [1972, (4.1)] and the q-Vandermonde sum. Some sums of q-multinomial coefficients are considered in Bressoud [1978, 1981c]. See also Agarwal [1953a].

Ex. 1.8 Jain [1980c] showed that the sum in this exercise is equivalent to the sum of a certain ${}_2\psi_2$ series, and summed some other ${}_2\psi_2$ series.

Ex. 1.10 Analogous recurrence relations for ${}_1\phi_1$ series are given in Slater [1954c].

Exercises 1.12 and 1.13 The notations Δ_q, ϑ_q, and D_q are also employed in the literature for this q-derivative operator. We employed the script $\mathcal{D}_q$ operator notation to distinguish this q-derivative operator from the q-derivative operator defined in (7.7.3) and the q-difference operator defined in Ex. 1.31. Additional results involving q-derivatives and q-difference equations are contained in Adams [1931], Agarwal [1953d], Andrews [1968, 1971a], Bowman [2002], Carmichael [1912], Di Vizio [2002, 2003], Faddeev and Kashaev [2002], Faddeev, Kashaev and Volkov [2001], Hahn [1949a,c, 1950, 1952, 1953], Ismail, Merkes and Styer [1990], Jackson [1905c, 1909a, 1910b,d,e], Miller [1970], Mimachi [1989], Sauloy [2003], Starcher [1931], and Trjitzinsky [1933]. For fractional q-derivatives and q-integrals see Agarwal [1969b] and Al-Salam and Verma [1975a,b]. Some "q-Taylor series" are considered in Jackson [1909b,c] and Wallisser [1985]. A q-Taylor theorem based on the sequence $\{\phi_n(x)\}_{n=0}^{\infty}$ with $\phi_n(x) = (ae^{i\theta}, ae^{-i\theta}; q)_n$, $x = \cos\theta$, was obtained by Ismail and Stanton [2003a,b] along with some interesting applications.

Ex. 1.14 For q-tangent and q-secant numbers and some of their properties, see Andrews and Foata [1980] and Foata [1981]. A discussion of q-trigonometry is given in Gosper [2001]. See also Bustoz and Suslov [1998] and Suslov [2003].

Exercises 1.20–1.23 Ismail and Muldoon [1994] studied some inequalities and monotonicity properties of the gamma and q-gamma functions.

Ex. 1.22 Also see Artin [1964, pp. 14–15]. A different characterization of Γ_q is presented in Kairies and Muldoon [1982].

Exercises 1.24–1.27 Other formulas involving q-Bessel functions are contained in Jackson [1904a–d, 1908], Ismail and Muldoon [1988], Rahman [1987, 1988c, 1989b,c], and Swarttouw and Meijer [1994]. It was pointed out by Ismail in an unpublished preprint in 1999 (rewritten for publication as Ismail [2003c]) that $J_\nu^{(3)}(x;q)$ was actually introduced by Jackson [1905a], contrary to the claim in Swarttouw [1992] that a special case of it was first discovered by Hahn [1953] and then in full generality by Exton [1978].

Ex. 1.28 See the generating functions for the continuous q-Hermite polynomials derived in Carlitz [1963b, 1972] and Bressoud [1980b], and the applications to modular forms in Bressoud [1986]. An extension of these q-Bessel functions to a q-quadratic grid is given in Ismail, Masson and Suslov [1999].

Ex. 1.32 Masuda *et al.* [1991] showed that the matrix elements that arise in the representations of certain quantum groups are expressible in terms of little q-Jacobi polynomials, and that this and a form of the Peter-Weyl theorem imply the orthogonality relation for these polynomials. Padé approximants for the moment generating function for the little q-Jacobi polynomials are employed in Andrews, Goulden and D.M. Jackson [1986] to explain and extend Shank's method for accelerating the convergence of sequences. Padé approximations for some q-hypergeometric functions are considered in Ismail, Perline and Wimp [1992].