Lectures on
Advanced Applied Mathematics

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Course Name: Advanced Applied Mathematics
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1. **The Riemann-Liouville Fractional Integrals:**

   Definition of the Fractional Integral, Some Examples of Fractional Integrals.

2. **Hankel Transforms and Their Applications:**

   The Hankel Transform and Examples, Operational Properties of the Hankel Transform, Applications of Hankel Transforms to Partial Differential Equations.

3. **Mellin Transforms and Their Applications:**

   Definition of the Mellin Transform and Examples, Basic Operational Properties of Mellin Transforms, Applications of Mellin Transforms to Partial Differential Equations.

4. **Power Series Solutions of Linear Differential Equations with Variable Coefficients:**

   Series solutions near a regular singular point, Method of Frobeius, General Solution.

References:

1. K.S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, John Wiley & Sons, Inc., New York, 1993.

2. L. Debnath and D. Bhatta, *Integral Transforms and Their Applications*, CRC Press Taylor & Francis Group,Boca Raton, 3$^{\text{nd}}$ ed., 2015.

3. R. Bronson and G.B. Costa, *Schaum's outline of Differential Equations*, McGraw-Hill Companies, Inc., 3$^{\text{nd}}$ ed., 2006.

III

THE RIEMANN–LIOUVILLE FRACTIONAL INTEGRAL

1. INTRODUCTION

After the lengthy justifications of Chapter II, we begin our mathematical development of the fractional calculus. We start with a formal definition of the Riemann–Liouville fractional integral, carefully delineating the class of functions to which this fractional operator may be applied. Numerous examples, some trivial and some not so elementary, are given and discussed. This analysis provides a convenient vehicle for introducing certain new functions such as $E_t(\nu, a)$, $C_t(\nu, a)$, $S_t(\nu, a)$ that play a forward role in the fractional calculus and fractional differential equations. (Properties of these functions are examined in some detail in Appendix C.)

Certain techniques are developed that enable us to find fractional integrals of more complicated functions. In Section III-4 we consider the Dirichlet formula and analyze some of its consequences. Most prominent is its use in the proof of the law of exponents for fractional integrals. That is, we shall show that ${}_0D_t^{-\mu}({}_0D_t^{-\nu}) = {}_0D_t^{-\mu-\nu}$ for all positive μ and ν (Theorem 1). It also will be used to obtain the fractional integrals of certain nonelementary functions.

In later sections we examine the relations that exist between (ordinary) derivatives of fractional integrals and fractional integrals of derivatives. Many ancillary results in the theory of the fractional calculus may be deduced from these theorems. The penultimate section is devoted to the problem of finding the Laplace transform of fractional integrals, together with the inevitable consequences. The Laplace transform frequently will be exploited in remaining chapters, especially in our study of fractional differential equations. In the final section we discuss Leibniz's formula for fractional integrals and give some interesting applications of this rule.

2. DEFINITION OF THE FRACTIONAL INTEGRAL

As we have stated before, our objective is to investigate various aspects of the Riemann–Liouville fractional integral. We begin with a formal definition (see Definition 1 below).

Let X be a positive number and let f be continuous on $[0, X]$. Then if $\nu \geqq 1$,

$$\int_0^t (t-\xi)^{\nu-1} f(\xi)\, d\xi \tag{2.1}$$

exists as a Riemann integral for all $t \in [0, X]$. Of course, (2.1) will exist under more general conditions. For example, if f is continuous on $(0, X]$ and behaves like t^λ for $-1 < \lambda < 0$ in a neighborhood of the origin and/or if $0 < \operatorname{Re} \nu < 1$, then (2.1) exists as an improper Riemann integral. The following definition, however, is sufficiently broad for our purposes.

Definition 1. Let $\operatorname{Re} \nu > 0$ and let f be piecewise continuous on $J' = (0, \infty)$ and integrable on any finite subinterval of $J = [0, \infty)$. Then for $t > 0$ we call

$$_0D_t^{-\nu} f(t) = \frac{1}{\Gamma(\nu)} \int_0^t (t-\xi)^{\nu-1} f(\xi)\, d\xi \tag{2.2}$$

the Riemann–Liouville fractional integral of f of order ν.

Let us discuss this definition. As we have observed above, (2.2) is an improper integral if $0 < \operatorname{Re} \nu < 1$. We require f to be piecewise continuous only on $J' = (0, \infty)$ (the interval J excluding the origin) to accommodate functions that behave like $\ln t$ or t^μ (for $-1 < \mu < 0$) in a neighborhood of the origin. We shall denote by $\mathbf{C}$ the class of functions described in Definition 1. [One readily may generalize $\mathbf{C}$ to include, for example, such functions as $f(\xi) = |\xi - a|^\lambda$, $\lambda > -1$, $0 < a < t$. We seldom shall have occasion to do so.]

For example, if $f(t) = t^{\mu}$ with $\mu > -1$, then [see (II-6.3), p. 36]

$$ {}_0D_t^{-\nu} t^{\mu} = \frac{\Gamma(\mu + 1)}{\Gamma(\mu + \nu + 1)} t^{\mu+\nu}, \qquad t > 0 \tag{2.3}$$

[since (2.2) is now essentially the beta function]. Because $\mu + \operatorname{Re}\nu$ may be negative, we see from this example why we must include the caveat $t > 0$ in our definition of the fractional integral. [Of course, if $\mu \geqq 0$, then (2.3) is continuous on J.] To avoid minor mathematical complications not related to the fractional calculus, and with little loss of generality, we shall, as a practical matter, assume that ν is real. Occasionally, we indicate that certain formulas are valid for $\operatorname{Re}\nu > 0$ rather than just for $\nu > 0$. A discussion of fractional operators when ν is purely imaginary may be found in [19].

If we write (2.2) as the Stieltjes integral

$$ {}_0D_t^{-\nu} f(t) = \frac{1}{\Gamma(\nu + 1)} \int_0^t f(\xi)\, d\alpha(\xi), $$

where

$$ \alpha(\xi) = -(t - \xi)^{\nu} \tag{2.4}$$

is a (continuous) monotonic increasing function of ξ on $[0, t]$, then if f is continuous on $[0, t]$, the first mean value theorem for integrals [45, p. 107] implies that

$$ \int_0^t f(\xi)\, d\alpha(\xi) = f(x) t^{\nu} $$

for some $x \in [0, t]$. Hence

$$ \lim_{t \to 0} {}_0D_t^{-\nu} f(t) = 0. \tag{2.5}$$

If f is not continuous (but still of class **C**), then (2.5) need not be true. In fact, we see from (2.3) with $\nu > 0$, $\mu > -1$, that

$$ \lim_{t \to 0} {}_0D_t^{-\nu} t^{\mu} = \begin{cases} 0, & \mu + \nu > 0 \\ \Gamma(\mu + 1), & \mu + \nu = 0 \\ \infty, & \mu + \nu < 0. \end{cases} $$

Furthermore, we also conclude from (2.3) that even the continuity of f at the origin does not guarantee the differentiability of ${}_0D_t^{-\nu}f(t)$ at $t = 0$. (For example, let $\mu > 0$ and $\mu + \nu < 1$.)

At times it may be expedient to consider certain subclasses of **C**. For instance, in Chapter IV we introduce a class of functions that includes functions of the form

$$t^{\lambda}\eta(t)$$

where $\lambda > -1$ and $\eta(t)$ is analytic. At other times we shall find it convenient to take the Laplace transform of the fractional integral. In such cases we require that f be of exponential order. Since we mainly shall be considering integrals of the form (2.2), the notation will be simplified by dropping the subscripts 0 and t on ${}_0D_t^{-\nu}$, as was done in Section II-7. Occasionally, we shall use them for emphasis, or if there is a possibility of ambiguity, or if we wish to consider a fractional integral whose lower limit is not zero.

3. SOME EXAMPLES OF FRACTIONAL INTEGRALS

Before we embark on a theoretical analysis of the fractional integral, let us calculate the fractional integrals of a few elementary functions. We already have shown in (2.3) that

$$D^{-\nu}t^{\mu} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\nu+1)}t^{\mu+\nu}, \qquad \nu > 0, \quad \mu > -1, \quad t > 0. \quad (3.1)$$

In particular, if $\mu = 0$, the fractional integral of a constant K of order ν is

$$D^{-\nu}K = \frac{K}{\Gamma(\nu+1)}t^{\nu}, \qquad \nu > 0. \quad (3.2)$$

Perhaps the reader may have wondered why we did not give a few additional examples of fractional integrals. The answer is simple—fractional integrals, even of such elementary functions as exponentials and sines and cosines, lead to higher transcendental functions—as we shall now demonstrate.

Suppose that

$$f(t) = e^{at}$$

where a is a constant. Certainly, e^{at} is of class **C**, and by Definition 1,

$$D^{-\nu}e^{at} = \frac{1}{\Gamma(\nu)}\int_0^t (t-\xi)^{\nu-1}e^{a\xi}\,d\xi, \qquad \nu > 0. \tag{3.3}$$

If we make the change of variable $x = t - \xi$, (3.3) becomes

$$D^{-\nu}e^{at} = \frac{e^{at}}{\Gamma(\nu)}\int_0^t x^{\nu-1}e^{-ax}\,dx, \qquad \nu > 0. \tag{3.4}$$

Clearly, (3.4) is not an elementary function. But it *is* closely related to the transcendental function known as the incomplete gamma function [(B-2.19), p. 300, Section C-2]. For $\operatorname{Re}\nu > 0$ the incomplete gamma function $\gamma^*(\nu, t)$ may be defined as

$$\gamma^*(\nu, t) = \frac{1}{\Gamma(\nu)t^\nu}\int_0^t \xi^{\nu-1}e^{-\xi}\,d\xi. \tag{3.5}$$

Thus we may write (3.4) as

$$D^{-\nu}e^{at} = t^\nu e^{at}\gamma^*(\nu, at). \tag{3.6}$$

Since the right-hand side of (3.6) is the fractional integral of an exponential, it is not surprising that this function frequently arises in the study of the fractional calculus. We shall call it $E_t(\nu, a)$,

$$E_t(\nu, a) = t^\nu e^{at}\gamma^*(\nu, at). \tag{3.7}$$

Some of the elementary properties of γ^* and E_t are examined in Appendix C.

A direct application of the definition of the fractional integral leads to

$$D^{-\nu}\cos at = \frac{1}{\Gamma(\nu)}\int_0^t \xi^{\nu-1}\cos a(t-\xi)\,d\xi, \qquad \nu > 0 \tag{3.8}$$

and

$$D^{-\nu}\sin at = \frac{1}{\Gamma(\nu)}\int_0^t \xi^{\nu-1}\sin a(t-\xi)\,d\xi, \qquad \nu > 0. \tag{3.9}$$

We find it convenient to define the right-hand sides of (3.8) and (3.9) as $C_t(\nu, a)$ and $S_t(\nu, a)$, respectively. Properties of these functions also are studied in Appendix C.

Thus from (3.7), (3.8), and (3.9) we have for $\nu > 0$ the compact formulas

$$\begin{aligned} D^{-\nu} e^{at} &= E_t(\nu, a) \\ D^{-\nu} \cos at &= C_t(\nu, a) \\ D^{-\nu} \sin at &= S_t(\nu, a). \end{aligned} \tag{3.10}$$

In the special case $\nu = \frac{1}{2}$,

$$\begin{aligned} D^{-1/2} e^{at} &= E_t(\tfrac{1}{2}, a) \\ &= a^{-1/2} e^{at} \operatorname{Erf}(at)^{1/2}, \end{aligned} \tag{3.11}$$

where Erf x is the error function (B-2.25), p. 301. Also,

$$\begin{aligned} D^{-1/2} \cos at &= C_t(\tfrac{1}{2}, a) \\ &= \sqrt{\frac{2}{a}}\,[(\cos at)C(x) + (\sin at)S(x)] \end{aligned} \tag{3.12}$$

and

$$\begin{aligned} D^{-1/2} \sin at &= S_t(\tfrac{1}{2}, a) \\ &= \sqrt{\frac{2}{a}}\,[(\sin at)C(x) - (\cos at)S(x)], \end{aligned} \tag{3.13}$$

where

$$x = \sqrt{\frac{2at}{\pi}}$$

and $C(x)$ and $S(x)$ are the Fresnel integrals (B-2.27) and (B-2.28), p. 301.

Simple trigonometric identities may be used to calculate other fractional integrals of trigonometric functions. For example, from $\cos 2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$,

$$D^{-\nu} \cos^2 at = \frac{t^\nu}{2\Gamma(\nu + 1)} + \frac{1}{2} C_t(\nu, 2a) \tag{3.14}$$

and

$$D^{-\nu} \sin^2 at = \frac{t^\nu}{2\Gamma(\nu+1)} - \frac{1}{2} C_t(\nu, 2a). \tag{3.15}$$

We consider some slightly more complicated functions. Suppose that

$$f(t) = (a-t)^\lambda, \qquad a > t > 0.$$

Then $f \in \mathbf{C}$, and by definition,

$$D^{-\nu}(a-t)^\lambda = \frac{1}{\Gamma(\nu)} \int_0^t (t-\xi)^{\nu-1}(a-\xi)^\lambda \, d\xi,$$

$$\operatorname{Re} \nu > 0, \quad a > t > 0. \tag{3.16}$$

If we make the bilinear transformation

$$x = \frac{t-\xi}{a-\xi}$$

in the integrand of (3.16), we obtain

$$D^{-\nu}(a-t)^\lambda = \frac{(a-t)^{\lambda+\nu}}{\Gamma(\nu)} \int_0^{t/a} x^{\nu-1}(1-x)^{-\nu-\lambda-1} \, dx.$$

But the integral above is just the incomplete beta function (B-2.24), p. 300. Thus

$$D^{-\nu}(a-t)^\lambda = \frac{1}{\Gamma(\nu)}(a-t)^{\lambda+\nu} \mathrm{B}_{t/a}(\nu, -\lambda-\nu). \tag{3.17}$$

If, in particular, $a = 1$, $\nu = \frac{1}{2}$, and $\lambda = -\frac{1}{2}$, direction integration leads to

$$D^{-1/2} \frac{1}{\sqrt{1-t}} = \frac{1}{\sqrt{\pi}} \ln \frac{1+\sqrt{t}}{1-\sqrt{t}}, \qquad 0 < t < 1. \tag{3.18}$$

As our next example we consider the logarithm. Certainly, $\ln t$ is of class $\mathbf{C}$, and its fractional integral of order ν is

$$D^{-\nu} \ln t = \frac{1}{\Gamma(\nu)} \int_0^t (t-\xi)^{\nu-1} \ln \xi \, d\xi, \qquad \nu > 0.$$

If we make the change of variable $\xi = tx$, then

$$D^{-\nu} \ln t = \frac{t^\nu}{\Gamma(\nu + 1)} \ln t + \frac{t^\nu}{\Gamma(\nu)} \int_0^1 (1 - x)^{\nu - 1} \ln x \, dx. \quad (3.19)$$

But from [12, p. 538],

$$\int_0^1 x^{\mu - 1}(1 - x)^{\nu - 1} \ln x \, dx = \mathrm{B}(\mu, \nu)[\psi(\mu) - \psi(\mu + \nu)],$$
$$\operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0, \quad (3.20)$$

where B is the beta function and ψ is the digamma function (B-2.11), p. 299. Thus if we let $\mu = 1$ in (3.20),

$$D^{-\nu} \ln t = \frac{t^\nu}{\Gamma(\nu + 1)} [\ln t - \gamma - \psi(\nu + 1)], \quad (3.21)$$

where γ is Euler's constant.

If in particular $\nu = \frac{1}{2}$, then

$$\psi\left(\tfrac{3}{2}\right) = 2 - \gamma - \ln 4$$

[21, p. 15] and

$$D^{-1/2} \ln t = \frac{2t^{1/2}}{\sqrt{\pi}} [\ln 4t - 2]. \quad (3.22)$$

More generally, from (3.20) we have

$$D^{-\nu}[t^\lambda \ln t] = \frac{\Gamma(\lambda + 1) t^{\lambda + \nu}}{\Gamma(\lambda + \nu + 1)} [\ln t + \psi(\lambda + 1) - \psi(\lambda + \nu + 1)],$$
$$\lambda > -1, \quad \nu > 0 \quad (3.23)$$

and with $\nu = \frac{1}{2}$ and $\lambda = -\frac{1}{2}$,

$$D^{-1/2}[t^{-1/2} \ln t] = \sqrt{\pi} \ln \frac{t}{4}$$

[see (B-2.13), p. 299].

Another function, which we shall encounter in our future work, is $f(t) = e^{-1/t}$. [If we define $f(0)$ as zero, all the derivatives of f vanish at the origin. Thus f is not analytic at $t = 0$.] We shall calculate the fractional integral in the more general case where $f(t) = t^{\lambda}e^{-a/t}$, $\lambda > -1$. By definition

$$D^{-\nu}\left[t^{\lambda}e^{-a/t}\right] = \frac{1}{\Gamma(\nu)}\int_0^t (t-\xi)^{\nu-1}\xi^{\lambda}e^{-a/\xi}\,d\xi$$

for $\nu > 0$ and $t > 0$. The change of variable of integration

$$\xi = \frac{t}{x+1}$$

immediately leads to

$$D^{-\nu}\left[t^{\lambda}e^{-a/t}\right] = t^{\lambda+\nu}e^{-a/t}U(\nu, -\lambda, a/t) \tag{3.24}$$

for $\nu > 0$, $\lambda > -1$, $t > 0$. If $\operatorname{Re} a > 0$, then U has the integral representation of (B-4.12), p. 305.

Our ability to calculate explicitly the fractional integral of a function f frequently depends on our proficiency in performing the integration

$$\int_0^t (t-\xi)^{\nu-1} f(\xi)\,d\xi, \qquad \nu > 0. \tag{3.25}$$

However, because of the nature of the kernel $(t-\xi)^{\nu-1}$ in (3.25), it is possible to develop certain analytical techniques that allow us to calculate the fractional integral of a large class of functions with minimal effort. We discuss one such technique now.

The procedure we have in mind will allow us to express the fractional integral of an integral power of t times a function $f(t)$ in terms of fractional integrals of f. Using this argument we may show, for example, that

$$D^{-\nu}\left[te^{at}\right] = tE_t(\nu, a) - \nu E_t(\nu+1, a). \tag{3.26}$$

If $f \in \mathbf{C}$, then from Definition 1, p. 45,

$$D^{-\nu}[tf(t)] = \frac{1}{\Gamma(\nu)}\int_0^t (t-\xi)^{\nu-1}[\xi f(\xi)]\,d\xi, \qquad \nu > 0. \tag{3.27}$$

If we replace the term in brackets in the integrand of (3.27) by the identity

$$[t-(t-\xi)]f(\xi)$$

(i.e., we have added and subtracted t), then (3.27) becomes

$$D^{-\nu}[tf(t)] = tD^{-\nu}f(t) - \nu D^{-\nu-1}f(t). \tag{3.28}$$

In the case $f(t) = e^{at}$, formula (3.28) becomes (3.26) [see (3.10)].
Similarly, (3.28) implies that

$$D^{-\nu}[t\cos at] = tC_t(\nu,a) - \nu C_t(\nu+1,a), \qquad \nu > 0 \tag{3.29}$$

and

$$D^{-\nu}[t\sin at] = tS_t(\nu,a) - \nu S_t(\nu+1,a), \qquad \nu > 0. \tag{3.30}$$

Equation (3.28) may readily be generalized. For if p is a nonnegative integer, then

$$D^{-\nu}[t^p f(t)] = \frac{1}{\Gamma(\nu)}\int_0^t (t-\xi)^{\nu-1}[\xi^p f(\xi)]\,d\xi, \qquad \nu > 0 \tag{3.31}$$

and

$$\xi^p = [t-(t-\xi)]^p = \sum_{k=0}^{p}(-1)^k\binom{p}{k}t^{p-k}(t-\xi)^k.$$

Substituting this expression in (3.31), we obtain

$$\begin{aligned} D^{-\nu}[t^p f(t)] &= \frac{1}{\Gamma(\nu)}\sum_{k=0}^{p}(-1)^k\binom{p}{k}t^{p-k}\int_0^t (t-\xi)^{\nu+k-1}f(\xi)\,d\xi \\ &= \frac{1}{\Gamma(\nu)}\sum_{k=0}^{p}(-1)^k\binom{p}{k}\Gamma(\nu+k)t^{p-k}D^{-(\nu+k)}f(t). \end{aligned} \tag{3.32}$$

Using (B-2.6), p. 298, we also may write (3.32) as

$$D^{-\nu}[t^p f(t)] = \sum_{k=0}^{p}\binom{-\nu}{k}[D^k t^p][D^{-\nu-k}f(t)]. \tag{3.33}$$

For example,

$$D^{-\nu}[t^p e^{at}] = \frac{1}{\Gamma(\nu)} \sum_{k=0}^{p} (-1)^k \binom{p}{k} \Gamma(\nu + k) t^{p-k} E_t(\nu + k, a). \quad (3.34)$$

As we develop further techniques we shall be able to find fractional integrals of still more complicated functions. For example, we show in the next section that for $\nu > 0$ and $\mu > -1$,

$$D^{-\nu} E_t(\mu, a) = E_t(\mu + \nu, a). \quad (3.35)$$

Now let us give a few examples of fractional integrals when the lower limit of integration is not necessarily zero. Consider, then,

$$ {}_cD_t^{-\nu} f(t) = \frac{1}{\Gamma(\nu)} \int_c^t (t - \xi)^{\nu - 1} f(\xi)\, d\xi, \qquad \nu > 0, \quad 0 \leqq c < t, \quad (3.36)$$

where f is of class **C** on $[c, \infty)$.

The change of variable

$$\xi = t(1 - x)$$

in (3.36) leads to

$$ {}_cD_t^{-\nu} f(t) = \frac{t^\nu}{\Gamma(\nu)} \int_0^\tau x^{\nu - 1} f(t - tx)\, dx, \quad (3.37)$$

where

$$\tau = \frac{t - c}{t}. \quad (3.38)$$

For example, suppose that

$$f(t) = t^\mu,$$

where $\mu > -1$ if $c = 0$, and μ is arbitrary if $c > 0$. Substitution in (3.37) leads to

$$ {}_cD_t^{-\nu} t^\mu = \frac{t^{\mu + \nu}}{\Gamma(\nu)} \int_0^\tau x^{\nu - 1} (1 - x)^\mu\, dx.$$

But the integral in the expression above is simply the incomplete beta function. Thus

$$ {}_cD_t^{-\nu} t^{\mu} = \frac{t^{\mu+\nu}}{\Gamma(\nu)} \mathrm{B}_{\tau}(\nu, \mu + 1), \tag{3.39} $$

and if we let $c = 0$, formula (3.39) reduces to (3.1), as it should.

Furthermore, if we let $f(t)$ be e^{at} or $\cos at$ or $\sin at$, then (3.37) yields

$$ \begin{aligned} {}_cD_t^{-\nu} e^{at} &= e^{ac} E_{t-c}(\nu, a) \\ {}_cD_t^{-\nu} \cos at &= (\cos ac) C_{t-c}(\nu, a) - (\sin ac) S_{t-c}(\nu, a) \\ {}_cD_t^{-\nu} \sin at &= (\sin ac) C_{t-c}(\nu, a) + (\cos ac) S_{t-c}(\nu, a), \end{aligned} \tag{3.40} $$

which reduce to our previous formulas, (3.10), when $c = 0$. For a table of Riemann–Liouville fractional integrals, see [9] and Appendix D.

We conclude this section with a theoretical result. Suppose that f is continuous on $[0, X]$. Then the Riemann–Liouville fractional integral of f of order ν is

$$ \begin{aligned} D^{-\nu} f(t) &= \frac{1}{\Gamma(\nu)} \int_0^t (t - \xi)^{\nu-1} f(\xi)\, d\xi, \qquad \nu > 0, \quad 0 < t \leqq X \\ &= \frac{1}{\Gamma(\nu)} \int_0^t x^{\nu-1} f(t - x)\, dx. \end{aligned} \tag{3.41} $$

If, furthermore, we require that $f(x)$ be analytic at $x = a$ for all $a \in [0, X]$, the power series

$$ f(t - x) = f(t) + \sum_{k=1}^{\infty} (-1)^k \frac{D^k f(t)}{k!} x^k. \tag{3.42} $$

converges for all x in an interval that properly contains $[0, t]$. Thus it converges uniformly on $[0, t]$.

Now substitute (3.42) in (3.41),

$$ \begin{aligned} D^{-\nu} f(t) &= \frac{1}{\Gamma(\nu)} f(t) \int_0^t x^{\nu-1}\, dx \\ &\quad + \frac{1}{\Gamma(\nu)} \int_0^t x^{\nu} \left[\sum_{k=1}^{\infty} (-1)^k \frac{D^k f(t)}{k!} x^{k-1} \right] dx. \end{aligned} \tag{3.43} $$

By the uniform convergence we may interchange the order of summation and integration to obtain

$$D^{-\nu}f(t) = \frac{1}{\Gamma(\nu)} \sum_{k=0}^{\infty} \frac{(-1)^k D^k f(t)}{k!(\nu+k)} t^{\nu+k}, \qquad 0 \leqq t \leqq X. \tag{3.44}$$

Thus we have expressed the fractional integral of an analytic function in terms of ordinary derivatives of that function. If we recall that

$$D^{-\nu-k}(1) = \frac{1}{\Gamma(\nu+k+1)} t^{\nu+k},$$

we also may write (3.44) as

$$\begin{aligned} D^{-\nu}f(t) &= \sum_{k=0}^{\infty} (-1)^k \binom{\nu+k-1}{k} [D^k f(t)][D^{-\nu-k}(1)] \\ &= \sum_{k=0}^{\infty} \binom{-\nu}{k} [D^k f(t)][D^{-\nu-k}(1)] \end{aligned} \tag{3.45}$$

[see (B-2.6), p. 298].

4. DIRICHLET'S FORMULA

If $G(x, y)$ is jointly continuous on $[a, b] \times [a, b]$, we know from the elementary theory of functions that

$$\int_a^b dx \int_a^x G(x, y)\, dy = \int_a^b dy \int_y^b G(x, y)\, dx. \tag{4.1}$$

If, however, G is not continuous, but the integrals $\int_a^x G\, dy$ and $\int_y^b G\, dx$ exist as ordinary or improper Riemann integrals, general conditions under which the order of integration may be interchanged are difficult to obtain. Dirichlet's formula [48, p. 77] furnishes an example of a function for which (4.1) is true even though G may not be continuous. Because of the form of the integrand, this formula is well suited to the fractional calculus.

Dirichlet's Formula. Let F be jointly continuous on the Euclidean plane, and let λ, μ, ν be positive numbers. Then

$$\int_a^t (t-x)^{\mu-1}\,dx \int_a^x (y-a)^{\lambda-1}(x-y)^{\nu-1}F(x,y)\,dy$$
$$= \int_a^t (y-a)^{\lambda-1}\,dy \int_y^t (t-x)^{\mu-1}(x-y)^{\nu-1}F(x,y)\,dx. \quad (4.2)$$

Certain special cases are of particular interest. If $a = 0$, $\lambda = 1$, and $F(x, y) = g(x)f(y)$, then (4.2) becomes

$$\int_0^t (t-x)^{\mu-1}g(x)\,dx \int_0^x (x-y)^{\nu-1}f(y)\,dy$$
$$= \int_0^t f(y)\,dy \int_y^t (t-x)^{\mu-1}(x-y)^{\nu-1}g(x)\,dx. \quad (4.3)$$

Furthermore, if $g(x) \equiv 1$, (4.3) assumes the form

$$\int_0^t (t-x)^{\mu-1}\,dx \int_0^x (x-y)^{\nu-1}f(y)\,dy$$
$$= \mathrm{B}(\mu,\nu)\int_0^t (t-y)^{\mu+\nu-1}f(y)\,dy, \quad (4.4)$$

where B is the beta function.

As an important illustration of the usefulness of Dirichlet's formula, we shall prove the law of exponents for fractional integrals.

Theorem 1. Let f be continuous on J, and let $\mu, \nu > 0$. Then for all t,

$$D^{-\nu}[D^{-\mu}f(t)] = D^{-(\mu+\nu)}f(t) = D^{-\mu}[D^{-\nu}f(t)]. \quad (4.5)$$

Proof. By definition of the fractional integral,

$$D^{-\nu}[D^{-\mu}f(t)] = \frac{1}{\Gamma(\nu)}\int_0^t (t-x)^{\nu-1}\left[\frac{1}{\Gamma(\mu)}\int_0^x (x-y)^{\mu-1}f(y)\,dy\right]dx$$

and

$$D^{-(\mu+\nu)}f(t) = \frac{1}{\Gamma(\mu+\nu)}\int_0^t (t-y)^{\mu+\nu-1} f(y)\,dy.$$

Equation (4.4) now implies the truth of (4.5). ■

An alternative proof of this important theorem may be given by noting that

$$D^{-\nu}[D^{-\mu}P(t)] = D^{-(\mu+\nu)}P(t)$$

for any polynomial P, and then applying the Weierstrass approximation theorem, see [45].

If we wish Theorem 1 to be true when μ (or ν) is zero (which we do), we see that D^0 must be defined as the identity operator I. We shall make this identification.

For any positive integer p and continuous function f, we have seen that

$$D^{-p}f(t) = \frac{1}{(p-1)!}\int_0^t (t-x)^{p-1} f(x)\,dx \tag{4.6}$$

is the p-fold integral of $f(t)$. Thus if we let $\mu = p$ in (4.5), we have

$$D^{-p}[D^{-\nu}f(t)] = D^{-(p+\nu)}f(t) = D^{-\nu}[D^{-p}f(t)]. \tag{4.7}$$

We see, therefore, that the p-fold integral of the fractional integral $D^{-\nu}f(t)$ is the fractional integral of f of order $p+\nu$, and that they are both equal to the fractional integral of the p-fold integral of f of order ν.

As we have observed before, the fractional integral of an elementary function need not be elementary. We thus may use Theorem 1 to find the fractional integral of certain nonelementary functions. For example, if $f(t) = e^{at}$, then since e^{at} is continuous, Theorem 1 implies that

$$D^{-\nu}\left[D^{-\mu}e^{at}\right] = D^{-(\mu+\nu)}e^{at} \tag{4.8}$$

for positive μ and ν. But from (3.10), $D^{-\mu}e^{at} = E_t(\mu, a)$ and

$$D^{-(\mu+\nu)}e^{at} = E_t(\mu+\nu, a).$$

Thus with little effort we have established the formula

$$D^{-\nu}E_t(\mu, a) = E_t(\mu + \nu, a), \qquad \mu > -1, \quad \nu > 0 \tag{4.9}$$

[see (3.35)]. Similar arguments yield

$$D^{-\nu}C_t(\mu, a) = C_t(\mu + \nu, a), \qquad \mu > -1, \quad \nu > 0 \tag{4.10}$$

and

$$D^{-\nu}S_t(\mu, a) = S_t(\mu + \nu, a), \qquad \mu > -2, \quad \nu > 0. \tag{4.11}$$

Further formulas may be obtained by the use of (3.28) and (3.32) [or (3.33)]. For example, if we apply (3.28) to (4.9),

$$D^{-\nu}[tE_t(\mu, a)] = tE_t(\mu + \nu, a) - \nu E_t(\mu + \nu + 1, a),$$
$$\mu > -2, \quad \nu > 0. \tag{4.12}$$

5. DERIVATIVES OF THE FRACTIONAL INTEGRAL AND THE FRACTIONAL INTEGRAL OF DERIVATIVES

In Section III-4 we showed that the integral of the fractional integral was the fractional integral of the integral. We now develop similar formulas involving derivatives. Unfortunately, the relations are not quite as simple. The basic rules for manipulating these quantities are given below in Theorem 2. Some examples of $D^p[D^{-\nu}f(t)]$ and $D^{-\nu}[D^p f(t)]$ (where p is a positive integer) will be given.

Theorem 2. Let f be continuous on J and let $\nu > 0$. Then:
(a) If Df is of class **C**, then

$$D^{-\nu-1}[Df(t)] = D^{-\nu}f(t) - \frac{f(0)}{\Gamma(\nu+1)}t^{\nu}$$

and
(b) If Df is continuous on J, then for $t > 0$,

$$D[D^{-\nu}f(t)] = D^{-\nu}[Df(t)] + \frac{f(0)}{\Gamma(\nu)}t^{\nu-1}.$$

Proof. To prove part (a), let $\epsilon > 0$, $\eta > 0$ be assigned. Then $(t - \xi)^{\nu-1}$ and $f(\xi)$ are continuously differentiable on $[\eta, t - \epsilon]$. Thus an integration by parts establishes

$$\int_\eta^{t-\epsilon} (t - \xi)^\nu [Df(\xi)] \, d\xi = \nu \int_\eta^{t-\epsilon} (t - \xi)^{\nu-1} f(\xi) \, d\xi$$
$$+ \epsilon^\nu f(t - \epsilon) - (t - \eta)^\nu f(\eta).$$

Now take the limit as ϵ and η independently approach zero and divide by $\Gamma(\nu + 1)$ to obtain part (a).

To prove part (b), make the change of variable

$$\xi = t - x^\lambda \tag{5.1}$$

(where $\lambda = 1/\nu$) in

$$D^{-\nu} f(t) = \frac{1}{\Gamma(\nu)} \int_0^t (t - \xi)^{\nu-1} f(\xi) \, d\xi$$

to obtain

$$D^{-\nu} f(t) = \frac{1}{\Gamma(\nu + 1)} \int_0^{t^\nu} f(t - x^\lambda) \, dx.$$

Then for $t > 0$,

$$D[D^{-\nu} f(t)] = \frac{1}{\Gamma(\nu + 1)} \left[f(0)(\nu t^{\nu-1}) + \int_0^{t^\nu} \frac{\partial}{\partial t} f(t - x^\lambda) \, dx \right].$$

Now reversing the transformation (5.1), that is, letting $t - x^\lambda = \xi$, proves part (b). ■

If we apply Theorem 2 to the special case

$$f(t) = t^\mu, \qquad \mu > 0,$$

then both parts (a) and (b) yield identities.

Now let $f(t) = e^{at}$. Then part (a) implies that

$$D^{-\nu-1}[ae^{at}] = D^{-\nu}[e^{at}] - \frac{t^\nu}{\Gamma(\nu + 1)},$$

and using (3.10),

$$aE_t(\nu+1,a) = E_t(\nu,a) - \frac{t^\nu}{\Gamma(\nu+1)}, \tag{5.2}$$

a recursion formula for the E_t function that may be found in Appendix C. If we apply part (b) to e^{at}, then

$$DE_t(\nu,a) = aE_t(\nu,a) + \frac{t^{\nu-1}}{\Gamma(\nu)}$$

and using (5.2) we see that

$$DE_t(\nu,a) = E_t(\nu-1,a), \tag{5.3}$$

a differentiation formula for E_t that also may be found in Appendix C. Thus we see that an application of Theorem 2 results in a painless derivation of such formulas as (5.2) and (5.3).

If $f(t) = \cos at$, then using (3.10), p. 49, we see that parts (a) and (b) of Theorem 2 yield

$$-aS_t(\nu+1,a) = C_t(\nu,a) - \frac{t^\nu}{\Gamma(\nu+1)} \tag{5.4}$$

and

$$DC_t(\nu,a) = -aS_t(\nu,a) + \frac{t^{\nu-1}}{\Gamma(\nu)},$$

respectively. Replacing ν by $\nu-1$ in (5.4) and substituting in the equation above yields the differentiation formula

$$DC_t(\nu,a) = C_t(\nu-1,a). \tag{5.5}$$

Similarly, we see that if we apply Theorem 2 to $f(t) = \sin at$, we obtain

$$aC_t(\nu+1,a) = S_t(\nu,a) \tag{5.6}$$

and

$$DS_t(\nu,a) = S_t(\nu-1,a). \tag{5.7}$$

Formulas (5.4), (5.5), (5.6), and (5.7) also may be found in Appendix C.

Using (5.4), we may write (3.15) in the neat form

$$D^{-\nu}\sin^2 at = aS_t(\nu+1, 2a). \tag{5.8}$$

We may generalize Theorem 2 to derivatives of higher order.

Theorem 3. Let p be a positive integer. Let $D^{p-1}f$ be continuous on J, and let $\nu > 0$. Then:

(a) If $D^p f$ is of class **C**, then

$$D^{-\nu}f(t) = D^{-\nu-p}[D^p f(t)] + Q_p(t,\nu)$$

and

(b) if $D^p f$ is continuous on J, then for $t > 0$

$$D^p[D^{-\nu}f(t)] = D^{-\nu}[D^p f(t)] + Q_p(t, \nu - p),$$

where

$$Q_p(t,\nu) = \sum_{k=0}^{p-1} \frac{t^{\nu+k}}{\Gamma(\nu+k+1)} D^k f(0). \tag{5.9}$$

Proof. Replacing ν by $\nu+1$ and f by Df in part (a) of Theorem 2 yields

$$D^{-\nu-2}[D^2 f(t)] = D^{-\nu-1}[Df(t)] - \frac{Df(0)}{\Gamma(\nu+2)} t^{\nu+1}.$$

Now replace $D^{-\nu-1}[Df(t)]$ in the expression above by part (a) of Theorem 2 to obtain

$$D^{-\nu-2}[D^2 f(t)] = D^{-\nu}f(t) - \frac{f(0)}{\Gamma(\nu+1)} t^{\nu} - \frac{Df(0)}{\Gamma(\nu+2)} t^{\nu+1}.$$

Repeated iterations establish part (a).

To prove part (b), differentiate part (b) of Theorem 2 to obtain (for $t > 0$)

$$D^2[D^{-\nu}f(t)] = D\{D^{-\nu}[Df(t)]\} + \frac{f(0)}{\Gamma(\nu-1)} t^{\nu-2}.$$

Now the term in braces is given by part (b) of Theorem 2 with f

replaced by Df. Hence

$$D^2[D^{-\nu}f(t)] = D^{-\nu}[D^2f(t)] + \frac{f(0)}{\Gamma(\nu-1)}t^{\nu-2} + \frac{Df(0)}{\Gamma(\nu)}t^{\nu-1}.$$

Repeated iterations establish part (b). ■

Since $Q_p(t,\nu)$ may be expressed as a fractional integral, that is,

$$Q_p(t,\nu) = D^{-\nu}[R_p(t)], \tag{5.10}$$

where

$$R_p(t) = \sum_{k=0}^{p-1} \frac{D^kf(0)}{k!}t^k, \tag{5.11}$$

we may write part (a) of Theorem 3 as

$$D^{-\nu}[f(t) - R_p(t)] = D^{-\nu-p}[D^pf(t)]. \tag{5.12}$$

As a corollary to Theorem 3, we see that if $D^kf(0) = 0$, $k = 0, 1, \ldots, p-1$, then

$$D^{-\nu}f(t) = D^{-\nu-p}[D^pf(t)] \tag{5.13}$$

and

$$D^p[D^{-\nu}f(t)] = D^{-\nu}[D^pf(t)]. \tag{5.14}$$

These formulas are generalized in Chapter IV.

Before continuing our theoretical development, let us consider some consequences of Theorem 3. If we apply part (a) to the function $f(t) = e^{at}$, then

$$D^{-\nu}[e^{at}] = a^pD^{-\nu-p}[e^{at}] + Q_p(t,\nu), \tag{5.15}$$

where

$$Q_p(t,\nu) = \sum_{k=0}^{p-1} a^k \frac{t^{\nu+k}}{\Gamma(\nu+k+1)}.$$

Thus from (3.10) we see that (5.15) reduces to the recursion formula

$$E_t(\nu,a) = a^pE_t(\nu+p,a) + \sum_{k=0}^{p-1} a^k \frac{t^{\nu+k}}{\Gamma(\nu+k+1)} \tag{5.16}$$

[see (C-3.4), p. 315]. On the other hand, part (b) implies that

$$D^p E_t(\nu, a) = a^p E_t(\nu, a) + \sum_{k=0}^{p-1} a^k \frac{t^{\nu+k-p}}{\Gamma(\nu+k+1-p)} \tag{5.17}$$

see (C-3.5), p. 316]. If we replace ν by $\nu - p$ in (5.16) and substitute in (5.17), we have the elegant formula

$$D^p E_t(\nu, a) = E_t(\nu - p, a), \qquad p = 0, 1, \ldots \tag{5.18}$$

[which also could have been obtained by iterating (5.3)].

Similar arguments, of course, establish that

$$C_t(\nu, a) = (-1)^{p/2} a^p C_t(\nu + p, a) + \sum_{j=0}^{(1/2)p-1} (-1)^j a^{2j} \frac{t^{\nu+2j}}{\Gamma(\nu+2j+1)} \tag{5.19}$$

if p is even, and

$$C_t(\nu, a) = (-1)^{(1/2)(p+1)} a^{p+1} C_t(\nu + p + 1, a) + \sum_{j=0}^{(1/2)(p-1)} (-1)^j a^{2j} \frac{t^{\nu+2j}}{\Gamma(\nu+2j+1)} \tag{5.20}$$

if p is odd, and

$$S_t(\nu, a) = (-1)^{p/2} a^p S_t(\nu + p, a) + \sum_{j=0}^{(1/2)p-1} (-1)^j a^{2j+1} \frac{t^{\nu+2j+1}}{\Gamma(\nu+2j+2)} \tag{5.21}$$

if p is even, and

$$S_t(\nu, a) = (-1)^{(1/2)(p+1)} a^{p+1} S_t(\nu + p + 1, a) + \sum_{j=0}^{(1/2)(p-1)} (-1)^j a^{2j+1} \frac{t^{\nu+2j+1}}{\Gamma(\nu+2j+2)} \tag{5.22}$$

if p is odd; while

$$D^p C_t(\nu, a) = C_t(\nu - p, a) \tag{5.23}$$

and

$$D^p S_t(\nu, a) = S_t(\nu - p, a) \tag{5.24}$$

for $p = 0, 1, \ldots$.

In the spirit of Theorems 2 and 3 and (5.13) and (5.14) we shall prove a theorem that expresses the derivative of a fractional integral of a function as a fractional integral of that function.

Theorem 4. Let f have a continuous derivative on J. Let p be a positive integer and let $\nu > p$. Then for all $t \in J$,

$$D^p[D^{-\nu} f(t)] = D^{-(\nu - p)} f(t). \tag{5.25}$$

Proof. By Definition 1,

$$D^{-\nu} f(t) = \frac{1}{\Gamma(\nu)} \int_0^t (t - \xi)^{\nu - 1} f(\xi)\, d\xi$$

and

$$D^{p-1}[D^{-\nu} f(t)] = D^{-(\nu - p) - 1} f(t) \tag{5.26}$$

since $\nu > p$. Differentiation of the expression above leads to

$$D^p[D^{-\nu} f(t)] = D[D^{p-1-\nu} f(t)].$$

If we replace ν by $\nu - p + 1$ in part (b) of Theorem 2, and then substitute this result for the right-hand side of the formula above, we get

$$D^p[D^{-\nu} f(t)] = D^{p-1-\nu}[Df(t)] + \frac{f(0)}{\Gamma(\nu + 1 - p)} t^{\nu - p}. \tag{5.27}$$

Now replace ν by $\nu - p$ in part (a) of Theorem 2 and substitute in (5.27). ■

Suppose that q is a positive integer and let $\mu > q$. Then from Theorem 4,

$$D^q[D^{-\mu} f(t)] = D^{-(\mu - q)} f(t). \tag{5.28}$$

Suppose further that

$$p - \nu = q - \mu. \tag{5.29}$$

Then we have the interesting corollary that

$$D^p[D^{-\nu}f(t)] = D^q[D^{-\mu}f(t)]. \tag{5.30}$$

In the next theorem we generalize this result by showing that (5.30) is true even if $p > \nu$ and $q > \mu$, and also exhibit the relation between $D^{-\nu}[D^p f(t)]$ and $D^{-\mu}[D^q f(t)]$.

Theorem 5. Let p and q be positive integers and let μ and ν be positive numbers such that

$$p - \nu = q - \mu. \tag{5.31}$$

Let f have r continuous derivatives on J where $r = \max(p, q)$. Then for all $t \in J$,

$$D^{-\nu}[D^p f(t)] = D^{-\mu}[D^q f(t)] + \operatorname{sgn}(q - p) \sum_{k=s}^{r-1} \frac{t^{\nu-p+k}}{\Gamma(\nu - p + k + 1)} D^k f(0), \tag{5.32}$$

where $s = \min(p, q)$, and for all $t \in J'$,

$$D^q[D^{-\mu}f(t)] = D^p[D^{-\nu}f(t)]. \tag{5.33}$$

Proof. If $p = q$, the theorem is trivial. Suppose then that $q > p$. Let $\sigma = q - p > 0$. Then from (5.31) we have

$$\mu = \nu + \sigma > 0.$$

From part (a) of Theorem 3

$$D^{-\nu}[D^p f(t)] = D^{-\nu-\sigma}[D^{\sigma+p}f(t)] + \sum_{k=0}^{\sigma-1} \frac{t^{\nu+k}}{\Gamma(\nu + k + 1)} D^{k+p}f(0).$$

Now recall that $\nu + \sigma = \mu$ and $\sigma + p = q$. Thus we have proved (5.32).

To prove (5.33) we have from Theorem 4 that

$$D^{\sigma}[D^{-\nu-\sigma}f(t)] = D^{-\nu}f(t). \tag{5.34}$$

If we differentiate (5.34) p times,

$$D^{p+\sigma}[D^{-\nu-\sigma}f(t)] = D^{p}[D^{-\nu}f(t)].$$

But $p + \sigma = q$ and $\nu + \sigma = \mu$. Thus we have established (5.33). ■

6. LAPLACE TRANSFORM OF THE FRACTIONAL INTEGRAL

The Laplace transform will prove to be an indispensable tool, especially in our study of fractional differential equations. We briefly inaugurate our discussion of this powerful method in the present section. In future chapters as well as in Appendix C we consider additional information about, and applications of, this important technique.

We recall that a function $f(t)$ defined on J' is said to be of exponential order α if there exist positive constants M and T such that

$$e^{-\alpha t}|f(t)| \leqq M$$

for all $t \geqq T$. If $f(t)$ is of class **C** and of exponential order α, then

$$\int_0^\infty f(t)e^{-st}\,dt \tag{6.1}$$

exists for all s with $\operatorname{Re} s > \alpha$. We shall call (6.1) the *Laplace transform* of $f(t)$ and write

$$\mathscr{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}\,dt.$$

Sometimes it is convenient to denote the Laplace transform of f by F,

$$F(s) = \mathscr{L}\{f(t)\}.$$

We shall also have occasion to write

$$f(t) = \mathscr{L}^{-1}\{F(s)\}$$

to indicate that f is the (unique) inverse Laplace transform of F.

If f and g are of exponential order, then clearly $f(t)g(t)$ is of exponential order. We also assert that if f is continuous on J and Df is of class **C**, then if Df is of exponential order, so is f. To demonstrate this we first note that if $\epsilon > 0$, then

$$\int_\epsilon^t [Df(\xi)]\, d\xi = f(t) - f(\epsilon),$$

and since f is continuous on J,

$$f(t) = f(0) + \int_0^t [Df(\xi)]\, d\xi.$$

By hypothesis Df is of exponential order. Hence there exists an α (which we shall assume to be positive) and constants T and M such that

$$|e^{-\alpha t} Df(t)| < M \tag{6.2}$$

for all $t > T$. Now if we write

$$f(t) = f(0) + \int_0^t e^{\alpha\xi}[e^{-\alpha\xi} Df(\xi)]\, d\xi$$

(i.e., we have multiplied the integrand by $1 = e^{\alpha\xi}e^{-\alpha\xi}$), then

$$f(t) = f(0) + \int_0^T Df(\xi)\, d\xi + \int_T^t e^{\alpha\xi}[e^{-\alpha\xi} Df(\xi)]\, d\xi, \qquad t > T,$$

and by (6.2),

$$|f(t)| \leqq A + M\int_T^t e^{\alpha\xi}\, d\xi,$$

where A is a positive constant. But

$$\int_T^t e^{\alpha\xi}\, d\xi < \frac{e^{\alpha t}}{\alpha}.$$

Thus for all $t \geqq T$,

$$|f(t)| < M'e^{\alpha t}$$

for some M'. Hence $f(t)$ is of exponential order.

If a function of class **C** has compact support, then the condition that f be of exponential order is vacuous.

The functions t^{μ} $(\mu > -1)$, e^{at}, $t^{\mu-1}e^{at}$ $(\mu > 0)$, $\cos at$, and $\sin at$ all are of class **C** and of exponential order. Some elementary calculus then shows that

$$\mathscr{L}\{t^{\mu}\} = \frac{\Gamma(\mu+1)}{s^{\mu+1}}, \qquad \mu > -1 \tag{6.3a}$$

$$\mathscr{L}\{e^{at}\} = \frac{1}{s-a} \tag{6.3b}$$

$$\mathscr{L}\{t^{\mu-1}e^{at}\} = \frac{\Gamma(\mu)}{(s-a)^{\mu}}, \qquad \mu > 0 \tag{6.3c}$$

$$\mathscr{L}\{\cos at\} = \frac{s}{s^2+a^2} \tag{6.3d}$$

$$\mathscr{L}\{\sin at\} = \frac{a}{s^2+a^2}. \tag{6.3e}$$

One of the most useful properties of the Laplace transform is embodied in the *convolution* theorem (see [7]). The theorem states that the Laplace transform of the convolution of two functions is the product of their Laplace transforms. Thus if $F(s)$ and $G(s)$ are the Laplace transforms of $f(t)$ and $g(t)$, respectively, then

$$\mathscr{L}\left\{\int_0^t f(t-\xi)g(\xi)\,d\xi\right\} = F(s)G(s). \tag{6.4}$$

Now if f is of class **C**, the fractional integral of f of order ν is

$$D^{-\nu}f(t) = \frac{1}{\Gamma(\nu)}\int_0^t (t-\xi)^{\nu-1} f(\xi)\,d\xi, \qquad \nu > 0,$$

which is a convolution integral. Thus if f is of exponential order

$$\mathscr{L}\{D^{-\nu}f(t)\} = \frac{1}{\Gamma(\nu)}\mathscr{L}\{t^{\nu-1}\}\mathscr{L}\{f(t)\} \tag{6.5a}$$

$$= s^{-\nu}F(s), \qquad \nu > 0, \tag{6.5b}$$

where F is the Laplace transform of f. We observe that (6.5b) is valid even if $\nu = 0$, but that (6.5a) is indeterminate. However,

$$\lim_{\nu \to 0} \mathscr{L}\left\{ \frac{t^{\nu-1}}{\Gamma(\nu)} \right\} = 1. \tag{6.6}$$

As examples of (6.5) we see from (6.3) that

$$\mathscr{L}\{D^{-\nu} t^{\mu}\} = \frac{\Gamma(\mu + 1)}{s^{\mu+\nu+1}}, \qquad \nu > 0, \quad \mu > -1 \tag{6.7a}$$

$$\mathscr{L}\{D^{-\nu} e^{at}\} = \frac{1}{s^{\nu}(s - a)}, \qquad \nu > 0 \tag{6.7b}$$

$$\mathscr{L}\{D^{-\nu} t^{\mu-1} e^{at}\} = \frac{\Gamma(\mu)}{s^{\nu}(s - a)^{\mu}}, \qquad \nu > 0, \quad \mu > 0 \tag{6.7c}$$

$$\mathscr{L}\{D^{-\nu} \cos at\} = \frac{1}{s^{\nu-1}(s^2 + a^2)}, \quad \nu > 0 \tag{6.7d}$$

$$\mathscr{L}\{D^{-\nu} \sin at\} = \frac{a}{s^{\nu}(s^2 + a^2)}, \qquad \nu > 0. \tag{6.7e}$$

We turn now to the problem of finding the Laplace transform of the fractional integral of the derivative and the Laplace transform of the derivative of the fractional integral. Suppose then that f is continuous on J and Df is of class **C** and of exponential order. Then, by (6.5),

$$\begin{aligned} \mathscr{L}\{D^{-\nu}[Df(t)]\} &= s^{-\nu}\mathscr{L}\{Df(t)\} \\ &= s^{-\nu}[sF(s) - f(0)], \qquad \nu > 0, \end{aligned} \tag{6.8}$$

where $F(s)$ is the Laplace transform of $f(t)$. Since $f(t)$ is continuous on J by hypothesis, $f(0)$ exists. Thus we have found the Laplace transform of the fractional integral of the derivative. This formula is obviously also valid if $\nu = 0$.

Now we consider the problem of finding the Laplace transform of the derivative of the fractional integral. From part (b) of Theorem 2, p. 60,

$$\begin{aligned} \mathscr{L}\{D[D^{-\nu}f(t)]\} &= \mathscr{L}\{D^{-\nu}[Df(t)]\} + f(0)\mathscr{L}\left\{ \frac{t^{\nu-1}}{\Gamma(\nu)} \right\} \\ &= s^{-\nu}[sF(s) - f(0)] + s^{-\nu}f(0) \\ &= s^{1-\nu}F(s), \qquad \nu > 0 \end{aligned} \tag{6.9}$$

[where we have used (6.8)]. Now we know that if $\nu = 0$,

$$\mathscr{L}\{Df(t)\} = sF(s) - f(0). \tag{6.10}$$

But this is not the same result we would get if we let $\nu = 0$ in (6.9). This "discontinuity" arises from the fact that

$$\lim_{\nu \to 0} \frac{t^{\nu-1}}{\Gamma(\nu)} = 0, \tag{6.11}$$

and comparing with (6.6) we see that "$\mathscr{L}$" and "lim" do not commute.

Returning to (6.7) and recalling (3.10), we see that

$$\begin{aligned}
\mathscr{L}\{E_t(\nu, a)\} &= \frac{1}{s^\nu(s-a)}, & \nu > 0 \\
\mathscr{L}\{C_t(\nu, a)\} &= \frac{1}{s^{\nu-1}(s^2+a^2)}, & \nu > 0 \\
\mathscr{L}\{S_t(\nu, a)\} &= \frac{a}{s^\nu(s^2+a^2)}, & \nu > 0.
\end{aligned} \tag{6.12}$$

We elaborate on these formulas in Section C-4. Thus we see that with the aid of the fractional calculus, we have found, with little effort, the Laplace transforms of some nonelementary functions.

For completeness, from (6.7c),

$$\mathscr{L}^{-1}\left\{\frac{\Gamma(\mu)}{s^\nu(s-a)^\mu}\right\} = \frac{1}{\Gamma(\nu)}\int_0^t (t-\xi)^{\nu-1}\xi^{\mu-1}e^{a\xi}\,d\xi.$$

Thus from (B-4.8), p. 305, we have

$$D^{-\nu}\left[t^{\mu-1}e^{at}\right] = \frac{\Gamma(\mu)}{\Gamma(\mu+\nu)}t^{\mu+\nu-1}\,{}_1F_1(\mu, \mu+\nu; at),$$

$$\nu > 0, \quad \mu > 0. \tag{6.13}$$

[If μ is a positive integer, see (3.34), p. 54 and (C-4.5), p. 323.]

Finally, we wish to mention a phenomenon that some readers might not have noticed. Although this phenomenon is prevalent in all of mathematics, we wish to emphasize it in our dealings with the Laplace transform. Depending on the method used, it is sometimes possible to weaken a set of hypotheses and still arrive at the same conclusion. For example, if

$$F(s) = \frac{1}{s}$$

and

$$G(s) = \frac{1}{s^{\nu}}$$

then $G(s)$ is meaningful only if $\nu > 0$, but $F(s)G(s)$ is meaningful if $\nu > -1$. Thus if we find the inverse transform of $F(s)G(s)$ directly, namely

$$\mathscr{L}^{-1}\left\{\frac{1}{s^{1+\nu}}\right\} = \frac{t^{\nu}}{\Gamma(\nu+1)}, \tag{6.14}$$

we need require only the weaker hypothesis, $\nu > -1$. But if we use the convolution approach, namely,

$$\int_0^t \frac{(t-\xi)^{\nu-1}}{\Gamma(\nu)} \xi^0 \, d\xi = \frac{t^{\nu}}{\Gamma(\nu+1)}, \tag{6.15}$$

then since the integral is meaningful only for $\nu > 0$, we have proved our result only with the stronger hypothesis $\nu > 0$ (even though we know that the result is valid for $\nu > -1$).

As another more subtle example, let

$$x(t) = t^{\lambda-1}$$

and as our problem let it be required to find the inverse Laplace transform of

$$Y(s) = \frac{s^2 X(s)}{s^2+1}.$$

Now

$$X(s) = \frac{\Gamma(\lambda)}{s^{\lambda}} \tag{6.16}$$

provided that $\lambda > 0$ and

$$Y(s) = \frac{\Gamma(\lambda)}{s^{\lambda-2}(s^2+1)}$$

is meaningful if $\lambda > 0$. Thus

$$y(t) = \Gamma(\lambda) S_t(\lambda - 2, 1), \qquad \lambda > 0. \tag{6.17}$$

On the other hand, if we write

$$s^2 X(s) = \mathscr{L}\{D^2 x(t)\} + sx(0) + Dx(0), \tag{6.18}$$

then $Y(s)$ may be expressed as

$$Y(s) = \frac{\mathscr{L}\{D^2 x(t)\}}{s^2+1} + \frac{sx(0) + Dx(0)}{x^2+1}.$$

But from (6.16),

$$s^2 X(s) = \frac{\Gamma(\lambda)}{s^{\lambda-2}},$$

which is meaningful only for $\lambda > 2$. If this is the case, $x(0) = 0 = Dx(0)$ and by the convolution theorem

$$\begin{aligned} y(t) &= \int_0^t \sin(t-\xi) D^2 x(\xi)\, d\xi \\ &= (\lambda-1)(\lambda-2)\int_0^t \sin(t-\xi)\xi^{\lambda-3}\, d\xi \\ &= \Gamma(\lambda) S_t(\lambda-2, 1) \end{aligned}$$

[see (C-3.20), p. 320]. Thus we have proved our result only for $\lambda > 2$, while we know from (6.17) that it is valid for $\lambda > 0$.

7. LEIBNIZ'S FORMULA FOR FRACTIONAL INTEGRALS

A Leibniz-type formula expresses the result of operating on the product of two functions as a sum of products of operations performed on each function. The classical Leibniz rule or formula of elementary calculus is

$$D^n[f(t)g(t)] = \sum_{k=0}^{n} \binom{n}{k} [D^k g(t)][D^{n-k} f(t)], \tag{7.1}$$

where f and g are assumed to be n-fold differentiable on some interval. Now we wish to extend (7.1) to fractional operators.

We have seen in Section III-3 that if f is of class **C** and $g(t) = t^p$, where p is a positive integer, then the fractional integral of the product fg of order $\nu > 0$ may be written as

$$D^{-\nu}[f(t)g(t)] = \sum_{k=0}^{p} \binom{-\nu}{k} [D^k g(t)][D^{-\nu-k} f(t)] \tag{7.2}$$

[see (3.33), p. 53]. The resemblance of this formula to (7.1) is obvious. The immediate problem we wish to address is the extension of (7.2) to the case where g is not just a simple polynomial. Later, in Chapter IV, we extend these formulas to fractional derivatives.

Suppose then that f is continuous on $[0, X]$ and that g is analytic at a for all $a \in [0, X]$. Then fg is certainly of class **C**, and for $\nu > 0$, the fractional integral

$$D^{-\nu}[f(t)g(t)] = \frac{1}{\Gamma(\nu)} \int_0^t (t-\xi)^{\nu-1} [f(\xi)g(\xi)]\, d\xi, \qquad 0 < t \leqq X \tag{7.3}$$

exists. We may write

$$\begin{aligned} g(\xi) &= \sum_{k=0}^{\infty} (-1)^k \frac{D^k g(t)}{k!} (t-\xi)^k \\ &= g(t) + \sum_{k=1}^{\infty} (-1)^k \frac{D^k g(t)}{k!} (t-\xi)^k. \end{aligned} \tag{7.4}$$

The series (7.4) converges for all ξ in an interval that properly contains $[0, t]$, and hence uniformly on $[0, t]$.

Now substitute (7.4) into (7.3) to obtain

$$\begin{aligned} D^{-\nu}[f(t)g(t)] &= g(t) D^{-\nu} f(t) + \frac{1}{\Gamma(\nu)} \int_0^t (t-\xi)^{\nu} f(\xi) \\ &\quad \times \sum_{k=1}^{\infty} (-1)^k \frac{D^k g(t)}{k!} (t-\xi)^{k-1}\, d\xi. \end{aligned} \tag{7.5}$$

Since f is continuous on $[0, X]$ and $\nu > 0$,

$$(t-\xi)^{\nu} f(\xi)$$

is bounded on $[0, t]$. Hence we may interchange the order of integration and summation in (7.5) to obtain

$$\begin{aligned} D^{-\nu}[f(t)g(t)] &= \sum_{k=0}^{\infty} (-1)^k \frac{\Gamma(\nu+k)}{k!\,\Gamma(\nu)} [D^k g(t)][D^{-\nu-k} f(t)] \\ &= \sum_{k=0}^{\infty} \binom{-\nu}{k} [D^k g(t)][D^{-\nu-k} f(t)] \qquad (7.6) \end{aligned}$$

[see (B-2.6), p. 298].

Thus we have shown:

Theorem 6. Let f be continuous on $[0, X]$, and let g be analytic at a for all $a \in [0, X]$. Then for $\nu > 0$ and $0 < t \leqq X$,

$$D^{-\nu}[f(t)g(t)] = \sum_{k=0}^{\infty} \binom{-\nu}{k} [D^k g(t)][D^{-\nu-k} f(t)]. \qquad (7.7)$$

We call (7.7) the Leibniz formula for fractional integrals. Equation (7.2) is a special case.

Note: The only reason we assumed g analytic for all points a in $[0, X]$ was to guarantee the uniform convergence of (7.4) for $\xi \in [0, t]$.

As our first application of the Leibniz rule, let $f(t) = t^{\lambda}$, $\lambda \geqq 0$, and let $g(t) = e^t$. Then from Theorem 6

$$\begin{aligned} D^{-\nu}[t^{\lambda} e^t] &= \sum_{k=0}^{\infty} (-1)^k \frac{\Gamma(\nu+k)}{k!\,\Gamma(\nu)} [e^t] \left[\frac{\Gamma(\lambda+1)}{\Gamma(\lambda+\nu+k+1)} t^{\lambda+\nu+k} \right] \\ &= \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+\nu+1)} t^{\lambda+\nu} e^t {}_1F_1(\nu, \lambda+\nu+1; -t). \qquad (7.8) \end{aligned}$$

Using this result we may deduce a useful identity involving the confluent hypergeometric functions [see (7.11)]. For, from the definition of the fractional integral,

$$\begin{aligned} D^{-\nu}[t^{\lambda} e^t] &= \frac{1}{\Gamma(\nu)} \int_0^t (t-\xi)^{\nu-1} \xi^{\lambda} e^{\xi}\, d\xi \\ &= \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+\nu+1)} t^{\lambda+\nu} {}_1F_1(\lambda+1, \lambda+\nu+1; t) \qquad (7.9) \end{aligned}$$

by (6.13) [see also (3.34), p. 54]. Comparing (7.8) and (7.9) establishes the identity

$$e^{t}{}_1F_1(\nu, \lambda+\nu+1; -t) = {}_1F_1(\lambda+1, \lambda+\nu+1; t). \tag{7.10}$$

Or if we let

$$\begin{aligned} a &= \lambda + 1 \\ c &= \lambda + \nu + 1, \end{aligned}$$

we have, in more conventional notation,

$${}_1F_1(a, c; t) = e^{t}{}_1F_1(c-a, c; -t) \tag{7.11}$$

[see (B-4.10), p. 305].

As a second example, let $f(t) = t^{\lambda}$, $\lambda \geqq 0$, and let $g(t) = (1-t)^{-\alpha}$. Let X be a fixed positive number less than 1. Then $(1-t)^{-\alpha}$ is analytic at every point of $[0, X]$, and by Theorem 6,

$$D^{-\nu}\left[t^{\lambda}(1-t)^{-\alpha}\right] = \sum_{k=0}^{\infty}(-1)^{k}\frac{\Gamma(\nu+k)}{k!\,\Gamma(\nu)}\left[D^{k}(1-t)^{-\alpha}\right]\left[D^{-\nu-k}t^{\lambda}\right]$$

for $0 < t \leqq X$. But

$$D^{k}(1-t)^{-\alpha} = \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)}(1-t)^{-\alpha-k}$$

and

$$D^{-\nu-k}t^{\lambda} = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+\nu+k+1)}t^{\lambda+\nu+k}.$$

Thus

$$\begin{aligned} D^{-\nu}\left[t^{\lambda}(1-t)^{-\alpha}\right] &= \frac{\Gamma(\lambda+1)}{\Gamma(\nu)\Gamma(\alpha)}t^{\lambda+\nu}(1-t)^{-\alpha} \\ &\quad\times\sum_{k=0}^{\infty}\frac{\Gamma(\nu+k)\Gamma(\alpha+k)}{\Gamma(\lambda+\nu+k+1)k!}\left(\frac{t}{t-1}\right)^{k} \\ &= \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+\nu+1)}t^{\lambda+\nu}(1-t)^{-\alpha} \\ &\quad\times{}_2F_1\left(\nu, \alpha, \lambda+\nu+1; \frac{t}{t-1}\right). \end{aligned} \tag{7.12}$$

On the other hand, since

$$(1-t)^{-\alpha} = {}_1F_0(\alpha; t) = \frac{1}{\Gamma(\alpha)} \sum_{k=0}^{\infty} \frac{\Gamma(\alpha+k)}{k!} t^k$$

for $|t| < 1$ [see (B-4.13), p. 305],

$$\begin{aligned} D^{-\nu}\left[t^{\lambda}(1-t)^{-\alpha}\right] &= \frac{1}{\Gamma(\alpha)} D^{-\nu}\left[\sum_{k=0}^{\infty} \frac{\Gamma(\alpha+k)}{k!} t^{k+\lambda}\right] \\ &= \frac{1}{\Gamma(\nu)\Gamma(\alpha)} \int_0^t (t-\xi)^{\nu-1} \sum_{k=0}^{\infty} \frac{\Gamma(\alpha+k)}{k!} \xi^{k+\lambda}\, d\xi \\ &= \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+\nu+1)} t^{\lambda+\nu} {}_2F_1(\lambda+1, \alpha, \lambda+\nu+1; t). \end{aligned} \tag{7.13}$$

Comparing (7.12) and (7.13) leads to

$$(1-t)^{-\alpha} {}_2F_1\left(\nu, \alpha, \lambda+\nu+1; \frac{t}{t-1}\right) = {}_2F_1(\lambda+1, \alpha, \lambda+\nu+1; t).$$

Or in more conventional notation with

$$\begin{aligned} a &= \lambda + 1 \\ b &= \alpha \\ c &= \lambda + \nu + 1, \end{aligned}$$

we have established the identity

$$(1-t)^{-b} {}_2F_1\left(c-a, b, c; \frac{t}{t-1}\right) = {}_2F_1(a, b, c; t) \tag{7.14}$$

between hypergeometric functions [see (B-4.6), p. 304].

Another interesting result that we may establish using the Leibniz rule is the identity

$${}_2F_1(a, b, c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \tag{7.15}$$

[sometimes called Laurent's formula; see (B-4.4), p. 304].

To prove (7.15) we start with the trivial identity

$$t^{\lambda+\mu} = t^{\lambda}t^{\mu}, \qquad t > 0. \tag{7.16}$$

Now for $\nu > 0$ and $\lambda + \mu > -1$,

$$D^{-\nu}t^{\lambda+\mu} = \frac{\Gamma(\lambda+\mu+1)}{\Gamma(\lambda+\mu+\nu+1)}t^{\lambda+\mu+\nu}. \tag{7.17}$$

We shall show that if $\lambda, \mu \geqq 0$, we may apply Leibniz's formula to the product of $f(t) = t^{\lambda}$ and $g(t) = t^{\mu}$. This result may then be compared with (7.17) to establish (7.15).

We begin by expanding $g(\xi)$ in powers of $(\xi - t)$. By the binomial theorem

$$\begin{aligned} g(\xi) = \xi^{\mu} &= [t + (\xi - t)]^{\mu} \\ &= t^{\mu}\left(1 + \frac{\xi - t}{t}\right)^{\mu} \\ &= t^{\mu}\sum_{k=0}^{\infty}\binom{\mu}{k}\left(\frac{\xi - t}{t}\right)^{k}. \end{aligned} \tag{7.18}$$

Considered as a power series in $(\xi - t)/t$, the radius of convergence is 1. Using Raabe's test we see that the series converges absolutely for

$$\frac{\xi - t}{t} = \pm 1.$$

Furthermore, it converges to ξ^{μ}. Since

$$\left|\binom{\mu}{k}\left(\frac{\xi - t}{t}\right)^{k}\right| \leqq \left|\binom{\mu}{k}\right|$$

for all $(\xi - t)/t$ in $[-1, 1]$, the Weierstrass M-test implies that the convergence is uniform in the closed interval $[-1, 1]$. Thus (7.18) converges uniformly for $\xi \in [0, t]$.

It therefore follows (see the note after Theorem 6, p. 75) that

$$D^{-\nu}[t^{\lambda}t^{\mu}] = \sum_{k=0}^{\infty}(-1)^{k}\frac{\Gamma(\nu + k)}{k!\,\Gamma(\nu)}[D^{k}t^{\lambda}][D^{-\nu-k}t^{\mu}] \tag{7.19}$$

is valid for $\nu > 0$, $t > 0$, $\lambda, \mu \geqq 0$. Thus

$$D^{-\nu}[t^\lambda t^\mu] = t^{\lambda+\mu+\nu} \frac{\Gamma(\mu+1)}{\Gamma(-\lambda)\Gamma(\nu)} \sum_{k=0}^{\infty} \frac{\Gamma(-\lambda+k)\Gamma(\nu+k)}{\Gamma(\mu+\nu+k+1)} \frac{1}{k!}$$

$$= t^{\lambda+\mu+\nu} \frac{\Gamma(\mu+1)}{\Gamma(\mu+\nu+1)} {}_2F_1(-\lambda, \nu, \mu+\nu+1; 1).$$

If we equate this result to (7.17), we obtain

$$\frac{\Gamma(\lambda+\mu+1)}{\Gamma(\lambda+\mu+\nu+1)} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\nu+1)} {}_2F_1(-\lambda, \nu, \mu+\nu+1; 1). \tag{7.20}$$

In more conventional notation let $a = -\lambda$, $b = \nu$, $c = \mu + \nu + 1$. Then (7.20) becomes

$$\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} = {}_2F_1(a, b, c; 1) \tag{7.21}$$

for

$$a \leqq 0, \qquad c - 1 \geqq b > 0. \tag{7.22}$$

Now (7.21) is the same as (7.15). And we know that this formula is valid for

$$c - a - b > 0 \tag{7.23}$$

with c unequal to a nonpositive integer. Thus we have established (7.21) only under the more restrictive conditions of (7.22). But we have encountered this phenomenon before (see pp. 71–73).

7

Hankel Transforms and Their Applications

> "In most sciences one generation tears down what another has built, and what one has established, another undoes. In mathematics alone each generation adds a new storey to the old structure."
>
> Hermann Hankel

> "I have always regarded mathematics as an object of amusement rather than of ambition, and I can assure you that I enjoy the works of others much more than my own."
>
> Joseph-Louis Lagrange

7.1 Introduction

Hermann Hankel (1839–1873), a German mathematician, is remembered for his numerous contributions to mathematical analysis including the Hankel transformation, which occurs in the study of functions which depend only on the distance from the origin. He also studied functions, now named Hankel functions or Bessel functions of the third kind. The Hankel transform involving Bessel functions as the kernel arises naturally in axisymmetric problems formulated in cylindrical polar coordinates. This chapter deals with the definition and basic operational properties of the Hankel transform. A large number of axisymmetric problems in cylindrical polar coordinates are solved with the aid of the Hankel transform. The use of the joint Laplace and Hankel transforms is illustrated by several examples of applications to partial differential equations.

7.2 The Hankel Transform and Examples

We introduce the definition of the *Hankel transform* from the two-dimensional Fourier transform and its inverse given by

$$\mathscr{F}\{f(x,y)\} = F(k,l) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\{-i(\boldsymbol{\kappa}\cdot\mathbf{r})\}f(x,y)\,dx\,dy, \tag{7.2.1}$$

$$\mathscr{F}^{-1}\{F(k,l)\} = f(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\{i(\boldsymbol{\kappa}\cdot\mathbf{r})\}F(k,l)\,dk\,dl, \tag{7.2.2}$$

where $\mathbf{r}=(x,y)$ and $\boldsymbol{\kappa}=(k,l)$. Introducing polar coordinates $(x,y)=r(\cos\theta, \sin\theta)$ and $(k,l)=\kappa(\cos\phi,\sin\phi)$, we find $\boldsymbol{\kappa}\cdot\mathbf{r}=\kappa r\cos(\theta-\phi)$ and then

$$F(\kappa,\phi) = \frac{1}{2\pi}\int_0^{\infty} r\,dr\int_0^{2\pi}\exp[-i\kappa r\,\cos(\theta-\phi)]f(r,\theta)d\theta. \tag{7.2.3}$$

We next assume $f(r,\theta)=\exp(in\theta)f(r)$, which is not a very severe restriction, and make a change of variable $\theta-\phi=\alpha-\frac{\pi}{2}$ to reduce (7.2.3) to the form

$$\begin{aligned} F(\kappa,\phi) &= \frac{1}{2\pi}\int_0^{\infty} rf(r)dr \\ &\quad\times\int_{\phi_0}^{2\pi+\phi_0}\exp\left[in\left(\phi-\frac{\pi}{2}\right)+i(n\alpha-\kappa r\sin\alpha)\right]d\alpha, \end{aligned} \tag{7.2.4}$$

where $\phi_0=\left(\dfrac{\pi}{2}-\phi\right)$.

Using the integral representation of the Bessel function of order n

$$J_n(\kappa r) = \frac{1}{2\pi}\int_{\phi_0}^{2\pi+\phi_0}\exp[i(n\alpha-\kappa r\sin\alpha)]d\alpha \tag{7.2.5}$$

integral (7.2.4) becomes

$$F(\kappa,\phi) = \exp\left[in\left(\phi-\frac{\pi}{2}\right)\right]\int_0^{\infty} rJ_n(\kappa r)f(r)dr \tag{7.2.6}$$

$$= \exp\left[in\left(\phi-\frac{\pi}{2}\right)\right]\tilde{f}_n(\kappa), \tag{7.2.7}$$

where $\tilde{f}_n(\kappa)$ is called the *Hankel transform* of $f(r)$ and is defined formally by

$$\mathscr{H}_n\{f(r)\} = \tilde{f}_n(\kappa) = \int_0^\infty r J_n(\kappa r) f(r)\, dr. \tag{7.2.8}$$

Similarly, in terms of the polar variables with the assumption $f(x,y) = f(r,\theta) = e^{in\theta} f(r)$ with (7.2.7), the inverse Fourier transform (7.2.2) becomes

$$\begin{aligned}
e^{in\theta} f(r) &= \frac{1}{2\pi}\int_0^\infty \kappa\, d\kappa \int_0^{2\pi} \exp[i\kappa r\cos(\theta-\phi)] F(\kappa,\phi) d\phi \\
&= \frac{1}{2\pi}\int_0^\infty \kappa\, \tilde{f}_n(\kappa) d\kappa \int_0^{2\pi} \exp\left[in\left(\phi - \frac{\pi}{2}\right) + i\kappa r\cos(\theta-\phi)\right] d\phi,
\end{aligned}$$

which is, by the change of variables $\theta - \phi = -\left(\alpha + \frac{\pi}{2}\right)$ and $\theta_0 = -\left(\theta + \frac{\pi}{2}\right)$,

$$\begin{aligned}
&= \frac{1}{2\pi}\int_0^\infty \kappa \tilde{f}_n(\kappa) d\kappa \int_{\theta_0}^{2\pi+\theta_0} \exp[in(\theta+\alpha) - i\kappa r\sin\alpha] d\alpha \\
&= e^{in\theta}\int_0^\infty \kappa J_n(\kappa r) \tilde{f}_n(\kappa) d\kappa, \qquad \text{by (7.2.5).}
\end{aligned} \tag{7.2.9}$$

Thus, the *inverse Hankel transform* is defined by

$$\mathscr{H}_n^{-1}\left[\tilde{f}_n(\kappa)\right] = f(r) = \int_0^\infty \kappa J_n(\kappa r) \tilde{f}_n(\kappa) d\kappa. \tag{7.2.10}$$

Instead of $\tilde{f}_n(\kappa)$, we often simply write $\tilde{f}(\kappa)$ for the Hankel transform specifying the order. Integrals (7.2.8) and (7.2.10) exist for certain large classes of functions, which usually occur in physical applications.

Alternatively, the famous Hankel integral formula (Watson, 1944, p. 453)

$$f(r) = \int_0^\infty \kappa\, J_n(\kappa r) d\kappa \int_0^\infty p J_n(\kappa p) f(p) dp, \tag{7.2.11}$$

can be used to define the Hankel transform (7.2.8) and its inverse (7.2.10).

In particular, the Hankel transforms of the zero order $(n = 0)$ and of order one $(n = 1)$ are often useful for the solution of problems involving Laplace's equation in an axisymmetric cylindrical geometry.

Example 7.2.1 Obtain the zero-order Hankel transforms of

(a) $r^{-1}\exp(-ar)$, (b) $\dfrac{\delta(r)}{r}$, (c) $H(a-r)$,

where $H(r)$ is the Heaviside unit step function.

We have

$$\text{(a)}\ \tilde{f}(\kappa) = \mathscr{H}_0\left\{\frac{1}{r}\exp(-ar)\right\} = \int_0^\infty \exp(-ar)\, J_0(\kappa r)dr = \frac{1}{\sqrt{\kappa^2+a^2}}.$$

$$\text{(b)}\ \tilde{f}(\kappa) = \mathscr{H}_0\left\{\frac{\delta(r)}{r}\right\} = \int_0^\infty \delta(r) J_0(\kappa r)dr = 1.$$

$$\begin{aligned}\text{(c)}\ \tilde{f}(\kappa) = \mathscr{H}_0\{H(a-r)\} &= \int_0^a r J_0(\kappa r)dr = \frac{1}{\kappa^2}\int_0^{a\kappa} pJ_0(p)dp \\ &= \frac{1}{\kappa^2}[pJ_1(p)]_0^{a\kappa} = \frac{a}{\kappa}J_1(a\kappa).\end{aligned}$$

$\square$

Example 7.2.2 Find the first-order Hankel transforms of

(a) $f(r) = e^{-ar}$, (b) $f(r) = \dfrac{1}{r}\, e^{-ar}$, (c) $f(r) = \dfrac{\sin ar}{r}$.

We can write

$$\text{(a)}\ \tilde{f}(\kappa) = \mathscr{H}_1\{e^{-ar}\} = \int_0^\infty re^{-ar} J_1(\kappa r)dr = \frac{\kappa}{(a^2+\kappa^2)^{\frac{3}{2}}}.$$

$$\text{(b)}\ \tilde{f}(\kappa) = \mathscr{H}_1\left\{\frac{a^{-ar}}{r}\right\} = \int_0^\infty e^{-ar} J_1(\kappa r)dr = \frac{1}{\kappa}\left[1 - a(\kappa^2+a^2)^{-\frac{1}{2}}\right].$$

$$\text{(c)}\ \tilde{f}(\kappa) = \mathscr{H}_1\left\{\frac{\sin ar}{r}\right\} = \int_0^\infty \sin ar\, J_1(\kappa r)\, dr = \frac{a\, H(\kappa-a)}{\kappa(\kappa^2-a^2)^{\frac{1}{2}}}.$$

$\square$

Example 7.2.3 Find the nth $(n > -1)$ order Hankel transforms of

(a) $f(r) = r^n H(a-r)$, (b) $f(r) = r^n \exp(-ar^2)$.

Here we have, for $n > -1$,

$$\text{(a) } \tilde{\mathrm{f}}(\kappa) = \mathscr{H}_n\left[r^n H(a-r)\right] = \int_0^a r^{n+1} J_n(\kappa r)dr = \frac{a^{n+1}}{\kappa} J_{n+1}(a\kappa).$$

$$\text{(b) } \tilde{\mathrm{f}}(\kappa) = \mathscr{H}_n\left[r^n \exp(-ar^2)\right] = \int_0^\infty r^{n+1} J_n(\kappa r)\exp(-ar^2)dr$$

$$= \frac{\kappa^n}{(2a)^{n+1}} \exp\left(-\frac{\kappa^2}{4a}\right).$$

□

7.3 Operational Properties of the Hankel Transform

THEOREM 7.3.1 (**Scaling**).
If $\mathscr{H}_n\{f(r)\} = \tilde{f}_n(\kappa)$, then

$$\mathscr{H}_n\{f(ar)\} = \frac{1}{a^2}\tilde{f}_n\left(\frac{\kappa}{a}\right), \qquad a > 0. \tag{7.3.1}$$

PROOF We have, by definition,

$$\mathscr{H}_n\{f(ar)\} = \int_0^\infty r J_n(\kappa r) f(ar)dr$$

$$= \frac{1}{a^2}\int_0^\infty s J_n\left(\frac{\kappa}{a}s\right) f(s)ds = \frac{1}{a^2}\tilde{f}_n\left(\frac{\kappa}{a}\right).$$

■

THEOREM 7.3.2 (**Parseval's Relation**).
If $\tilde{f}(\kappa) = \mathscr{H}_n\{f(r)\}$ and $\tilde{g}(\kappa) = \mathscr{H}_n\{g(r)\}$, then

$$\int_0^\infty r f(r)g(r)dr = \int_0^\infty \kappa \tilde{f}(\kappa)\tilde{g}(\kappa)d\kappa. \tag{7.3.2}$$

PROOF We proceed formally to obtain

$$\int_0^\infty \kappa \tilde{f}(\kappa)\tilde{g}(\kappa)d\kappa = \int_0^\infty \kappa \tilde{f}(\kappa)d\kappa \int_0^\infty r J_n(\kappa r) g(r) dr,$$

which is, interchanging the order of integration,

$$= \int_0^\infty r g(r) dr \int_0^\infty \kappa J_n(\kappa r)\tilde{f}(\kappa)d\kappa$$

$$= \int_0^\infty r g(r) f(r) dr.$$

▮

THEOREM 7.3.3 ***(****Hankel Transforms of Derivatives****).***
If $\tilde{f}_n(\kappa) = \mathscr{H}_n\{f(r)\}$, then

$$\mathscr{H}_n\{f'(r)\} = \frac{\kappa}{2n}\left[(n-1)\tilde{f}_{n+1}(\kappa) - (n+1)\tilde{f}_{n-1}(\kappa)\right], \quad n \geq 1, \tag{7.3.3}$$

$$\mathscr{H}_1\{f'(r)\} = -\kappa \tilde{f}_0(\kappa), \tag{7.3.4}$$

provided $[rf(r)]$ vanishes as $r \to 0$ and $r \to \infty$.

PROOF We have, by definition,

$$\mathscr{H}_n\{f'(r)\} = \int_0^\infty r J_n(\kappa r) f'(r) dr$$

which is, integrating by parts,

$$= [rf(r)J_n(\kappa r)]_0^\infty - \int_0^\infty f(r)\frac{d}{dr}[rJ_n(\kappa r)]dr. \tag{7.3.5}$$

We now use the properties of the Bessel function

$$\begin{aligned}\frac{d}{dr}[rJ_n(\kappa r)] &= J_n(\kappa r) + r\kappa J_n'(\kappa r) = J_n(\kappa r) + r\kappa J_{n-1}(\kappa r) - nJ_n(\kappa r) \\ &= (1-n)J_n(\kappa r) + r\kappa J_{n-1}(\kappa r).\end{aligned} \tag{7.3.6}$$

In view of the given condition, the first term of (7.3.5) vanishes as $r \to 0$ and $r \to \infty$, and the derivative within the integral in (7.3.5) can be replaced by (7.3.6) so that (7.3.5) becomes

$$\mathscr{H}_n\{f'(r)\} = (n-1)\int_0^\infty f(r)J_n(\kappa r)dr - \kappa\tilde{f}_{n-1}(\kappa). \tag{7.3.7}$$

We next use the standard recurrence relation for the Bessel function

$$J_n(\kappa r) = \frac{\kappa r}{2n}[J_{n-1}(\kappa r) + J_{n+1}(\kappa r)]. \tag{7.3.8}$$

Thus, (7.3.7) can be rewritten as

$$\begin{aligned}
\mathscr{H}_n[f'(r)] &= -\kappa\tilde{f}_{n-1}(\kappa) + \kappa\left(\frac{n-1}{2n}\right)\left[\int_0^\infty rf(r)\{J_{n-1}(\kappa r) + J_{n+1}(\kappa r)\}dr\right] \\
&= -\kappa\tilde{f}_{n-1}(\kappa) + \kappa\left(\frac{n-1}{2n}\right)\left[\tilde{f}_{n-1}(\kappa) + \tilde{f}_{n+1}(\kappa)\right] \\
&= \left(\frac{\kappa}{2n}\right)\left[(n-1)\tilde{f}_{n+1}(\kappa) - (n+1)\tilde{f}_{n-1}(\kappa)\right].
\end{aligned}$$

In particular, when $n=1$, (7.3.4) follows immediately.

Similarly, repeated applications of (7.3.3) lead to the following result

$$\begin{aligned}
\mathscr{H}_n\{f''(r)\} &= \frac{\kappa}{2n}\left[(n-1)\mathscr{H}_{n+1}\{f'(r)\} - (n+1)\mathscr{H}_{n-1}\{f'(r)\}\right] \\
&= \frac{\kappa^2}{4}\left[\left(\frac{n+1}{n-1}\right)\tilde{f}_{n-2}(\kappa) - 2\left(\frac{n^2-3}{n^2-1}\right)\tilde{f}_n(\kappa)\right. \\
&\qquad \left. + \left(\frac{n-1}{n+1}\right)\tilde{f}_{n+2}(\kappa)\right].
\end{aligned} \tag{7.3.9}$$

∎

THEOREM 7.3.4 If $\mathscr{H}_n\{f(r)\} = \tilde{f}_n(\kappa)$**,** then

$$\mathscr{H}_n\left\{\left(\nabla^2 - \frac{n^2}{r^2}\right)f(r)\right\} = \mathscr{H}_n\left\{\frac{1}{r}\frac{d}{dr}\left(r\frac{df}{dr}\right) - \frac{n^2}{r^2}f(r)\right\} = -\kappa^2\tilde{f}_n(\kappa), \tag{7.3.10}$$

provided both $rf'(r)$ and $rf(r)$ vanish as $r\to 0$ and $r\to\infty$.

PROOF We have, by definition (7.2.8),

$$\mathscr{H}_n\left\{\frac{1}{r}\frac{d}{dr}\left(r\frac{df}{dr}\right) - \frac{n^2}{r^2}f(r)\right\} = \int_0^\infty J_n(\kappa r)\left[\frac{d}{dr}\left(r\frac{df}{dr}\right)\right]dr$$
$$- \int_0^\infty \frac{n^2}{r^2}[rJ_n(\kappa r)]f(r)dr,$$

which is, invoking integration by parts,

$$= \left[\left(r\frac{df}{dr}\right) J_n(\kappa r)\right]_0^\infty - \kappa \int_0^\infty r\frac{df}{dr} J_n'(\kappa r)dr - \int_0^\infty \frac{n^2}{r^2}[rJ_n(\kappa r)]f(r)dr,$$

which is, by replacing the first term with zero because of the given assumption, and by invoking integration by parts again,

$$= -\left[\kappa r f(r) J_n'(\kappa r)\right]_0^\infty + \int_0^\infty \frac{d}{dr}\left[\kappa r J_n'(\kappa r)\right] f(r)dr - \int_0^\infty \frac{n^2}{r^2}[rJ_n(\kappa r)]f(r)dr.$$

We use the given assumptions and Bessel's differential equation,

$$\frac{d}{dr}\left[\kappa r\, J_n'(\kappa r)\right] + r\left(\kappa^2 - \frac{n^2}{r^2}\right) J_n(\kappa r) = 0, \tag{7.3.11}$$

to obtain

$$\begin{aligned}\mathscr{H}_n\left\{\left(\nabla^2 - \frac{n^2}{r^2}\right) f(r)\right\} &= -\int_0^\infty \left(\kappa^2 - \frac{n^2}{r^2}\right) r f(r) J_n(\kappa r) dr \\ &\qquad - \int_0^\infty \frac{n^2}{r^2}[r f(r)]\, J_n(\kappa r)\, dr \\ &= -\kappa^2 \int_0^\infty r J_n(\kappa r) f(r) dr = -\kappa^2 \mathscr{H}_n[f(r)] = -\kappa^2 \tilde{f}_n(\kappa).\end{aligned}$$

This proves the theorem.

In particular, when $n=0$ and $n=1$, we obtain

$$\mathscr{H}_0\left\{\frac{1}{r}\frac{d}{dr}\left(r\frac{df}{dr}\right)\right\} = -\kappa^2 \tilde{f}_0(\kappa), \tag{7.3.12}$$

$$\mathscr{H}_1\left\{\frac{1}{r}\frac{d}{dr}\left(r\frac{df}{dr}\right) - \frac{1}{r^2}f(r)\right\} = -\kappa^2 \tilde{f}_1(\kappa). \tag{7.3.13}$$

Results (7.3.10), (7.3.12), and (7.3.13) are widely used for finding solutions of partial differential equations in axisymmetric cylindrical configurations. We illustrate this point by considering several examples of applications. ∎

7.4 Applications of Hankel Transforms to Partial Differential Equations

The Hankel transforms are extremely useful in solving a variety of partial differential equations in cylindrical polar coordinates. The following examples illustrate applications of the Hankel transforms. The examples given here are only representative of a whole variety of physical problems that can be solved in a similar way.

Example 7.4.1 (*Free Vibration of a Large Circular Membrane*).
Obtain the solution of the free vibration of a large circular elastic membrane governed by the initial value problem

$$c^2\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}\right)=\frac{\partial^2 u}{\partial t^2}, \quad 0<r<\infty, \quad t>0, \tag{7.4.1}$$

$$u(r,0)=f(r), \quad u_t(r,0)=g(r), \quad \text{for } 0\le r<\infty, \tag{7.4.2ab}$$

where $c^2=(T/\rho)=$ constant, T is the tension in the membrane, and ρ is the surface density of the membrane.

Application of the zero-order Hankel transform with respect to r

$$\tilde{u}(\kappa,t)=\int_0^\infty r\,J_0(\kappa r)u(r,t)dr, \tag{7.4.3}$$

to (7.4.1)–(7.4.2ab) gives

$$\frac{d^2\tilde{u}}{dt^2}+c^2\kappa^2\tilde{u}=0, \tag{7.4.4}$$

$$\tilde{u}(\kappa,0)=\tilde{f}(\kappa), \qquad \tilde{u}_t(\kappa,0)=\tilde{g}(\kappa). \tag{7.4.5ab}$$

The general solution of this transformed system is

$$\tilde{u}(\kappa,t)=\tilde{f}(\kappa)\cos(c\kappa t)+(c\kappa)^{-1}\,\tilde{g}(\kappa)\sin(c\kappa t). \tag{7.4.6}$$

The inverse Hankel transform leads to the solution

$$\begin{aligned} u(r,t) = &\int_0^\infty \kappa\,\tilde{f}(\kappa)\cos(c\kappa t)J_0(\kappa r)d\kappa \\ &+\frac{1}{c}\int_0^\infty \tilde{g}(\kappa)\sin(c\kappa t)J_0(\kappa r)d\kappa. \end{aligned} \tag{7.4.7}$$

In particular, we consider

$$u(r,0)=f(r)=Aa(r^2+a^2)^{-\frac{1}{2}}, \quad u_t(r,0)=g(r)=0, \tag{7.4.8ab}$$

so that $\tilde{g}(\kappa)\equiv 0$ and

$$\tilde{f}(\kappa)=Aa\int_0^\infty r(a^2+r^2)^{-\frac{1}{2}}J_0(\kappa r)dr=\frac{Aa}{\kappa}e^{-a\kappa}, \qquad \text{by Example 7.2.1(a).}$$

Thus, the formal solution (7.4.7) becomes

$$u(r,t) = Aa\int_0^\infty e^{-a\kappa} J_0(\kappa r)\cos(c\kappa t)d\kappa = Aa\,\mathrm{Re}\int_0^\infty \exp[-\kappa(a+ict)]J_0(\kappa r)d\kappa$$
$$= Aa\,\mathrm{Re}\left\{r^2 + (a+ict)^2\right\}^{-\frac{1}{2}}, \qquad \text{by Example 7.2.1(a).} \tag{7.4.9}$$

▯

Example 7.4.2 (*Steady Temperature Distribution in a Semi-Infinite Solid with a Steady Heat Source*).
Find the solution of the Laplace equation for the steady temperature distribution $u(r,z)$ with a steady and symmetric heat source $Q_0 q(r)$:

$$u_{rr} + \frac{1}{r}u_r + u_{zz} = -Q_0 q(r), \quad 0 < r < \infty, \quad 0 < z < \infty, \tag{7.4.10}$$
$$u(r,0) = 0, \quad 0 < r < \infty, \tag{7.4.11}$$

where Q_0 is a constant. This boundary condition represents zero temperature at the boundary $z=0$.

Application of the zero-order Hankel transform to (7.4.10) and (7.4.11) gives

$$\frac{d^2\tilde{u}}{dz^2} - \kappa^2\tilde{u} = -Q_0\tilde{q}(\kappa), \quad \tilde{u}(\kappa,0) = 0.$$

The bounded general solution of this system is

$$\tilde{u}(\kappa,z) = A\exp(-\kappa z) + \frac{Q_0}{\kappa^2}\tilde{q}(\kappa),$$

where A is a constant to be determined from the transformed boundary condition. In this case

$$A = -\frac{Q_0}{\kappa^2}\tilde{q}(\kappa).$$

Thus, the formal solution is

$$\tilde{u}(\kappa,z) = \frac{Q_0\,\tilde{q}(\kappa)}{\kappa^2}(1 - e^{-\kappa z}). \tag{7.4.12}$$

The inverse Hankel transform yields the exact integral solution

$$u(r,z) = Q_0\int_0^\infty \frac{\tilde{q}(\kappa)}{\kappa}(1 - e^{-\kappa z})J_0(\kappa r)d\kappa. \tag{7.4.13}$$

▯

Example 7.4.3 (*Axisymmetric Diffusion Equation*).
Find the solution of the axisymmetric diffusion equation

$$u_t = \kappa \left(u_{rr} + \frac{1}{r} u_r \right), \qquad 0 < r < \infty, \quad t > 0, \tag{7.4.14}$$

where $\kappa(>0)$ is a diffusivity constant and

$$u(r,0) = f(r), \qquad \text{for} \quad 0 < r < \infty. \tag{7.4.15}$$

We apply the zero-order Hankel transform defined by (7.4.3) to obtain

$$\frac{d\tilde{u}}{dt} + k^2 \kappa \tilde{u} = 0, \qquad \tilde{u}(k,0) = \tilde{f}(k),$$

where k is the Hankel transform variable. The solution of this transformed system is

$$\tilde{u}(k,t) = \tilde{f}(k) \exp(-\kappa k^2 t). \tag{7.4.16}$$

Application of the inverse Hankel transform gives

$$u(r,t) = \int_0^\infty k \tilde{f}(k) J_0(kr) e^{-\kappa k^2 t} dk = \int_0^\infty k \left[\int_0^\infty l J_0(kl) f(l) dl \right] e^{-\kappa k^2 t} J_0(kr) dk$$

which is, interchanging the order of integration,

$$= \int_0^\infty l\, f(l) dl \int_0^\infty k\, J_0(kl) J_0(kr) \exp(-\kappa k^2 t) dk. \tag{7.4.17}$$

Using a standard table of integrals involving Bessel functions, we state

$$\int_0^\infty k J_0(kl)\, J_0(kr) \exp(-k^2 \kappa t) dk = \frac{1}{2\kappa t} \exp\left[-\frac{(r^2 + l^2)}{4\kappa t} \right] I_0 \left(\frac{rl}{2\kappa t} \right), \tag{7.4.18}$$

where $I_0(x)$ is the modified Bessel function and $I_0(0) = 1$. In particular, when $l = 0$, $J_0(0) = 1$ and integral (7.4.18) becomes

$$\int_0^\infty k J_0(kr) \exp(-k^2 \kappa t) dk = \frac{1}{2\kappa t} \exp\left(-\frac{r^2}{4\kappa t} \right). \tag{7.4.19}$$

We next use (7.4.18) to rewrite (7.4.17) as

$$u(r,t) = \frac{1}{2\kappa t} \int_0^\infty l f(l) I_0 \left(\frac{rl}{2\kappa t} \right) \exp\left[-\frac{(r^2 + l^2)}{4\kappa t} \right] dl. \tag{7.4.20}$$

We now assume $f(r)$ to represent a heat source concentrated in a circle of radius a and allow $a \to 0$ so that the heat source is concentrated at $r=0$ and

$$\lim_{a\to 0} 2\pi \int_0^a r f(r) dr = 1.$$

Or, equivalently,

$$f(r) = \frac{1}{2\pi} \frac{\delta(r)}{r},$$

where $\delta(r)$ is the Dirac delta function.

Thus, the final solution due to the concentrated heat source at $r=0$ is

$$\begin{aligned} u(r,t) &= \frac{1}{4\pi\kappa t} \int_0^\infty \delta(l) I_0\left(\frac{rl}{2\kappa t}\right) \exp\left[-\frac{r^2+l^2}{4\kappa t}\right] dl \\ &= \frac{1}{4\pi\kappa t} \exp\left(-\frac{r^2}{4\kappa t}\right). \end{aligned} \tag{7.4.21}$$

□

Example 7.4.4 (*Axisymmetric Acoustic Radiation Problem*).
Obtain the solution of the wave equation

$$c^2\left(u_{rr} + \frac{1}{r}u_r + u_{zz}\right) = u_{tt}, \quad 0<r<\infty, \quad z>0, \quad t>0, \tag{7.4.22}$$

$$u_z = F(r,t) \quad \text{on } z=0, \tag{7.4.23}$$

where $F(r,t)$ is a given function and c is a constant. We also assume that the solution is bounded and behaves as outgoing spherical waves.

We seek a steady-state solution for the acoustic radiation potential $u = e^{i\omega t}\phi(r,z)$ with $F(r,t) = e^{i\omega t} f(r)$, so that ϕ satisfies the Helmholtz equation

$$\phi_{rr} + \frac{1}{r}\phi_r + \phi_{zz} + \left(\frac{\omega^2}{c^2}\right)\phi = 0, \quad 0<r<\infty, \quad z>0, \tag{7.4.24}$$

with the boundary condition

$$\phi_z = f(r) \qquad \text{on } \ z=0, \tag{7.4.25}$$

where $f(r)$ is a given function of r.

Application of the Hankel transform $\mathscr{H}_0\{\phi(r,z)\} = \tilde{\phi}(k,z)$ to (7.4.24)–(7.4.25) gives

$$\tilde{\phi}_{zz} = \kappa^2 \tilde{\phi}, \qquad z>0,$$

$$\tilde{\phi}_z = \tilde{f}(k), \qquad \text{on } z=0,$$

where

$$\kappa = \left(k^2 - \frac{\omega^2}{c^2}\right)^{\frac{1}{2}}.$$

The solution of this differential system is

$$\tilde{\phi}(k,z) = -\frac{1}{\kappa}\tilde{f}(k)\exp(-\kappa z), \tag{7.4.26}$$

where κ is real and positive for $k > \omega/c$, and purely imaginary for $k < \omega/c$.

The inverse Hankel transform yields the formal solution

$$\phi(r,z) = -\int_0^\infty \frac{k}{\kappa}\tilde{f}(k)\,J_0(kr)\exp(-\kappa z)dk. \tag{7.4.27}$$

Since the exact evaluation of this integral is difficult for an arbitrary $\tilde{f}(k)$, we choose a simple form of $f(r)$ as

$$f(r) = AH(a-r), \tag{7.4.28}$$

where A is a constant, and hence, $\tilde{f}(k) = \frac{Aa}{k}J_1(ak)$.

Thus, the solution (7.4.27) takes the form

$$\phi(r,z) = -Aa\int_0^\infty \frac{1}{\kappa}J_1(ak)J_0(kr)\exp(-\kappa z)dk. \tag{7.4.29}$$

For an asymptotic evaluation of this integral, it is convenient to express (7.4.29) in terms of R which is the distance from the z-axis so that $R^2 = r^2 + z^2$ and $z = R\cos\theta$. Using the asymptotic result for the Bessel function

$$J_0(kr) \sim \left(\frac{2}{\pi kr}\right)^{\frac{1}{2}}\cos\left(kr - \frac{\pi}{4}\right) \qquad \text{as} \quad r\to\infty, \tag{7.4.30}$$

where $r = R\sin\theta$. Consequently, (7.4.29) combined with $u = \exp(i\omega t)\phi$ becomes

$$u \sim -\frac{Aa\sqrt{2}e^{i\omega t}}{\sqrt{\pi R\sin\theta}}\int_0^\infty \frac{1}{\kappa\sqrt{k}}J_1(ak)\cos\left(kR\sin\theta - \frac{\pi}{4}\right)\exp(-\kappa z)dk.$$

This integral can be evaluated asymptotically for $R\to\infty$ using the stationary phase approximation formula to obtain the final result

$$u \sim -\frac{Aac}{\omega R\sin\theta}J_1(ak_1)\exp\left[i\left(\omega t - \frac{\omega R}{c}\right)\right], \tag{7.4.31}$$

where $k_1 = \omega/(c \sin\theta)$ is the stationary point. Physically, this solution represents outgoing spherical waves with constant velocity c and decaying amplitude as $R \to \infty$. $\Box$

Example 7.4.5 (*Axisymmetric Biharmonic Equation*).
We solve the axisymmetric boundary value problem

$$\nabla^4 u(r,z) = 0, \qquad 0 \le r < \infty, \quad z > 0\,, \tag{7.4.32}$$

with the boundary data

$$\begin{aligned} u(r,0) &= f(r), \quad 0 \le r < \infty, && (7.4.33)\\ \frac{\partial u}{\partial z} &= 0 \qquad \text{on } z = 0, \quad 0 \le r < \infty, && (7.4.34)\\ u(r,z) &\to 0 \qquad \text{as } r \to \infty, && (7.4.35) \end{aligned}$$

where the axisymmetric biharmonic operator is

$$\nabla^4 = \nabla^2(\nabla^2) = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}\right)\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}\right). \tag{7.4.36}$$

The use of the Hankel transform $\mathscr{H}_0\{u(r,z)\} = \tilde{u}(k,z)$ to this problem gives

$$\left(\frac{d^2}{dz^2} - k^2\right)^2 \tilde{u}(k,z) = 0, \quad z > 0, \tag{7.4.37}$$

$$\tilde{u}(k,0) = \tilde{f}(k), \quad \frac{d\tilde{u}}{dz} = 0 \qquad \text{on } z = 0. \tag{7.4.38}$$

The bounded solution of (7.4.37) is

$$\tilde{u}(k,z) = (A + zB)\exp(-kz), \tag{7.4.39}$$

where A and B are integrating constants to be determined by (7.4.38) as $A = \tilde{f}(k)$ and $B = k\tilde{f}(k)$. Thus, solution (7.4.39) becomes

$$\tilde{u}(k,z) = (1 + kz)\tilde{f}(k)\exp(-kz). \tag{7.4.40}$$

The inverse Hankel transform gives the formal solution

$$u(r,z) = \int_0^\infty k(1 + kz)\tilde{f}(k) J_0(kr)\exp(-kz)dk. \tag{7.4.41}$$

$\Box$

Example 7.4.6 (*The Axisymmetric Cauchy-Poisson Water Wave Problem*).
We consider the initial value problem for an inviscid water of finite depth h with a free horizontal surface at $z = 0$, and the z-axis positive upward. We

assume that the liquid has constant density ρ with no surface tension. The surface waves are generated in water, which is initially at rest for $t<0$ by the prescribed free surface elevation. In cylindrical polar coordinates (r,θ,z), the axisymmetric water wave equations for the velocity potential $\phi(r,z,t)$ and the free surface elevation $\eta(r,t)$ are

$$\nabla^2\phi = \phi_{rr} + \frac{1}{r}\phi_r + \phi_{zz} = 0, \quad 0 \le r < \infty, \quad -h \le z \le 0, \quad t > 0, \tag{7.4.42}$$

$$\left.\begin{aligned} \phi_z - \eta_t &= 0 \\ \phi_t + g\eta &= 0 \end{aligned}\right\} \quad \text{on } z=0, \quad t>0, \tag{7.4.43ab}$$

$$\phi_z = 0 \qquad \text{on } z=-h, \ \ t>0. \tag{7.4.44}$$

The initial conditions are

$$\phi(r,0,0)=0 \quad \text{and} \quad \eta(r,0)=\eta_0(r), \qquad \text{for } 0 \le r < \infty, \tag{7.4.45}$$

where g is the acceleration due to gravity and $\eta_0(r)$ is the given free surface elevation.

We apply the joint Laplace and the zero-order Hankel transform defined by

$$\bar{\tilde{\phi}}(k,z,s) = \int_0^\infty e^{-st}dt \int_0^\infty rJ_0(kr)\phi(r,z,t)dr, \tag{7.4.46}$$

to (7.4.42)–(7.4.44) so that these equations reduce to

$$\left(\frac{d^2}{dz^2} - k^2\right)\bar{\tilde{\phi}} = 0,$$

$$\left.\begin{aligned} \frac{d\bar{\tilde{\phi}}}{dz} - s\,\bar{\tilde{\eta}} &= -\tilde{\eta}_0(k) \\ s\,\bar{\tilde{\phi}} + g\,\bar{\tilde{\eta}} &= 0 \end{aligned}\right\} \text{ on } z=0,$$

$$\bar{\tilde{\phi}}_z = 0 \qquad\qquad \text{on } z=-h,$$

where $\tilde{\eta}_0(k)$ is the Hankel transform of $\eta_0(r)$ of order zero.

The solutions of this system are

$$\bar{\tilde{\phi}}(k,z,s) = -\frac{g\,\tilde{\eta}_0(k)}{(s^2+\omega^2)}\frac{\cosh k(z+h)}{\cosh kh}, \tag{7.4.47}$$

$$\bar{\tilde{\eta}}(k,s) = \frac{s\,\tilde{\eta}_0(k)}{(s^2+\omega^2)}, \tag{7.4.48}$$

where

$$\omega^2 = gk\,\tanh(kh), \tag{7.4.49}$$

is the famous *dispersion relation* between frequency ω and wavenumber k for water waves in a liquid of depth h. Physically, this dispersion relation describes the interaction between the inertial and gravitational forces.

Application of the inverse transforms gives the integral solutions

$$\phi(r,z,t) = -g\int_0^\infty kJ_0(kr)\tilde{\eta}_0(k)\left(\frac{\sin\omega t}{\omega}\right)\frac{\cosh k(z+h)}{\cosh kh}dk, \quad (7.4.50)$$

$$\eta(r,t) = \int_0^\infty kJ_0(kr)\tilde{\eta}_0(k)\cos\omega t\, dk. \quad (7.4.51)$$

These wave integrals represent exact solutions for ϕ and η at any r and t, but the physical features of the wave motions cannot be described by them. In general, the exact evaluation of the integrals is almost a formidable task. In order to resolve this difficulty, it is necessary and useful to resort to asymptotic methods. It will be sufficient for the determination of the basic features of the wave motions to evaluate (7.4.50) or (7.4.51) asymptotically for a large time and distance with (r/t) held fixed. We now replace $J_0(kr)$ by its asymptotic formula (7.4.30) for $kr\to\infty$, so that (7.4.51) gives

$$\begin{aligned}\eta(r,t) &\sim \left(\frac{2}{\pi r}\right)^{\frac{1}{2}}\int_0^\infty \sqrt{k}\,\tilde{\eta}_0(k)\cos\left(kr-\frac{\pi}{4}\right)\cos\omega t\,dk \\ &= (2\pi r)^{-\frac{1}{2}}\,\text{Re}\int_0^\infty \sqrt{k}\,\tilde{\eta}_0(k)\exp\left[i\left(\omega t - kr + \frac{\pi}{4}\right)\right]dk. \quad (7.4.52)\end{aligned}$$

Application of the stationary phase method to (7.4.52) yields the solution

$$\eta(r,t) \sim \left[\frac{k_1}{rt|\omega''(k_1)|}\right]^{\frac{1}{2}}\tilde{\eta}_0(k_1)\cos[t\,\omega(k_1) - k_1 r], \quad (7.4.53)$$

where the stationary point $k_1 = \left(gt^2/4r^2\right)$ is the root of the equation

$$\omega'(k) = \frac{r}{t}. \quad (7.4.54)$$

For sufficiently deep water, $kh\to\infty$, the dispersion relation becomes

$$\omega^2 = gk. \quad (7.4.55)$$

The solution of the axisymmetric Cauchy-Poisson problem is based on a prescribed initial displacement of unit volume that is concentrated at the origin, which means that $\eta_0(r) = (a/2\pi r)\delta(r)$ so that $\tilde{\eta}_0(k) = \dfrac{a}{2\pi}$. Thus, the asymptotic solution is obtained from (7.4.53) in the form

$$\eta(r,t) \sim \frac{agt^2}{4\pi\sqrt{2}\,r^3}\cos\left(\frac{gt^2}{4r}\right), \quad gt^2 >> 4r. \quad (7.4.56)$$

It is noted that solution (7.4.53) is no longer valid when $\omega''(k_1)=0$. This case can be handled by a modification of the asymptotic evaluation (see Debnath, 1994, p. 91).

A wide variety of other physical problems solved by the Hankel transform, and/or by the joint Hankel and Laplace transform are given in books by Sneddon (1951, 1972) and by Debnath (1994), and in research papers by Debnath (1969, 1983, 1989), Mohanti (1979), and Debnath and Rollins (1992) listed in the Bibliography. $\square$

7.5 Exercises

1. Show that

(a) $\mathscr{H}_0\{(a^2-r^2)H(a-r)\}=\dfrac{4a}{\kappa^3}J_1(\kappa a)-\dfrac{2a^2}{\kappa^2}J_0(a\kappa)$,

(b) $\mathscr{H}_n\{r^n e^{-ar}\}=\dfrac{a}{\sqrt{\pi}}\cdot 2^{n+1}\Gamma\left(n+\dfrac{3}{2}\right)\kappa^n(a^2+\kappa^2)^{-\left(n+\frac{3}{2}\right)}$,

(c) $\mathscr{H}_n\left\{\dfrac{2n}{r}f(r)\right\}=k\,\mathscr{H}_{n-1}\{f(r)\}+k\,\mathscr{H}_{n+1}\{f(r)\}$.

2. (a) Show that the solution of the boundary value problem

$$u_{rr}+\frac{1}{r}u_r+u_{zz}=0,\quad 0<r<\infty,\quad 0<z<\infty,$$
$$u(r,z)=\frac{1}{\sqrt{a^2+r^2}}\quad \text{on z}=0,\quad 0<\text{r}<\infty,$$

is

$$u(r,z)=\int_0^\infty e^{-\kappa(z+a)}J_0(\kappa r)d\kappa=\frac{1}{\sqrt{(z+a)^2+r^2}}.$$

(b) Obtain the solution of the equation in 2(a) with $u(r,0)=f(r)=H(a-r),\quad 0<r<\infty$.

3. (a) The axisymmetric initial value problem is governed by

$$u_t=\kappa\left(u_{rr}+\frac{1}{r}u_r\right)+\delta(t)f(r),\ 0<r<\infty,\ t>0,$$
$$u(r,0)=0\quad\text{for}\quad 0<r<\infty.$$

Show that the formal solution of this problem is

$$u(r,t)=\int_0^\infty kJ_0(kr)\tilde{f}(k)\exp(-k^2\kappa t)dk.$$

(b) For the special case when $f(r) = \left(\frac{Q}{\pi a^2}\right) H(a-r)$, show that the solution is

$$u(r,t) = \left(\frac{Q}{\pi a}\right) \int_0^\infty J_0(kr) J_1(ak) \exp(-k^2 \kappa t) dk.$$

4. If $f(r) = A(a^2 + r^2)^{-\frac{1}{2}}$ where A is a constant, show that the solution of the biharmonic equation described in Example 7.4.5 is

$$u(r,z) = A \frac{\{r^2 + (z+a)(2z+a)\}}{[r^2 + (z+a)^2]^{3/2}}.$$

5. Show that the solution of the boundary value problem

$$u_{rr} + \frac{1}{r} u_r + u_{zz} = 0, \quad 0 \le r < \infty, \quad z > 0,$$
$$u(r,0) = u_0 \qquad \text{for } 0 \le r \le a, \qquad u_0 \text{ is a constant},$$
$$u(r,z) \to 0 \qquad \text{as } z \to \infty,$$

is

$$u(r,z) = a\, u_0 \int_0^\infty J_1(ak) J_0(kr) \exp(-kz) dk.$$

Find the solution of the problem when u_0 is replaced by an arbitrary function $f(r)$, and a by infinity.

6. Solve the axisymmetric biharmonic equation for the small-amplitude free vibration of a thin elastic disk

$$b^2 \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)^2 u + u_{tt} = 0, \quad 0 < r < \infty, \quad t > 0,$$
$$u(r,0) = f(r), \quad u_t(r,0) = 0 \qquad \text{for } 0 < r < \infty,$$

where $b^2 = \left(\frac{D}{2\sigma h}\right)$ is the ratio of the flexural rigidity of the disk and its mass $2h\sigma$ per unit area.

7. Show that the zero-order Hankel transform solution of the axisymmetric Laplace equation

$$u_{rr} + \frac{1}{r} u_r + u_{zz} = 0, \quad 0 < r < \infty, \quad -\infty < z < \infty,$$

with the boundary data

$$\lim_{r\to 0}(r^2 u) = 0, \quad \lim_{t\to 0}(2\pi r) u_r = -f(z), \quad -\infty < z < \infty,$$

is

$$\tilde{u}(k,z) = \frac{1}{4\pi k} \int_{-\infty}^{\infty} \exp\{-k|z-\zeta|\} f(\zeta) d\zeta.$$

Hence, show that

$$u(r,z) = \frac{1}{4\pi} \int_{-\infty}^{\infty} \left\{r^2 + (z-\zeta)^2\right\}^{-\frac{1}{2}} f(\zeta) d\zeta.$$

8. Solve the nonhomogeneous diffusion problem

$$\begin{aligned} &u_t = \kappa \left(u_{rr} + \frac{1}{r} u_r \right) + Q(r,t), \quad 0 < r < \infty, \quad t > 0, \\ &u(r,0) = f(r) \qquad\qquad \text{for } \ 0 < r < \infty, \end{aligned}$$

where κ is a constant.

9. Solve the problem of the electrified unit disk in the $x-y$ plane with center at the origin. The electric potential $u(r,z)$ is axisymmetric and satisfies the boundary value problem

$$\begin{aligned} &u_{rr} + \frac{1}{r} u_r + u_{zz} = 0, \quad 0 < r < \infty, \quad 0 < z < \infty, \\ &u(r,0) = u_0, \quad 0 \le r < a, \\ &\frac{\partial u}{\partial z} = 0, \quad \text{on } z = 0 \quad \text{for } a < r < \infty, \\ &u(r,z) \to 0 \quad \text{as } z \to \infty \ \text{ for all } \ r, \end{aligned}$$

where u_0 is constant. Show that the solution is

$$u(r,z) = \left(\frac{2au_0}{\pi} \right) \int_0^{\infty} J_0(kr) \left(\frac{\sin ak}{k} \right) e^{-kz} dk.$$

10. Solve the axisymmetric surface wave problem in deep water due to an oscillatory surface pressure. The governing equations are

$$\nabla^2 \phi = \phi_{rr} + \frac{1}{r} \phi_r + \phi_{zz} = 0, \quad 0 \le r < \infty, \quad -\infty < z \le 0,$$

$$\left. \begin{aligned} &\phi_t + g\eta = -\frac{P}{\rho} p(r) \exp(i\omega t) \\ &\phi_z - \eta_t = 0 \end{aligned} \right\} \text{on } z = 0, \quad t > 0,$$

$$\phi(r,z,0) = 0 = \eta(r,0), \quad \text{for } 0 \le r < \infty, \quad -\infty < z \le 0.$$

11. Solve the Neumann problem for the Laplace equation

$$u_{rr} + \frac{1}{r}\,u_r + u_{zz} = 0, \quad 0 < r < \infty, \quad 0 < z < \infty$$
$$u_z(r,0) = -\frac{1}{\pi a^2} H(a-r), \quad 0 < r < \infty$$
$$u(r,z) \to 0 \quad \text{as } z \to \infty \quad \text{for } 0 < r < \infty.$$

Show that

$$\lim_{a \to 0} u(r,z) = \frac{1}{2\pi}(r^2 + z^2)^{-\frac{1}{2}}.$$

12. Solve the Cauchy problem for the wave equation in a dissipating medium

$$u_{tt} + 2\kappa u_t = c^2 \left(u_{rr} + \frac{1}{r}\,u_r\right), \quad 0 < r < \infty, \quad t > 0,$$
$$u(r,0) = f(r), \quad u_t(r,0) = g(r) \quad \text{for } 0 < r < \infty,$$

where κ is a constant.

13. Use the joint Laplace and Hankel transform to solve the initial-boundary value problem

$$c^2 \left(u_{rr} + \frac{1}{r} u_r + u_{zz}\right) = u_{tt}, \quad 0 < r < \infty, \quad 0 < z < \infty, \quad t > 0,$$
$$u_z(r,0,t) = H(a-r)H(t), \quad 0 < r < \infty, \quad t > 0,$$
$$u(r,z,t) \to 0 \quad \text{as } r \to \infty \text{ and } u(r,z,t) \to 0 \quad \text{as } z \to \infty,$$
$$u(r,z,0) = 0 = u_t(r,z,0),$$

and show that

$$u_t(r,z,t) = -ac\, H\left(t - \frac{z}{c}\right) \int_0^\infty J_1(ak)\, J_0\left\{ck\sqrt{t^2 - \frac{z^2}{c^2}}\right\} J_0(kr)dk.$$

14. Find the steady temperature $u(r,z)$ in a beam $0 \le r < \infty$, $0 \le z \le a$ when the face $z = 0$ is kept at temperature $u(r,0) = 0$, and the face $z = a$ is insulated except that heat is supplied through a circular hole such that

$$u_z(r,a) = H(b-r).$$

The temperature $u(r,z)$ satisfies the axisymmetric equation

$$u_{rr} + \frac{1}{r}\,u_r + u_{zz} = 0, \quad 0 \le r < \infty, \quad 0 \le z \le a.$$

15. Find the integral solution of the initial-boundary value problem

$$u_{rr} + \frac{1}{r}\, u_r + u_{zz} = u_t, \quad 0 \le r < \infty, \quad 0 \le z < \infty, \quad t > 0,$$
$$u(r,z,0) = 0 \qquad \text{for all } r \text{ and } z,$$
$$\left(\frac{\partial u}{\partial r}\right)_{r=0} = 0, \qquad \text{for} \quad 0 \le z < \infty, \quad t > 0,$$
$$\left(\frac{\partial u}{\partial z}\right)_{z=0} = -\frac{H(a-r)}{\sqrt{a^2+r^2}}, \qquad \text{for} \quad 0 < r < \infty, \quad 0 < t < \infty,$$
$$u(r,z,t) \to 0 \quad \text{as } r \to \infty \quad \text{or} \quad z \to \infty.$$

16. Heat is supplied at a constant rate Q per unit area per unit time over a circular area of radius a in the plane $z = 0$ to an infinite solid of thermal conductivity K, the rest of the plane is kept at zero temperature. Solve for the steady temperature field $u(r,z)$ that satisfies the Laplace equation

$$u_{rr} + \frac{1}{r}\, u_r + u_{zz} = 0, \quad 0 < r < \infty, \quad -\infty < z < \infty,$$

with the boundary conditions

$$u \to 0 \text{ as } r \to \infty, \quad u \to 0 \quad \text{as } |z| \to \infty,$$
$$-K u_z = \left(\frac{2Q}{\pi\, a^2}\right) H(a-r) \text{ when } z = 0.$$

17. The velocity potential $\phi(r,z)$ for the flow of an inviscid fluid through a circular aperture of unit radius in a plane rigid screen satisfies the Laplace equation

$$\phi_{rr} + \frac{1}{r}\phi_r + \phi_{zz} = 0, \; 0 < r < \infty$$

with the boundary conditions

$$\left.\begin{array}{lll} \phi = 1 & \text{for} & 0 < r < 1 \\ \phi_z = 0 & \text{for} & r > 1 \end{array}\right\} \text{ on } z = 0.$$

Obtain the solution of this boundary value problem.

18. Solve the Cauchy-Poisson wave problem (Debnath, 1989) for a viscous liquid of finite or infinite depth governed by the equations, free surface, boundary, and initial conditions

$$\phi_{rr} + \frac{1}{r}\phi_r + \phi_{zz} = 0,$$
$$\psi_t = \nu\left(\psi_{rr} + \frac{1}{r}\psi_r - \frac{1}{r^2}\psi + \psi_{zz}\right),$$

where $\phi(r,z,t)$ and $\psi(r,z,t)$ represent the potential and stream functions, respectively, $0 \le r < \infty$, $-h \le z \le 0$ (or $-\infty < z \le 0$) and $t > 0$.

The free surface conditions are

$$\left.\begin{aligned} &\eta_t - w = 0 \\ &\mu(u_z + w_r) = 0 \\ &\phi_t + g\eta + 2\nu\, w_z = 0 \end{aligned}\right\} \quad \text{on} \quad z = 0,\ t > 0$$

where $\eta = \eta(r,t)$ is the free surface elevation, $u = \phi_r + \psi_z$ and $w = \phi_z - \dfrac{\psi}{r} - \psi_r$ are the radial and vertical velocity components of liquid particles, $\mu = \rho\nu$ is the dynamic viscosity, ρ is the density, and ν is the kinematic viscosity of the liquid.

The boundary conditions at the rigid bottom are

$$\left.\begin{aligned} &u = \phi_r + \psi_z = 0 \\ &w = \phi_z - \frac{1}{r}(r\psi)_r = 0 \end{aligned}\right\} \text{ on } z = -h.$$

The initial conditions are

$$\eta = a\,\frac{\delta(r)}{r}, \quad \phi = \psi = 0 \quad \text{at } t = 0,$$

where a is a constant and $\delta(r)$ is the Dirac delta function.

If the liquid is of infinite depth, the bottom boundary conditions are

$$(\phi, \psi) \to (0,0) \qquad \text{as } z \to -\infty.$$

19. Use the joint Hankel and Laplace transform method to solve the initial-boundary value problem

$$u_{rr} + \frac{1}{r}\,u_r - u_{tt} - 2\varepsilon\, u_t = a\,\frac{\delta(r)}{r}\,\delta(t), \quad 0 < r < \infty, \quad t > 0,$$

$$u(r,t) \to 0 \qquad \text{as } r \to \infty,$$

$$u(0,t) \text{ is finite for } t > 0,$$

$$u(r,0) = 0 = u_t(r,0) \qquad \text{for } 0 < r < \infty.$$

20. Surface waves are generated in an inviscid liquid of infinite depth due to an explosion (Sen, 1963) above it, which generates the pressure field $p(r,t)$. The velocity potential $u = \phi(r,z,t)$ satisfies the Laplace equation

$$u_{rr} + \frac{1}{r}\,u_r + u_{zz} = 0, \quad 0 < r < \infty, \quad t > 0,$$

and the free surface condition

$$u_{tt} + g\,u_z = \frac{1}{\rho}\left(\frac{\partial p}{\partial t}\right)[H(r) - H\{r, r_0(t)\}] \qquad \text{on } z = 0,$$

where ρ is the constant density of the liquid, $r_0(t)$ is the extent of the blast, and the liquid is initially at rest.

Solve this problem.

21. The electrostatic potential $u(r, z)$ generated in the space between two horizontal disks at $z = \pm\, a$ by a point charge q at $r = z = 0$ is described by a singular function at $r = z = 0$ is

$$u(r, z) = \phi(r, z) + q(r^2 + z^2)^{-\frac{1}{2}},$$

where $\phi(r, z)$ satisfies the Laplace equation

$$\phi_{rr} + \frac{1}{r}\,\phi_r + \phi_{zz} = 0, \quad 0 < r < \infty$$

with the boundary conditions

$$\phi(r, z) = -q(r^2 + z^2)^{-\frac{1}{2}} \text{ at } z = \pm\, a.$$

Obtain the solution for $\phi(r, z)$ and then $u(r, z)$.

22. Show that

(a) $\mathscr{H}_n\left[e^{-ar} f(r)\right] = \mathscr{L}\left\{r f(r) J_n(\kappa r)\right\},$

(b) $\mathscr{H}_0\left[e^{-ar^2} J_0(br)\right] = \dfrac{a}{2}\exp\left(\dfrac{\kappa^2 - b^2}{4a}\right) I_0\left(\dfrac{b\kappa}{2a}\right),$

(c) $\mathscr{H}_n\left[r^{n-1} e^{-ar}\right] = \dfrac{(2\kappa)^n (n - \frac{1}{2})!}{\sqrt{\pi}(\kappa^2 + a^2)^{n+\frac{1}{2}}},$

(d) $\mathscr{H}_n\left[\dfrac{f(r)}{r}\right] = \left(\dfrac{\kappa}{2n}\right)\left[\tilde{f}_{n-1}(\kappa) + \tilde{f}_{n+1}(\kappa)\right],$

(e) $\mathscr{H}_n\left[r^{n-1}\dfrac{d}{dr}\left\{r^{1-n} f(r)\right\}\right] = -\kappa \tilde{f}_{n-1}(\kappa),$

(f) $\mathscr{H}_n\left[r^{-(n+1)}\dfrac{d}{dr}\left\{r^{n+1} f(r)\right\}\right] = \kappa \tilde{f}_{n+1}(\kappa).$

23. Show that

(a) $\mathscr{H}_0\left[e^{-\frac{r^2}{2}}\right] = e^{-\frac{\kappa^2}{2}}$ (Self-reciprocal).

(b) $\mathscr{H}_0\left[\delta(r - a)\right] = a\, J_0(a\kappa).$

(c) $\mathscr{H}_0\left[\frac{1}{r}\right]=\frac{1}{\kappa}.$

24. Using the Parseval relation (7.3.2), show that

$$I(a,b)=\int_0^\infty \frac{1}{\kappa}\,J_{n+1}(a\kappa)\,J_{n+1}(b\kappa)\,d\kappa=\frac{1}{2(n+1)}\,\left(\frac{a}{b}\right)^{n+1},$$

$$0<a<b,\;n+\frac{1}{2}>0.$$

25. (a) Solve the axisymmetric Dirichlet problem in a half space described by Laplace equation

$$\begin{aligned}
&u_{rr}+\frac{1}{r}u_r+u_{zz}=0, \qquad 0<r<\infty, \quad z>0,\\
&u(r,0)=f(r), \qquad 0<r<\infty,\\
&u(r,z)\to 0 \quad \text{as}\, r\to\infty,\; z\to\infty.
\end{aligned}$$

(b) Find the solution of (a) when $f(r)=H(c-r)$.

(c) Find the solution of (a) when $f(r)=\dfrac{1}{\sqrt{r^2+a^2}},\; a>0.$

26. Solve the axisymmetric small-amplitude vibration of a thin elastic plate governed by the equation

$$a^2\left(\frac{\partial^2}{\partial r^2}+\frac{\partial}{r\,\partial r}\right)^2 u(r,t)+\frac{\partial^2 u}{\partial t^2}=0, \qquad 0<r<\infty,\; t>0$$

with the initial conditions

$$u(r,0)=f(r), \quad u_t(r,0)=0, \qquad 0<r<\infty,$$

where $a=\frac{D}{2\rho h}$, D is the flexural rigidity, ρ is the density, and $2h$ is the thickness of the plate.

27. Solve the forced vibration problem of an elastic membrane described by the non-homogeneous boundary value problem

$$\begin{aligned}
&u_{rr}+\frac{1}{r}u_r-\frac{1}{c^2}u_{tt}=-\frac{1}{T}p(r,t), \qquad 0<r<\infty, \quad t>0,\\
&u(r,0)=f(r), \quad u_t(r,0)=g(r), \qquad 0<r<\infty,\\
&u(r,t)\,\text{is bounded at}\,\infty\;(r\to\infty),
\end{aligned}$$

where T is the tension of the membrane and $c^2=\frac{T}{\rho}$.

8

Mellin Transforms and Their Applications

"One cannot understand ... the universality of laws of nature, the relationship of things, without an understanding of mathematics. There is no other way to do it."

Richard P. Feynman

"The research worker, in his efforts to express the fundamental laws of Nature in mathematical form, should strive mainly for mathematical beauty. He should take simplicity into consideration in a subordinate way to beauty. ... It often happens that the requirements of simplicity and beauty are the same, but where they clash the latter must take precedence."

Paul Dirac

8.1 Introduction

This chapter deals with the theory and applications of the Mellin transform. We derive the Mellin transform and its inverse from the complex Fourier transform. This is followed by several examples and the basic operational properties of Mellin transforms. We discuss several applications of Mellin transforms to boundary value problems and to summation of infinite series. The Weyl transform and the Weyl fractional derivatives with examples are also included.

Historically, Riemann (1876) first recognized the *Mellin transform* in his famous memoir on prime numbers. Its explicit formulation was given by Cahen (1894). Almost simultaneously, Mellin (1896, 1902) gave an elaborate discussion of the Mellin transform and its inversion formula.

8.2 Definition of the Mellin Transform and Examples

We derive the Mellin transform and its inverse from the complex Fourier transform and its inverse, which are defined respectively by

$$\mathscr{F}\{g(\xi)\} = G(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ik\xi} g(\xi) d\xi, \tag{8.2.1}$$

$$\mathscr{F}^{-1}\{G(k)\} = g(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ik\xi} G(k) dk. \tag{8.2.2}$$

Making the changes of variables $\exp(\xi) = x$ and $ik = c - p$, where c is a constant, in results (8.2.1) and (8.2.2) we obtain

$$G(ip - ic) = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} x^{p-c-1} g(\log x) dx, \tag{8.2.3}$$

$$g(\log x) = \frac{1}{\sqrt{2\pi}} \int_{c-i\infty}^{c+i\infty} x^{c-p} G(ip - ic) dp. \tag{8.2.4}$$

We now write $\frac{1}{\sqrt{2\pi}} x^{-c} g(\log x) \equiv f(x)$ and $G(ip - ic) \equiv \tilde{f}(p)$ to define the *Mellin transform* of $f(x)$ and the *inverse Mellin transform* as

$$\mathscr{M}\{f(x)\} = \tilde{f}(p) = \int_{0}^{\infty} x^{p-1} f(x) dx, \tag{8.2.5}$$

$$\mathscr{M}^{-1}\{\tilde{f}(p)\} = f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} x^{-p} \tilde{f}(p) dp, \tag{8.2.6}$$

where $f(x)$ is a real valued function defined on $(0, \infty)$ and the Mellin transform variable p is a complex number. Sometimes, the Mellin transform of $f(x)$ is denoted explicitly by $\tilde{f}(p) = \mathscr{M}[f(x), p]$. Obviously, $\mathscr{M}$ and $\mathscr{M}^{-1}$ are linear integral operators.

Example 8.2.1 (a) If $f(x) = e^{-nx}$, where $n > 0$, then

$$\mathscr{M}\{e^{-nx}\} = \tilde{f}(p) = \int_{0}^{\infty} x^{p-1} e^{-nx} dx,$$

which is, by putting $nx = t$,

$$= \frac{1}{n^p} \int_0^\infty t^{p-1} e^{-t} dt = \frac{\Gamma(p)}{n^p}. \tag{8.2.7}$$

(b) If $f(x) = \dfrac{1}{1+x}$, then

$$\mathscr{M}\left\{\frac{1}{1+x}\right\} = \tilde{f}(p) = \int_0^\infty x^{p-1} \cdot \frac{dx}{1+x},$$

which is, by substituting $x = \dfrac{t}{1-t}$ or $t = \dfrac{x}{1+x}$,

$$= \int_0^1 t^{p-1}(1-t)^{(1-p)-1} dt = B(p, 1-p) = \Gamma(p)\Gamma(1-p),$$

which is, by a well-known result for the gamma function,

$$= \pi \operatorname{cosec}(p\pi), \qquad 0 < \operatorname{Re}(p) < 1. \tag{8.2.8}$$

(c) If $f(x) = (e^x - 1)^{-1}$, then

$$\mathscr{M}\left\{\frac{1}{e^x - 1}\right\} = \tilde{f}(p) = \int_0^\infty x^{p-1} \frac{1}{e^x - 1} dx,$$

which is, by using $\sum_{n=0}^{\infty} e^{-nx} = \dfrac{1}{1 - e^{-x}}$ and hence, $\sum_{n=1}^{\infty} e^{-nx} = \dfrac{1}{e^x - 1}$,

$$= \sum_{n=1}^{\infty} \int_0^\infty x^{p-1} e^{-nx} dx = \sum_{n=1}^{\infty} \frac{\Gamma(p)}{n^p} = \Gamma(p)\zeta(p), \tag{8.2.9}$$

where $\zeta(p) = \sum_{n=1}^{\infty} \dfrac{1}{n^p}$, $(\operatorname{Re} p > 1)$ is the famous *Riemann zeta function.*

(d) If $f(x) = \dfrac{2}{e^{2x} - 1}$, then

$$\mathscr{M}\left\{\frac{2}{e^{2x} - 1}\right\} = \tilde{f}(p) = 2\int_0^\infty x^{p-1} \frac{dx}{e^{2x} - 1} = 2\sum_{n=1}^{\infty} \int_0^\infty x^{p-1} e^{-2nx} dx$$

$$= 2\sum_{n=1}^{\infty} \frac{\Gamma(p)}{(2n)^p} = 2^{1-p}\,\Gamma(p) \sum_{n=1}^{\infty} \frac{1}{n^p} = 2^{1-p}\,\Gamma(p)\zeta(p). \tag{8.2.10}$$

(e) If $f(x) = \dfrac{1}{e^x + 1}$, then

$$\mathscr{M}\left\{\frac{1}{e^x+1}\right\} = (1-2^{1-p})\Gamma(p)\,\zeta(p). \tag{8.2.11}$$

This follows from the result

$$\left[\frac{1}{e^x-1} - \frac{1}{e^x+1}\right] = \frac{2}{e^{2x}-1}$$

combined with (8.2.9) and (8.2.10).

(f) If $f(x) = \dfrac{1}{(1+x)^n}$, then

$$\mathscr{M}\left\{\frac{1}{(1+x)^n}\right\} = \int_0^\infty x^{p-1}(1+x)^{-n}\,dx,$$

which is, by putting $x = \dfrac{t}{1-t}$ or $t = \dfrac{x}{1+x}$,

$$\begin{aligned} &= \int_0^1 t^{p-1}(1-t)^{n-p-1}\,dt \\ &= B(p, n-p) = \frac{\Gamma(p)\Gamma(n-p)}{\Gamma(n)}, \end{aligned} \tag{8.2.12}$$

where $B(p,q)$ is the standard beta function.

Hence,

$$\mathscr{M}^{-1}\{\Gamma(p)\Gamma(n-p)\} = \frac{\Gamma(n)}{(1+x)^n}.$$

(g) Find the Mellin transform of $\cos kx$ and $\sin kx$.

It follows from Example 8.2.1(a) that

$$\mathscr{M}\left[e^{-ikx}\right] = \frac{\Gamma(p)}{(ik)^p} = \frac{\Gamma(p)}{k^p}\left(\cos\frac{p\pi}{2} - i\sin\frac{p\pi}{2}\right).$$

Separating real and imaginary parts, we find

$$\mathscr{M}\left[\cos kx\right] = k^{-p}\,\Gamma(p)\cos\left(\frac{\pi p}{2}\right), \tag{8.2.13}$$

$$\mathscr{M}\left[\sin kx\right] = k^{-p}\,\Gamma(p)\sin\left(\frac{\pi p}{2}\right). \tag{8.2.14}$$

These results can be used to calculate the Fourier cosine and Fourier sine transforms of x^{p-1}. Result (8.2.13) can be written as

$$\int_0^\infty x^{p-1}\cos kx\,dx = \frac{\Gamma(p)}{k^p}\cos\left(\frac{\pi p}{2}\right).$$

Or, equivalently,

$$\mathscr{F}_c\left\{\sqrt{\frac{\pi}{2}}\,x^{p-1}\right\}=\frac{\Gamma(p)}{k^p}\cos\left(\frac{\pi p}{2}\right).$$

Or,

$$\mathscr{F}_c\left\{x^{p-1}\right\}=\sqrt{\frac{2}{\pi}}\,\frac{\Gamma(p)}{k^p}\cos\left(\frac{\pi p}{2}\right). \tag{8.2.15}$$

Similarly,

$$\mathscr{F}_s\left\{x^{p-1}\right\}=\sqrt{\frac{2}{\pi}}\,\frac{\Gamma(p)}{k^p}\sin\left(\frac{\pi p}{2}\right). \tag{8.2.16}$$

□

8.3 Basic Operational Properties of Mellin Transforms

If $\mathscr{M}\{f(x)\}=\tilde{f}(p)$, then the following operational properties hold:

(a) (*Scaling Property*).

$$\mathscr{M}\left\{f(ax)\right\}=a^{-p}\tilde{f}(p),\;\; a>0. \tag{8.3.1}$$

PROOF By definition, we have,

$$\mathscr{M}\left\{f(ax)\right\}=\int_0^\infty x^{p-1}f(ax)dx,$$

which is, by substituting $ax=t$,

$$=\frac{1}{a^p}\int_0^\infty t^{p-1}f(t)dt=\frac{\tilde{f}(p)}{a^p}.$$

■

(b) (*Shifting Property*).

$$\mathscr{M}\left[x^a\,f(x)\right]=\tilde{f}(p+a). \tag{8.3.2}$$

Its proof follows from the definition.

(c)

$$\mathscr{M}\left\{f(x^a)\right\}=\frac{1}{a}\,\tilde{f}\left(\frac{p}{a}\right), \tag{8.3.3}$$

$$\mathscr{M}\left\{\frac{1}{x}\,f\left(\frac{1}{x}\right)\right\}=\tilde{f}(1-p), \tag{8.3.4}$$

$$\mathscr{M}\left\{(\log x)^n\,f(x)\right\}=\frac{d^n}{dp^n}\tilde{f}(p), \quad n=1,2,3,\ldots. \tag{8.3.5}$$

The proofs of (8.3.3) and (8.3.4) are easy and hence, left to the reader.
Result (8.3.5) can easily be proved by using the result

$$\frac{d}{dp}x^{p-1}=(\log x)x^{p-1}. \tag{8.3.6}$$

(d) (*Mellin Transforms of Derivatives*).

$$\mathscr{M}\left[f'(x)\right]=-(p-1)\tilde{f}(p-1), \tag{8.3.7}$$

provided $[x^{p-1}f(x)]$ vanishes as $x\to 0$ and as $x\to\infty$.

$$\mathscr{M}\left[f''(x)\right]=(p-1)(p-2)\tilde{f}(p-2). \tag{8.3.8}$$

More generally,

$$\begin{aligned}\mathscr{M}\left[f^{(n)}(x)\right] &= (-1)^n\frac{\Gamma(p)}{\Gamma(p-n)}\tilde{f}(p-n)\\ &= (-1)^n\frac{\Gamma(p)}{\Gamma(p-n)}\,\mathscr{M}\left[f(x),p-n\right],\end{aligned} \tag{8.3.9}$$

provided $x^{p-r-1}f^{(r)}(x)=0$ as $x\to 0$ for $r=0,1,2,\ldots,(n-1)$.

PROOF We have, by definition,

$$\mathscr{M}\left[f'(x)\right]=\int_0^\infty x^{p-1}f'(x)\,dx,$$

which is, integrating by parts,

$$\begin{aligned}&= \left[x^{p-1}f(x)\right]_0^\infty-(p-1)\int_0^\infty x^{p-2}f(x)\,dx\\ &= -(p-1)\tilde{f}(p-1).\end{aligned}$$

∎

The proofs of (8.3.8) and (8.3.9) are similar and left to the reader.

(e) If $\mathscr{M}\left\{f(x)\right\}=\tilde{f}(p)$, then

$$\mathscr{M}\left\{xf'(x)\right\}=-p\tilde{f}(p), \tag{8.3.10}$$

provided $x^p f(x)$ vanishes at $x=0$ and as $x\to\infty$.

$$\mathscr{M}\left\{x^2 f''(x)\right\} = (-1)^2 p(p+1)\tilde{f}(p). \tag{8.3.11}$$

More generally,

$$\mathscr{M}\left\{x^n f^{(n)}(x)\right\} = (-1)^n \frac{\Gamma(p+n)}{\Gamma(p)}\tilde{f}(p). \tag{8.3.12}$$

PROOF We have, by definition,

$$\mathscr{M}\left\{x f'(x)\right\} = \int_0^\infty x^p f'(x)dx,$$

which is, integrating by parts,

$$= \left[x^p f(x)\right]_0^\infty - p\int_0^\infty x^{p-1} f(x)dx = -p\tilde{f}(p).$$

∎

Similar arguments can be used to prove results (8.3.11) and (8.3.12).

(f) (*Mellin Transforms of Differential Operators*).
If $\mathscr{M}\left\{f(x)\right\} = \tilde{f}(p)$, then

$$\mathscr{M}\left[\left(x\frac{d}{dx}\right)^2 f(x)\right] = \mathscr{M}\left[x^2 f''(x) + x f'(x)\right] = (-1)^2\, p^2 \tilde{f}(p), \tag{8.3.13}$$

and more generally,

$$\mathscr{M}\left[\left(x\frac{d}{dx}\right)^n f(x)\right] = (-1)^n p^n \tilde{f}(p). \tag{8.3.14}$$

PROOF We have, by definition,

$$\begin{aligned}
\mathscr{M}\left[\left(x\frac{d}{dx}\right)^2 f(x)\right] &= \mathscr{M}\left[x^2 f''(x) + x\, f'(x)\right] \\
&= \mathscr{M}\left[x^2 f''(x)\right] + \mathscr{M}\left[x\, f'(x)\right] \\
&= -p\tilde{f}(p) + p(p+1)\tilde{f}(p) \qquad \text{by (8.3.10) and (8.3.11)} \\
&= (-1)^2\, p^2\, \tilde{f}(p).
\end{aligned}$$

∎

Similar arguments can be used to prove the general result (8.3.14).

(g) (*Mellin Transforms of Integrals*).

$$\mathscr{M}\left\{\int_0^x f(t)dt\right\} = -\frac{1}{p}\,\tilde{f}(p+1). \tag{8.3.15}$$

In general,

$$\mathscr{M}\{I_n\, f(x)\} = \mathscr{M}\left\{\int_0^x I_{n-1} f(t)dt\right\} = (-1)^n \frac{\Gamma(p)}{\Gamma(p+n)}\tilde{f}(p+n), \tag{8.3.16}$$

where $I_n\, f(x)$ is the nth repeated integral of $f(x)$ defined by

$$I_n f(x) = \int_0^x I_{n-1} f(t)dt. \tag{8.3.17}$$

PROOF We write

$$F(x) = \int_0^x f(t)dt$$

so that $F'(x) = f(x)$ with $F(0) = 0$. Application of (8.3.7) with $F(x)$ as defined gives

$$\mathscr{M}\,\{f(x) = F'(x), p\} = -(p-1)\mathscr{M}\left\{\int_0^x f(t)dt,\, p-1\right\},$$

which is, replacing p by $p+1$,

$$\mathscr{M}\left\{\int_0^x f(t)\,dt,\, p\right\} = -\frac{1}{p}\mathscr{M}\,\{f(x), p+1\} = -\frac{1}{p}\,\tilde{f}(p+1).$$

An argument similar to this can be used to prove (8.3.16). ∎

(h) (*Convolution Type Theorems*).
If $\mathscr{M}\,\{f(x)\} = \tilde{f}(p)$ and $\mathscr{M}\{g(x)\} = \tilde{g}(p)$, then

$$\mathscr{M}\,[f(x) * g(x)] = \mathscr{M}\left[\int_0^\infty f(\xi)\, g\left(\frac{x}{\xi}\right)\frac{d\xi}{\xi}\right] = \tilde{f}(p)\tilde{g}(p), \tag{8.3.18}$$

$$\mathscr{M}\,[f(x) \circ g(x)] = \mathscr{M}\left[\int_0^\infty f(x\xi)\, g(\xi)d\xi\right] = \tilde{f}(p)\tilde{g}(1-p). \tag{8.3.19}$$

PROOF We have, by definition,

$$\begin{aligned}
\mathscr{M}\left[f(x)*g(x)\right] &= \mathscr{M}\left[\int_0^\infty f(\xi)\,g\left(\frac{x}{\xi}\right)\frac{d\xi}{\xi}\right]\\
&= \int_0^\infty x^{p-1}dx\int_0^\infty f(\xi)\,g\left(\frac{x}{\xi}\right)\frac{d\xi}{\xi}\\
&= \int_0^\infty f(\xi)\,\frac{d\xi}{\xi}\int_0^\infty x^{p-1}g\left(\frac{x}{\xi}\right)\,dx, \quad \left(\frac{x}{\xi}=\eta\right),\\
&= \int_0^\infty f(\xi)\,\frac{d\xi}{\xi}\int_0^\infty (\xi\eta)^{p-1}g(\eta)\,\xi\,d\eta\\
&= \int_0^\infty \xi^{p-1}f(\xi)d\xi\int_0^\infty \eta^{p-1}g(\eta)d\eta = \tilde{f}(p)\tilde{g}(p).
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\mathscr{M}\left[f(x)\circ g(x)\right] &= \mathscr{M}\left[\int_0^\infty f(x\xi)\,g(\xi)d\xi\right]\\
&= \int_0^\infty x^{p-1}dx\int_0^\infty f(x\xi)\,g(\xi)d\xi, \quad (x\xi=\eta),\\
&= \int_0^\infty g(\xi)d\xi\int_0^\infty \eta^{p-1}\xi^{1-p}f(\eta)\frac{d\eta}{\xi}\\
&= \int_0^\infty \xi^{1-p-1}g(\xi)d\xi\int_0^\infty \eta^{p-1}f(\eta)d\eta = \tilde{g}(1-p)\tilde{f}(p).
\end{aligned}$$

∎

Note that, in this case, the operation $\circ$ is not commutative.
Clearly, putting $x=s$,

$$\mathscr{M}^{-1}\{\tilde{f}(1-p)\tilde{g}(p)\} = \int_0^\infty g(st)f(t)dt.$$

Putting $g(t)=e^{-t}$ and $\tilde{g}(p)=\Gamma(p)$, we obtain the Laplace transform of $f(t)$

$$\mathscr{M}^{-1}\left\{\tilde{f}(1-p)\Gamma(p)\right\} = \int_0^\infty e^{-st}f(t)dt = \mathscr{L}\left\{f(t)\right\} = \bar{f}(s). \tag{8.3.20}$$

(i) (*Parseval's Type Property*).
If $\mathscr{M}\{f(x)\} = \tilde{f}(p)$ and $\mathscr{M}\{g(x)\} = \tilde{g}(p)$, then

$$\mathscr{M}[f(x)g(x)] = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \tilde{f}(s)\tilde{g}(p-s)ds. \tag{8.3.21}$$

Or, equivalently,

$$\int_0^\infty x^{p-1} f(x)g(x)dx = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \tilde{f}(s)\tilde{g}(p-s)ds. \tag{8.3.22}$$

In particular, when $p=1$, we obtain the *Parseval formula* for the Mellin transform,

$$\int_0^\infty f(x)g(x)dx = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \tilde{f}(s)\tilde{g}(1-s)ds. \tag{8.3.23}$$

PROOF By definition, we have

$$\begin{aligned}
\mathscr{M}[f(x)g(x)] &= \int_0^\infty x^{p-1} f(x)g(x)dx \\
&= \frac{1}{2\pi i}\int_0^\infty x^{p-1} g(x)dx \int_{c-i\infty}^{c+i\infty} x^{-s}\tilde{f}(s)ds \\
&= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \tilde{f}(s)ds \int_0^\infty x^{p-s-1} g(x)dx \\
&= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \tilde{f}(s)\tilde{g}(p-s)ds.
\end{aligned}$$

When $p=1$, the above result becomes (8.3.23). ∎

8.4 Applications of Mellin Transforms

Example 8.4.1 Obtain the solution of the boundary value problem

$$x^2 u_{xx} + x u_x + u_{yy} = 0, \qquad 0 \le x < \infty, \qquad 0 < y < 1 \tag{8.4.1}$$

$$u(x,0) = 0, \quad u(x,1) = \left\{ \begin{array}{ll} A, & 0 \le x \le 1 \\ 0, & x > 1 \end{array} \right\}, \tag{8.4.2}$$

where A is a constant.

We apply the Mellin transform of $u(x,\, y)$ with respect to x defined by

$$\tilde{u}(p,y) = \int_0^\infty x^{p-1} u(x,y)\, dx$$

to reduce the given system into the form

$$\tilde{u}_{yy} + p^2 \tilde{u} = 0, \quad 0 < y < 1$$

$$\tilde{u}(p,0) = 0, \quad \tilde{u}(p,1) = A \int_0^1 x^{p-1} dx = \frac{A}{p}.$$

The solution of the transformed problem is

$$\tilde{u}(p,y) = \frac{A}{p} \frac{\sin\, py}{\sin\, p}, \qquad 0 < \operatorname{Re} p < 1.$$

The inverse Mellin transform gives

$$u(x,y) = \frac{A}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{x^{-p}}{p} \frac{\sin\, py}{\sin\, p}\, dp, \tag{8.4.3}$$

where $\tilde{u}(p,y)$ is analytic in the vertical strip $0 < \operatorname{Re}(p) = c < \pi$. The integrand of (8.4.3) has simple poles at $p = n\pi$, $n = 1, 2, 3, \ldots$ which lie inside a semicircular contour in the right half plane. Evaluating (8.4.3) by theory of residues gives the solution for $x > 1$ as

$$u(x,y) = \frac{A}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} (-1)^n\, x^{-n\pi} \sin\, n\pi y. \tag{8.4.4}$$

□

Example 8.4.2 (*Potential in an Infinite Wedge*).
Find the potential $\phi(r,\theta)$ that satisfies the Laplace equation

$$r^2\phi_{rr} + r\phi_r + \phi_{\theta\theta} = 0 \tag{8.4.5}$$

in an infinite wedge $0 < r < \infty$, $-\alpha < \theta < \alpha$ as shown in Figure 8.1 with the boundary conditions

$$\phi(r,\alpha) = f(r), \;\; \phi(r,\,-\alpha) = g(r) \quad 0 \le r < \infty, \tag{8.4.6ab}$$

$$\phi(r,\theta) \to 0 \;\text{ as }\; r \to \infty \;\text{ for all }\; \theta \;\text{ in }\; -\alpha < \theta < \alpha. \tag{8.4.7}$$

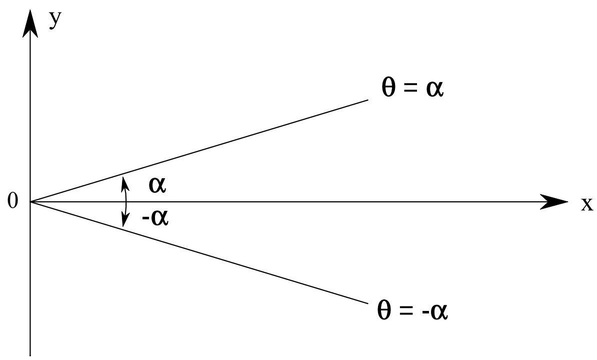

Figure 8.1 An infinite wedge.

We apply the Mellin transform of the potential $\phi(r,\theta)$ defined by

$$\mathscr{M}\left[\phi(r,\theta)\right] = \tilde{\phi}(p,\theta) = \int_0^\infty r^{p-1}\phi(r,\theta)\,dr$$

to the differential system (8.4.5)–(8.4.7) to obtain

$$\frac{d^2\tilde{\phi}}{d\theta^2} + p^2\tilde{\phi} = 0, \tag{8.4.8}$$

$$\tilde{\phi}(p,\alpha) = \tilde{f}(p), \qquad \tilde{\phi}(p,\,-\alpha) = \tilde{g}(p). \tag{8.4.9ab}$$

The general solution of the transformed equation is

$$\tilde{\phi}(p,\theta) = A\cos p\theta + B\,\sin\,p\theta, \tag{8.4.10}$$

where A and B are functions of p and α. The boundary conditions (8.4.9ab) determine A and B, which satisfy

$$A \cos p\alpha + B \sin p\alpha = \tilde{f}(p),$$
$$A \cos p\alpha - B \sin p\alpha = \tilde{g}(p).$$

These give $\quad A = \dfrac{\tilde{f}(p) + \tilde{g}(p)}{2\cos p\alpha}, \quad B = \dfrac{\tilde{f}(p) - \tilde{g}(p)}{2\sin p\alpha}.$

Thus, solution (8.4.10) becomes

$$\begin{aligned}\tilde{\phi}(p,\theta) &= \tilde{f}(p).\frac{\sin p(\alpha+\theta)}{\sin(2\,p\alpha)} + \tilde{g}(p)\frac{\sin p(\alpha-\theta)}{\sin(2\,p\alpha)} \\ &= \tilde{f}(p)\tilde{h}(p,\alpha+\theta) + \tilde{g}(p)\tilde{h}(p,\alpha-\theta),\end{aligned} \tag{8.4.11}$$

where

$$\tilde{h}(p,\theta) = \frac{\sin p\theta}{\sin(2\,p\alpha)}.$$

Or, equivalently,

$$h(r,\theta) = \mathscr{M}^{-1}\left\{\frac{\sin p\theta}{\sin 2\,p\alpha}\right\} = \left(\frac{1}{2\,\alpha}\right)\frac{r^n \sin n\theta}{(1 + 2\,r^n \cos n\theta + r^{2n})}, \tag{8.4.12}$$

where

$$n = \frac{\pi}{2\alpha} \quad \text{or,} \quad 2\alpha = \frac{\pi}{n}.$$

Application of the inverse Mellin transform to (8.4.11) gives

$$\phi(r,\theta) = \mathscr{M}^{-1}\left\{\tilde{f}(p)\tilde{h}(p,\alpha+\theta)\right\} + \mathscr{M}^{-1}\left\{\tilde{g}(p)\tilde{h}(p,\alpha-\theta)\right\},$$

which is, by the convolution property (8.3.18),

$$\begin{aligned}\phi(r,\theta) = \frac{r^n \cos n\theta}{2\alpha}\Bigg[&\int_0^\infty \frac{\xi^{n-1} f(\xi)d\xi}{\xi^{2n} - 2(r\xi)^n \sin n\theta + r^{2n}} \\ &+ \int_0^\infty \frac{\xi^{n-1} g(\xi)d\xi}{\xi^{2n} + 2(r\xi)^n \sin n\theta + r^{2n}}\Bigg], \quad |\alpha| < \frac{\pi}{2n}.\end{aligned} \tag{8.4.13}$$

This is the formal solution of the problem.

In particular, when $f(r) = g(r)$, solution (8.4.11) becomes

$$\tilde{\phi}(p,\theta) = \tilde{f}(p)\,\frac{\cos p\theta}{\cos p\alpha} = \tilde{f}(p)\tilde{h}(p,\theta), \tag{8.4.14}$$

where

$$\tilde{h}(p,\theta) = \frac{\cos p\theta}{\cos p\alpha} = \mathscr{M}\{h(r,\theta)\}.$$

Application of the inverse Mellin transform to (8.4.14) combined with the convolution property (8.3.18) yields the solution

$$\phi(r,\theta) = \int_0^\infty f(\xi)\, h\left(\frac{r}{\xi},\theta\right) \frac{d\xi}{\xi}, \tag{8.4.15}$$

where

$$h(r,\theta) = \mathscr{M}^{-1}\left\{\frac{\cos p\theta}{\cos p\alpha}\right\} = \left(\frac{r^n}{\alpha}\right) \frac{(1+r^{2n})\cos(n\theta)}{(1+2r^{2n}\cos 2n\theta + r^{2n})}, \tag{8.4.16}$$

and $n = \dfrac{\pi}{2\alpha}$. $\square$

Some applications of the Mellin transform to boundary value problems are given by Sneddon (1951) and Tranter (1966).

Example 8.4.3 Solve the integral equation

$$\int_0^\infty f(\xi)\, k(x\xi) d\xi = g(x), \qquad x > 0. \tag{8.4.17}$$

Application of the Mellin transform with respect to x to equation (8.4.17) combined with (8.3.19) gives

$$\tilde{f}(1-p)\tilde{k}(p) = \tilde{g}(p),$$

which gives, replacing p by $1-p$,

$$\tilde{f}(p) = \tilde{g}(1-p)\tilde{h}(p),$$

where

$$\tilde{h}(p) = \frac{1}{\tilde{k}(1-p)}.$$

The inverse Mellin transform combined with (8.3.19) leads to the solution

$$f(x) = \mathscr{M}^{-1}\left\{\tilde{g}(1-p)\tilde{h}(p)\right\} = \int_0^\infty g(\xi) h(x\xi) d\xi, \tag{8.4.18}$$

provided $h(x) = \mathscr{M}^{-1}\left\{\tilde{h}(p)\right\}$ exists. Thus, the problem is formally solved.

If, in particular, $\tilde{h}(p) = \tilde{k}(p)$, then the solution of (8.4.18) becomes

$$f(x) = \int_0^\infty g(\xi)\, k(x\xi) d\xi, \tag{8.4.19}$$

provided $\tilde{k}(p)\tilde{k}(1-p)=1$. □

Example 8.4.4 Solve the integral equation

$$\int_0^\infty f(\xi)\, g\left(\frac{x}{\xi}\right) \frac{d\xi}{\xi} = h(x), \tag{8.4.20}$$

where $f(x)$ is unknown and $g(x)$ and $h(x)$ are given functions.

Applications of the Mellin transform with respect to x gives

$$\tilde{f}(p) = \tilde{h}(p)\tilde{k}(p), \qquad \tilde{k}(p) = \frac{1}{\tilde{g}(p)}.$$

Inversion, by the convolution property (8.3.18), gives the solution

$$f(x) = \mathscr{M}^{-1}\left\{\tilde{h}(p)\tilde{k}(p)\right\} = \int_0^\infty h(\xi)\, k\left(\frac{x}{\xi}\right) \frac{d\xi}{\xi}. \tag{8.4.21}$$

□

8.5 Mellin Transforms of the Weyl Fractional Integral and the Weyl Fractional Derivative

DEFINITION 8.5.1 *The Mellin transform of the Weyl fractional integral of $f(x)$ is defined by*

$$W^{-\alpha}[f(x)] = \frac{1}{\Gamma(\alpha)} \int_x^\infty (t-x)^{\alpha-1} f(t)dt, \quad 0 < \mathrm{Re}\,\alpha < 1, \quad x > 0. \tag{8.5.1}$$

Often ${}_xW_\infty^{-\alpha}$ is used instead of $W^{-\alpha}$ to indicate the limits to integration. Result (8.5.1) can be interpreted as the Weyl transform of $f(t)$, defined by

$$W^{-\alpha}[f(t)] = F(x,\alpha) = \frac{1}{\Gamma(\alpha)} \int_x^\infty (t-x)^{\alpha-1} f(t)dt. \tag{8.5.2}$$

We first give some simple examples of the Weyl transform.

If $f(t) = \exp(-at)$, Re $a > 0$, then the Weyl transform of $f(t)$ is given by

$$W^{-\alpha}[\exp(-at)] = \frac{1}{\Gamma(\alpha)} \int_x^\infty (t-x)^{\alpha-1} \exp(-at)dt,$$

which is, by the change of variable $t-x=y$,

$$=\frac{e^{-ax}}{\Gamma(\alpha)}\int_0^\infty y^{\alpha-1}\exp(-ay)dy$$

which is, by letting $ay=t$,

$$W^{-\alpha}[f(t)]=\frac{e^{-ax}}{a^\alpha}\frac{1}{\Gamma(\alpha)}\int_0^\infty t^{\alpha-1}e^{-t}dt=\frac{e^{-ax}}{a^\alpha}. \tag{8.5.3}$$

Similarly, it can be shown that

$$W^{-\alpha}[t^{-\mu}]=\frac{\Gamma(\mu-\alpha)}{\Gamma(\mu)}x^{\alpha-\mu}, \qquad 0<\operatorname{Re}\alpha<\operatorname{Re}\mu. \tag{8.5.4}$$

Making reference to Gradshteyn and Ryzhik (2000, p. 424), we obtain

$$W^{-\alpha}[\sin at] = a^{-\alpha}\sin\left(ax+\frac{\pi\alpha}{2}\right), \tag{8.5.5}$$
$$W^{-\alpha}[\cos at] = a^{-\alpha}\cos\left(ax+\frac{\pi\alpha}{2}\right), \tag{8.5.6}$$

where $0<\operatorname{Re}\alpha<1$ and $a>0$.

It can be shown that, for any two positive numbers α and β, the Weyl fractional integral satisfies the laws of exponents

$$W^{-\alpha}[W^{-\beta}f(x)]=W^{-(\beta+\alpha)}[f(x)]=W^{-\beta}[W^{-\alpha}f(x)]. \tag{8.5.7}$$

Invoking a change of variable $t-x=y$ in (8.5.1), we obtain

$$W^{-\alpha}[f(x)]=\frac{1}{\Gamma(\alpha)}\int_0^\infty y^{\alpha-1}f(x+y)dy. \tag{8.5.8}$$

We next differentiate (8.5.8) to obtain, $D=\frac{d}{dx}$,

$$\begin{aligned} D[W^{-\alpha}f(x)] &= \frac{1}{\Gamma(\alpha)}\int_0^\infty t^{\alpha-1}\frac{\partial}{\partial x}f(x+t)dt \\ &= \frac{1}{\Gamma(\alpha)}\int_0^\infty t^{\alpha-1}Df(x+t)dt \\ &= W^{-\alpha}[Df(x)]. \end{aligned} \tag{8.5.9}$$

A similar argument leads to a more general result

$$D^n[W^{-\alpha}f(x)]=W^{-\alpha}[D^nf(x)], \tag{8.5.10}$$

where n is a positive integer.

Or, symbolically,

$$D^n W^{-\alpha} = W^{-\alpha} D^n. \tag{8.5.11}$$

We now calculate the Mellin transform of the Weyl fractional integral by putting $h(t) = t^\alpha f(t)$ and $g\left(\frac{x}{t}\right) = \frac{1}{\Gamma(\alpha)}\left(1 - \frac{x}{t}\right)^{\alpha-1} H\left(1 - \frac{x}{t}\right)$, where $H\left(1 - \frac{x}{t}\right)$ is the Heaviside unit step function so that (8.5.1) becomes

$$F(x,\alpha) = \int_0^\infty h(t)\, g\left(\frac{x}{t}\right)\frac{dt}{t}, \tag{8.5.12}$$

which is, by the convolution property (8.3.18),

$$\tilde{F}(p,\alpha) = \tilde{h}(p)\tilde{g}(p),$$

where

$$\tilde{h}(p) = \mathscr{M}\{x^\alpha f(x)\} = \tilde{f}(p+\alpha),$$

and

$$\begin{aligned}\tilde{g}(p) &= \mathscr{M}\left\{\frac{1}{\Gamma(\alpha)}(1-x)^{\alpha-1} H(1-x)\right\} \\ &= \frac{1}{\Gamma(\alpha)}\int_0^1 x^{p-1}(1-x)^{\alpha-1}dx = \frac{B(p,\alpha)}{\Gamma(\alpha)} = \frac{\Gamma(p)}{\Gamma(p+\alpha)}.\end{aligned}$$

Consequently,

$$\tilde{F}(p,\alpha) = \mathscr{M}\left[W^{-\alpha} f(x), p\right] = \frac{\Gamma(p)}{\Gamma(p+\alpha)}\,\tilde{f}(p+\alpha). \tag{8.5.13}$$

It is important to note that this result is an obvious extension of result 7(b) in Exercise 8.8

DEFINITION 8.5.2 *If β is a positive number and n is the smallest integer greater than β such that $n - \beta = \alpha > 0$, the Weyl fractional derivative of a function $f(x)$ is defined by*

$$\begin{aligned}W^\beta[f(x)] &= E^n\, W^{-(n-\beta)}[f(x)] \\ &= \frac{(-1)^n}{\Gamma(n-\beta)}\frac{d^n}{dx^n}\int_x^\infty (t-x)^{n-\beta-1} f(t)dt,\end{aligned} \tag{8.5.14}$$

where $E = -D$.

Or, symbolically,

$$W^\beta = E^n W^{-\alpha} = E^n W^{-(n-\beta)}. \tag{8.5.15}$$

It can be shown that, for any β,

$$W^{-\beta}W^{\beta} = I = W^{\beta}W^{-\beta}. \tag{8.5.16}$$

And, for any β and γ, the Weyl fractional derivative satisfies the laws of exponents

$$W^{\beta}[W^{\gamma}f(x)] = W^{\beta+\gamma}[f(x)] = W^{\gamma}[W^{\beta}f(x)]. \tag{8.5.17}$$

We now calculate the Weyl fractional derivative of some elementary functions.

If $f(x) = \exp(-ax)$, $a > 0$, then the definition (8.5.14) gives

$$W^{\beta}e^{-ax} = E^{n}[W^{-(n-\beta)}e^{-ax}]. \tag{8.5.18}$$

Writing $n - \beta = \alpha > 0$ and using (8.5.3) yields

$$\begin{aligned} W^{\beta}e^{-ax} &= E^{n}[W^{-\alpha}e^{-ax}] = E^{n}[a^{-\alpha}e^{-ax}] \\ &= a^{-\alpha}(a^{n}e^{-ax}) = a^{\beta}e^{-ax}. \end{aligned} \tag{8.5.19}$$

Replacing β by $-\alpha$ in (8.5.19) leads to result (8.5.3) as expected.

Similarly, we obtain

$$W^{\beta}x^{-\mu} = \frac{\Gamma(\beta+\mu)}{\Gamma(\mu)}\, x^{-(\beta+\mu)}. \tag{8.5.20}$$

It is easy to see that

$$W^{\beta}(\cos ax) = E[W^{-(1-\beta)}\cos ax],$$

which is, by (8.5.6),

$$= a^{\beta}\cos\left(ax - \frac{1}{2}\pi\beta\right). \tag{8.5.21}$$

Similarly,

$$W^{\beta}(\sin ax) = a^{\beta}\sin\left(ax - \frac{1}{2}\pi\beta\right), \tag{8.5.22}$$

provided α and β lie between 0 and 1.

If β is replaced by $-\alpha$, results (8.5.20)–(8.5.22) reduce to (8.5.4)–(8.5.6), respectively.

Finally, we calculate the Mellin transform of the Weyl fractional derivative with the help of (8.3.9) and find

$$\begin{aligned} \mathscr{M}[W^{\beta}f(x)] &= \mathscr{M}[E^{n}W^{-(n-\beta)}f(x)] = (-1)^{n}\mathscr{M}[D^{n}W^{-(n-\beta)}f(x)] \\ &= \frac{\Gamma(p)}{\Gamma(p-n)}\mathscr{M}[W^{-(n-\beta)}f(x),\, p-n], \end{aligned}$$

which is, by result (8.5.13),

$$= \frac{\Gamma(p)}{\Gamma(p-n)} \cdot \frac{\Gamma(p-n)}{\Gamma(p-\beta)} \tilde{f}(p-\beta)$$

$$= \frac{\Gamma(p)}{\Gamma(p-\beta)} \mathscr{M}[f(x), p-\beta]$$

$$= \frac{\Gamma(p)}{\Gamma(p-\beta)} \tilde{f}(p-\beta). \tag{8.5.23}$$

Example 8.5.1 (*The Fourier Transform of the Weyl Fractional Integral*).

$$\mathscr{F}\{W^{-\alpha} f(x)\} = \exp\left(-\frac{\pi i \alpha}{2}\right) k^{-\alpha} \mathscr{F}\{f(x)\}. \tag{8.5.24}$$

We have, by definition,

$$\mathscr{F}\{W^{-\alpha} f(x)\} = \frac{1}{\sqrt{2\pi}} \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{\infty} e^{-ikx} dx \int_{x}^{\infty} (t-x)^{\alpha-1} f(t) dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) dt \cdot \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{t} \exp(-ikx)(t-x)^{\alpha-1} dx.$$

Thus,

$$\mathscr{F}\{W^{-\alpha} f(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikt} f(t) dt \cdot \frac{1}{\Gamma(\alpha)} \int_{0}^{\infty} e^{ik\tau} \tau^{\alpha-1} d\tau, \quad (t-x=\tau)$$

$$= \mathscr{F}\{f(x)\} \frac{1}{\Gamma(\alpha)} \mathscr{M}\{e^{ik\tau}\}$$

$$= \exp\left(-\frac{\pi i \alpha}{2}\right) k^{-\alpha} \mathscr{F}\{f(x)\}.$$

In the limit as $\alpha \to 0$

$$\lim_{\alpha \to 0} \mathscr{F}\{W^{-\alpha} f(x)\} = \mathscr{F}\{f(x)\}.$$

This implies that

$$W^{0}\{f(x)\} = f(x).$$

We conclude this section by proving a general property of the Riemann-Liouville fractional integral operator $D^{-\alpha}$, and the Weyl fractional integral operator $W^{-\alpha}$. It follows from the definition (6.2.1) that $D^{-\alpha} f(t)$ can be expressed as the convolution

$$D^{-\alpha} f(x) = g_{\alpha}(t) * f(t), \tag{8.5.25}$$

where

$$g_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}, \; t > 0.$$

Similarly, $W^{-\alpha} f(x)$ can also be written in terms of the convolution

$$W^{-\alpha} f(x) = g_\alpha(-x) * f(x). \tag{8.5.26}$$

Then, under suitable conditions,

$$\mathscr{M}\left[D^{-\alpha} f(x)\right] = \frac{\Gamma(1-\alpha-p)}{\Gamma(1-p)}\,\tilde{f}(p+\alpha), \tag{8.5.27}$$

$$\mathscr{M}\left[W^{-\alpha} f(x)\right] = \frac{\Gamma(p)}{\Gamma(\alpha+p)}\,\tilde{f}(p+\alpha). \tag{8.5.28}$$

Finally, a formal computation gives

$$\begin{aligned}
\int_0^\infty \{D^{-\alpha} f(x)\} g(x)dx &= \frac{1}{\Gamma(\alpha)} \int_0^\infty g(x)dx \int_0^x (x-t)^{\alpha-1} f(t)dt \\
&= \int_0^\infty f(t)dt \cdot \frac{1}{\Gamma(\alpha)} \int_t^\infty (x-t)^{\alpha-1} g(x)dx \\
&= \int_0^\infty f(t)\left[W^{-\alpha} g(t)\right] dt,
\end{aligned}$$

which is, using the inner product notation,

$$\langle D^{-\alpha} f, \, g\rangle = \langle f, \, W^{-\alpha} g\rangle. \tag{8.5.29}$$

This show that $D^{-\alpha}$ and $W^{-\alpha}$ behave like adjoint operators. Obviously, this result can be used to define fractional integrals of distributions. This result is taken from Debnath and Grum (1988). $\square$

8.6 Application of Mellin Transforms to Summation of Series

In this section we discuss a method of summation of series that is particularly associated with the work of Macfarlane (1949).

THEOREM 8.6.1 If $\mathscr{M}\{f(x)\} = \tilde{f}(p)$, then

$$\sum_{n=0}^{\infty} f(n+a) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \tilde{f}(p)\, \xi(p,a) dp, \tag{8.6.1}$$

where $\xi(p,a)$ is the *Hurwitz zeta function* defined by

$$\xi(p,a) = \sum_{n=0}^{\infty} \frac{1}{(n+a)^p}, \qquad 0 \le a \le 1, \ \mathrm{Re}(p) > 1. \tag{8.6.2}$$

PROOF If follows from the inverse Mellin transform that

$$f(n+a) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \tilde{f}(p)(n+a)^{-p}\, dp. \tag{8.6.3}$$

Summing this over all n gives

$$\sum_{n=0}^{\infty} f(n+a) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \tilde{f}(p)\, \xi(p,a)\, dp.$$

This completes the proof.

Similarly, the scaling property (8.3.1) gives

$$f(nx) = \mathscr{M}^{-1}\{n^{-p}\, \tilde{f}(p)\} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} x^{-p}\, n^{-p} \tilde{f}(p) dp.$$

Thus,

$$\sum_{n=1}^{\infty} f(nx) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} x^{-p} \tilde{f}(p)\, \zeta(p) dp = \mathscr{M}^{-1}\{\tilde{f}(p)\, \zeta(p)\}, \tag{8.6.4}$$

where $\zeta(p) = \sum_{n=1}^{\infty} n^{-p}$ is the *Riemann zeta function.*

When $x = 1$, result (8.6.4) reduces to

$$\sum_{n=1}^{\infty} f(n) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \tilde{f}(p)\, \zeta(p) dp. \tag{8.6.5}$$

This can be obtained from (8.6.1) when $a = 0$. ∎

Example 8.6.1 Show that

$$\sum_{n=1}^{\infty} (-1)^{n-1} n^{-p} = (1 - 2^{1-p})\, \zeta(p). \tag{8.6.6}$$

Using Example 8.2.1(a), we can write the left-hand side of (8.6.6) multiplied by t^n as

$$
\begin{aligned}
\sum_{n=1}^{\infty}(-1)^{n-1}n^{-p}t^n &= \sum_{n=1}^{\infty}(-1)^{n-1}t^n \cdot \frac{1}{\Gamma(p)}\int_0^{\infty} x^{p-1}e^{-nx}dx \\
&= \frac{1}{\Gamma(p)}\int_0^{\infty} x^{p-1}dx \sum_{n=1}^{\infty}(-1)^{n-1}t^{nx}e^{-nx} \\
&= \frac{1}{\Gamma(p)}\int_0^{\infty} x^{p-1}\cdot\frac{te^{-x}}{1+te^{-x}}\cdot dx \\
&= \frac{1}{\Gamma(p)}\int_0^{\infty} x^{p-1}\cdot\frac{t}{e^x+t}\,dx.
\end{aligned}
$$

In the limit as $t \to 1$, the above result gives

$$
\begin{aligned}
\sum_{n=1}^{\infty}(-1)^{n-1}n^{-p} &= \frac{1}{\Gamma(p)}\int_0^{\infty} x^{p-1}\frac{1}{e^x+1}\,dx \\
&= \frac{1}{\Gamma(p)}\mathscr{M}\left\{\frac{1}{e^x+1}\right\} = \left(1-2^{1-p}\right)\zeta(p),
\end{aligned}
$$

in which result (8.2.11) is used. $\square$

Example 8.6.2 Show that

$$\sum_{n=1}^{\infty}\left(\frac{\sin an}{n}\right) = \frac{1}{2}(\pi - a), \qquad 0 < a < 2\pi. \tag{8.6.7}$$

The Mellin transform of $f(x) = \left(\dfrac{\sin ax}{x}\right)$ gives

$$
\begin{aligned}
\mathscr{M}\left[\frac{\sin ax}{x}\right] &= \int_0^{\infty} x^{p-2}\sin ax\,dx \\
&= \mathscr{F}_s\left\{\sqrt{\frac{\pi}{2}}\,x^{p-2}\right\} \\
&= -\frac{\Gamma(p-1)}{a^{p-1}}\cos\left(\frac{\pi p}{2}\right).
\end{aligned}
$$

Substituting this result into (8.6.5) gives

$$\sum_{n=1}^{\infty}\left(\frac{\sin an}{n}\right) = -\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\Gamma(p-1)}{a^{p-1}}\,\zeta(p)\,\cos\left(\frac{\pi p}{2}\right)\,dp. \tag{8.6.8}$$

We next use the well-known functional equation for the zeta function

$$(2\pi)^p\,\zeta(1-p)=2\Gamma(p)\,\zeta(p)\cos\left(\frac{\pi p}{2}\right) \tag{8.6.9}$$

in the integrand of (8.6.8) to obtain

$$\sum_{n=1}^{\infty}\left(\frac{\sin an}{n}\right)=-\frac{a}{2}\cdot\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left(\frac{2\pi}{a}\right)^p\frac{\zeta(1-p)}{p-1}\,dp.$$

The integral has two simple poles at $p=0$ and $p=1$ with residues 1 and $-\pi/a$, respectively, and the complex integral is evaluated by calculating the residues at these poles. Thus, the sum of the series is

$$\sum_{n=1}^{\infty}\left(\frac{\sin an}{n}\right)=\frac{1}{2}(\pi-a).$$

□

8.7 Generalized Mellin Transforms

In order to extend the applicability of the classical Mellin transform, Naylor (1963) generalized the method of Mellin integral transforms. This generalized Mellin transform is useful for finding solutions of boundary value problems in regions bounded by the natural coordinate surfaces of a spherical or cylindrical coordinate system. They can be used to solve boundary value problems in finite regions or in infinite regions bounded internally.

The *generalized Mellin transform* of a function $f(r)$ defined in $a<r<\infty$ is introduced by the integral

$$\mathscr{M}_-\{f(r)\}=F_-(p)=\int_a^{\infty}\left(r^{p-1}-\frac{a^{2p}}{r^{p+1}}\right)f(r)\,dr. \tag{8.7.1}$$

The inverse transform is given by

$$\mathscr{M}_-^{-1}\{F_-(p)\}=f(r)=\frac{1}{2\pi i}\int_L r^{-p}\,F(p)\,dp,\quad r>a, \tag{8.7.2}$$

where L is the line $\operatorname{Re} p=c$, and $F(p)$ is analytic in the strip $|\operatorname{Re}(p)|=|c|<\gamma$.

By integrating by parts, we can show that

$$\mathscr{M}_-\left[r^2\,\frac{\partial^2 f}{\partial r^2}+r\,\frac{\partial f}{\partial r}\right]=p^2\,F_-(p)+2\,p\,a^p f(a), \tag{8.7.3}$$

provided $f(r)$ is appropriately behaved at infinity. More precisely,

$$\lim_{r\to\infty}\left[(r^p - a^{2p}\,r^{-p})rf_r - p(r^p + a^{2p}r^{-p})f\right] = 0. \tag{8.7.4}$$

Obviously, this generalized transform seems to be very useful for finding the solution of boundary value problems in which $f(r)$ is prescribed on the internal boundary at $r=a$.

On the other hand, if the derivative of $f(r)$ is prescribed at $r=a$, it is convenient to define the associated integral transform by

$$\mathscr{M}_+[f(r)] = F_+(p) = \int_a^\infty \left(r^{p-1} + \frac{a^{2p}}{r^{p+1}}\right) f(r)\,dr, \quad |\mathrm{Re}(p)| < r, \tag{8.7.5}$$

and its inverse given by

$$\mathscr{M}_+^{-1}[f(p)] = f(r) = \frac{1}{2\pi i}\int_L r^{-p}\,F_+(p)dp, \quad r > a. \tag{8.7.6}$$

In this case, we can show by integration by parts that

$$\mathscr{M}_+\left[r^2\,\frac{\partial^2 f}{\partial r^2} + r\,\frac{\partial f}{\partial r}\right] = p^2 F_+(p) - 2\,a^{p+1} f'(a), \tag{8.7.7}$$

where $f'(r)$ exists at $r=a$.

***THEOREM 8.7.1** (Convolution).*

If $\mathscr{M}_+\{f(r)\} = F_+(p)$, and $\mathscr{M}_+\{g(r)\} = G_+(p)$, then

$$\mathscr{M}_+\{f(r)\,g(r)\} = \frac{1}{2\pi i}\int_L F_+(\xi)G_+(p-\xi)\,d\xi. \tag{8.7.8}$$

Or, equivalently,

$$f(r)g(r) = \mathscr{M}_+^{-1}\left[\frac{1}{2\pi i}\int_L F_+(\xi)\,G_+(p-\xi)d\xi\right]. \tag{8.7.9}$$

PROOF We assume that $F_+(p)$ and $G_+(p)$ are analytic in some strip

$|\mathrm{Re}(p)| < \gamma$. Then

$$\begin{aligned}
\mathscr{M}_+\{f(r)\,g(r)\} &= \int_a^\infty \left(r^{p-1} + \frac{a^{2p}}{r^{p+1}}\right) f(r)g(r)dr \\
&= \int_a^\infty r^{p-1} f(r)g(r)dr + \int_a^\infty \frac{a^{2p}}{r^{p+1}} f(r)g(r)dr \qquad (8.7.10) \\
&= \frac{1}{2\pi i}\int_L F_+(\xi)d\xi \int_a^\infty r^{p-\xi-1} g(r)dr \\
&\qquad + \frac{1}{2\pi}\int_a^\infty \frac{a^{2p}}{r^{p+1}}\, g(r)dr \int_L r^{-\xi} F_+(\xi)\, d\xi. \qquad (8.7.11)
\end{aligned}$$

Replacing ξ by $-\xi$ in the first integral term and using $F_+(\xi) = a^{2\xi} F_+(-\xi)$, which follows from the definition (8.7.5), we obtain

$$\int_L r^{-\xi}\, F_+(\xi)d\xi = \int_L r^{\xi}\, a^{-2\xi}\, F_+(\xi)d\xi. \tag{8.7.12}$$

The path of integration L, $\mathrm{Re}(\xi) = c$, becomes $\mathrm{Re}(\xi) = -c$, but these paths can be reconciled if $F(\xi)$ tends to zero for large $\mathrm{Im}(\xi)$.

In view of (8.7.11), we have rewritten

$$\int_a^\infty \frac{a^{2p}}{r^{p+1}} f(r)\, g(r)dr = \frac{1}{2\pi i}\int_L F_+(\xi)d\xi \int_a^\infty \frac{a^{2p-2\xi}}{r^{p-\xi+1}}\, g(r)\, dr. \tag{8.7.13}$$

This result is used to rewrite (8.7.10) as

$$\begin{aligned}
\mathscr{M}_+\{f(r)g(r)\} &= \int_a^\infty \left(r^{p-1} + \frac{a^{2p}}{r^{p+1}}\right) f(r)g(r)dr \\
&= \int_a^\infty r^{p-1} f(r)\, g(r)\, dr + \int_a^\infty \frac{a^{2p}}{r^{p+1}} f(r)\, g(r)\, dr \\
&= \frac{1}{2\pi i}\int_L F_+(\xi)\, d\xi \int_a^\infty r^{p-\xi-1} g(r)\, dr \\
&\qquad + \frac{1}{2\pi i}\int_L F_+(\xi)d\xi \int_a^\infty \frac{a^{2p-2\xi}}{r^{p-\xi+1}}\, g(r)\, dr \\
&= \frac{1}{2\pi i}\int_L F_+(\xi)\, G_+(p-\xi)\, d\xi.
\end{aligned}$$

This completes the proof. ∎

If the range of integration is finite, then we define the *generalized finite Mellin transform* by

$$\mathscr{M}_{-}^{a}\{f(r)\} = F_{-}^{a}(p) = \int_{0}^{a} \left(r^{p-1} - \frac{a^{2p}}{r^{p+1}}\right) f(r)dr, \tag{8.7.14}$$

where $\operatorname{Re} p < \gamma$.

The corresponding inverse transform is given by

$$f(r) = -\frac{1}{2\pi i}\int_{L} \left(\frac{r}{a^2}\right)^{p} F_{-}^{a}(p)dp, \quad 0 < r < a,$$

which is, by replacing p by $-p$ and using $F_{-}^{a}(-p) = -a^{-2p}\,F_{-}^{a}(p)$,

$$= \frac{1}{2\pi i}\int_{L} r^{-p}\,F_{-}^{a}(p)dp, \quad 0 < r < a, \tag{8.7.15}$$

where the path L is Re $p = -c$ with $|c| < \gamma$.

It is easy to verify the result

$$\begin{aligned}\mathscr{M}_{-}^{a}\{r^2 f_{rr} + r f_{-r}\} &= \int_{0}^{a} \left(r^{p-1} - \frac{a^{2p}}{r^{p+1}}\right) \{r^2 f_{rr} + r f_{r}\}dr \\ &= p^2\,F_{-}^{a}(p) - 2\,p\,a^{p}\,f(a).\end{aligned} \tag{8.7.16}$$

This is a useful result for applications.

Similarly, we define the generalized finite Mellin transform-pair by

$$\mathscr{M}_{+}^{a}\{f(r)\} = F_{+}^{a}(p) = \int_{0}^{a} \left(r^{p-1} + \frac{a^{2p}}{r^{p+1}}\right) f(r)\,dr, \tag{8.7.17}$$

$$f(r) = \left(\mathscr{M}_{+}^{a}\right)^{-1}\left[F_{+}^{a}(p)\right] = \frac{1}{2\pi i}\int_{L} r^{-p} F_{+}^{a}(p)\,dp, \tag{8.7.18}$$

where $|\operatorname{Re} p| < \gamma$.

For this finite transform, we can also prove

$$\begin{aligned}\mathscr{M}_{+}^{a}\left[r^2 f_{rr} + r\ f_{r}\right] &= \int_{0}^{a} \left(r^{p-1} + \frac{a^{2p}}{r^{p+1}}\right)\left(r^2 f_{rr} + r\ f_{r}\right)dr \\ &= p^2 F_{+}^{a}(p) + 2\,a^{p-1}\ f'(a).\end{aligned} \tag{8.7.19}$$

This result also seems to be useful for applications. The reader is referred to Naylor (1963) for applications of the above results to boundary value problems.

8.8 Exercises

1. Find the Mellin transform of each of the following functions:

(a) $f(x) = H(a-x),\ a > 0,$
(b) $f(x) = x^m e^{-nx}, \quad m, n > 0,$
(c) $f(x) = \frac{1}{1+x^2},$
(d) $f(x) = J_0^2(x),$
(e) $f(x) = x^z H(x - x_0),$
(f) $f(x) = [H(x-x_0) - H(x)]x^z,$
(g) $f(x) = Ei(x),$
(h) $f(x) = e^x Ei(x),$
(i) $f(x) = \exp\left(-ax^2\right),\ a > 0,$
(j) $f(x) = erfc(x),$
(k) $f(x) = Ci(x),$
(l) $f(x) = (1 + x^a)^{-b},$
(m) $f(x) = (1+x)^{-1},$

where the exponential integral is defined by

$$Ei(x) = \int_x^\infty t^{-1} e^{-t}\, dt = \int_1^\infty \xi^{-1} e^{-\xi x}\, d\xi.$$

2. Derive the Mellin transform-pairs from the bilateral Laplace transform and its inverse given by

$$\bar{g}(p) = \int_{-\infty}^{\infty} e^{-pt} g(t) dt, \qquad g(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{pt}\, \bar{g}(p) dp.$$

3. Show that

$$\mathscr{M}\left[\frac{1}{e^x + e^{-x}}\right] = \Gamma(p)\, L(p),$$

where $L(p) = \dfrac{1}{1^p} - \dfrac{1}{3^p} + \dfrac{1}{5^p} - \cdots$ is the *Dirichlet L-function.*

4. Show that

$$\mathscr{M}\left\{\frac{1}{(1+ax)^n}\right\} = \frac{\Gamma(p)\Gamma(n-p)}{a^p\,\Gamma(n)}.$$

5. Show that

$$\mathscr{M}\left\{x^{-n} J_n(ax)\right\} = \frac{1}{2}\left(\frac{a}{2}\right)^{n-p} \frac{\Gamma\left(\frac{p}{2}\right)}{\Gamma\left(n - \frac{p}{2} + 1\right)}, \quad a > 0,\ n > -\frac{1}{2}.$$

6. Show that

(a) $\mathscr{M}^{-1}\left[\cos\left(\frac{\pi p}{2}\right)\Gamma(p)\,\tilde{f}(1-p)\right]=\mathscr{F}_c\left\{\sqrt{\frac{\pi}{2}}\,f(x)\right\},$

(b) $\mathscr{M}^{-1}\left[\sin\left(\frac{\pi p}{2}\right)\Gamma(p)\,\tilde{f}(1-p)\right]=\mathscr{F}_s\left\{\sqrt{\frac{\pi}{2}}\,f(x)\right\}.$

7. If $I_n^\infty f(x)$ denotes the nth repeated integral of $f(x)$ defined by

$$I_n^\infty f(x)=\int_x^\infty I_{n-1}^\infty f(t)dt,$$

show that

(a) $\mathscr{M}\left[\int_x^\infty f(t)dt,\,p\right]=\frac{1}{p}\,\tilde{f}(p+1),$

(b) $\mathscr{M}\left[I_n^\infty\, f(x)\right]=\frac{\Gamma(p)}{\Gamma(p+n)}\,\tilde{f}(p+n).$

8. Show that the integral equation

$$f(x)=h(x)+\int_0^\infty g(x\xi)\ f(\xi)\ d\xi$$

has the formal solution

$$f(x)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left[\frac{\tilde{h}(p)+\tilde{g}(p)\,\tilde{h}(1-p)}{1-\tilde{g}(p)\,\tilde{g}(1-p)}\right]x^{-p}\,dp.$$

9. Find the solution of the Laplace integral equation

$$\int_0^\infty e^{-x\xi}\ f(\xi)\ d\xi=\frac{1}{(1+x)^n}.$$

10. Show that the integral equation

$$f(x)=h(x)+\int_0^\infty f(\xi)\ g\left(\frac{x}{\xi}\right)\ \frac{d\xi}{\xi}$$

has the formal solution

$$f(x)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{x^{-p}\,\tilde{h}(p)}{1-\tilde{g}(p)}\,dp.$$

11. Show that the solution of the integral equation

$$f(x) = e^{-ax} + \int_0^\infty \exp\left(-\frac{x}{\xi}\right) f(\xi)\,\frac{d\xi}{\xi}$$

is

$$f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} (ax)^{-p}\left\{\frac{\Gamma(p)}{1-\Gamma(p)}\right\} dp.$$

12. Assuming (see Harrington, 1967)

$$\mathscr{M}\left[f(re^{i\theta})\right] = \int_0^\infty r^{p-1} f(re^{i\theta})\, dr, \quad p \text{ is real,}$$

and putting $re^{i\theta} = \xi$, $\mathscr{M}\{f(\xi)\} = F(p)$ show that

(a) $\mathscr{M}[f(re^{i\theta});\, r \to p] = \exp(-ip\theta)\, F(p)$.

Hence, deduce

(b) $\mathscr{M}^{-1}\{F(p)\,\cos\, p\theta\} = \mathrm{Re}[f(re^{i\theta})]$,
(c) $\mathscr{M}^{-1}\{F(p)\,\sin\, p\theta\} = -\mathrm{Im}[f(re^{i\theta})]$.

13. (a) If $\mathscr{M}[\exp(-r)] = \Gamma(p)$, show that

$$\mathscr{M}\left[\exp(-re^{i\theta})\right] = \Gamma(p)\, e^{-ip\theta},$$

(b) If $\mathscr{M}[\log(1+r)] = \dfrac{\pi}{p\sin\pi p}$, then show that

$$\mathscr{M}\left[\mathrm{Re}\,\log(1+re^{i\theta})\right] = \frac{\pi\cos\, p\theta}{p\sin\pi p}.$$

14. Use $\mathscr{M}^{-1}\left\{\dfrac{\pi}{\sin p\pi}\right\} = \dfrac{1}{1+x} = f(x)$, and Exercises 12(b) and 12(c), respectively, to show that

(a) $\mathscr{M}^{-1}\left\{\dfrac{\pi\cos p\theta}{\sin p\pi};\, p \to r\right\} = \dfrac{1+r\cos\theta}{1+2r\cos\theta+r^2}$,

(b) $\mathscr{M}^{-1}\left\{\dfrac{\pi\sin p\theta}{\sin p\pi};\, p \to r\right\} = \dfrac{r\sin\theta}{1+2r\cos\theta+r^2}$.

15. Find the inverse Mellin transforms of

(a) $\Gamma(p)\cos p\theta$, where $-\dfrac{\pi}{2} < \theta < \dfrac{\pi}{2}$, (b) $\Gamma(p)\sin p\theta$.

16. Obtain the solution of Example 8.4.2 with the boundary data

    (a) $\phi(r,\,\alpha) = \phi(r,\,-\alpha) = H(a-r)$.

    (b) Solve equation (8.4.5) in $0 < r < \infty$, $0 < \theta < \alpha$ with the boundary conditions $\phi(r,\,0) = 0$ and $\phi(r,\,\alpha) = f(r)$.

17. Show that

    (a) $\displaystyle\sum_{n=1}^{\infty} \frac{\cos kn}{n^2} = \left[\frac{k^2}{4} - \frac{\pi k}{2} + \frac{\pi^2}{6}\right]$, and (b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

18. If $f(x) = \sum_{n=1}^{\infty} a_n e^{-nx}$, show that

$$\mathscr{M}\{f(x)\} = \tilde{f}(p) = \Gamma(p)\, g(p),$$

where $g(p) = \displaystyle\sum_{n=1}^{\infty} a_n\, n^{-p}$ is the Dirichlet series.
If $a_n = 1$ for all n, derive

$$\tilde{f}(p) = \Gamma(p)\,\zeta(p).$$

Show that

$$\mathscr{M}\left\{\frac{\exp(-ax)}{1-e^{-x}}\right\} = \Gamma(p)\,\xi(p, a).$$

19. Show that

    (a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^p} = (1-2^{1-p})\,\zeta(p)$.

    (b) $\displaystyle\mathscr{M}\left\{\sum_{n=1}^{\infty} (-1)^{n-1} f(nx)\right\} = (1-2^{1-p})\,\zeta(p)\tilde{f}(p)$.

Hence, deduce

    (c) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = \frac{\pi^2}{12}$, (d) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^4} = \left(\frac{7}{8}\right)\frac{\pi^4}{90}$.

20. Find the sum of the following series

    (a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} \cos kn$, (b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \sin kn$.

21. Show that the solution of the boundary value problem

$$r^2\phi_{rr} + r\phi_r + \phi_{\theta\theta} = 0, \quad 0 < r < \infty, \ 0 < \theta < \pi$$

$$\phi(r, 0) = \phi(r, \pi) = f(r),$$

is

$$\phi(r, \theta) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} r^{-p} \, \frac{\tilde{f}(p) \ \cos\left\{p\left(\theta - \frac{\pi}{2}\right)\right\} \, dp}{\cos\left(\frac{\pi p}{2}\right)}.$$

22. Evaluate

$$\sum_{n=1}^{\infty} \frac{\cos an}{n^3} = \frac{1}{12}(a^3 - 3\pi a^2 + 2\pi^2 a).$$

23. Prove the following results:

(a) $\mathscr{M}\left[\int_0^\infty \xi^n f(x\xi)\, g(\xi) d\xi\right] = \tilde{f}(p)\, \tilde{g}(1 + n - p),$

(b) $\mathscr{M}\left[\int_0^\infty \xi^n f\left(\frac{x}{\xi}\right) g(\xi) d\xi\right] = \tilde{f}(p)\, \tilde{g}(p + n + 1).$

24. Show that

(a) $W^{-\alpha}[e^{-x}] = e^{-x}, \quad \alpha > 0,$

(b) $W^{\frac{1}{2}}\left[\frac{1}{\sqrt{x}} \exp\left(-\sqrt{x}\right)\right] = \frac{K_1(\sqrt{x})}{\sqrt{\pi x}}, \quad x > 0,$

where $K_1(x)$ is the modified Bessel function of the second kind and order one.

25. (a) Show that the integral (Wong, 1989, pp. 186–187)

$$I(x) = \int_0^{\pi/2} J_\nu^2(x\cos\theta)\, d\theta, \qquad \nu > -\frac{1}{2},$$

can be written as a Mellin convolution

$$I(x) = \int_0^\infty f(x\xi)\, g(\xi)\, d\xi,$$

where

$$f(\xi) = J_\nu^2(\xi) \ \text{ and } \ \mathrm{g}(\xi) = \begin{Bmatrix} (1 - \xi^2)^{-\frac{1}{2}}, & 0 < \xi < 1 \\ 0, & \xi \geq 1 \end{Bmatrix}.$$

(b) Prove that the integration contour in the Parseval identity

$$I(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} x^{-p}\ \tilde{f}(p)\, \tilde{g}(1-p)\ dp, \qquad -2\nu < c < 1,$$

cannot be shifted to the right beyond the vertical line Re $p = 2$.

26. If $f(x) = \int_0^\infty \exp(-x^2 t^2) \cdot \frac{\sin t}{t^2}\, J_1(t) dt$, show that

$$\mathscr{M}\{f(x)\} = \frac{\Gamma\left(p + \frac{3}{2}\right)\Gamma\left(\frac{1-p}{2}\right)}{p\,\Gamma(p+3)}.$$

27. Prove the following relations to the Laplace and the Fourier transforms:

(a) $\mathscr{M}[f(x), p] = \mathscr{L}[f(e^{-t}), p]$,
(b) $\mathscr{M}[f(x); a + i\omega] = \mathscr{F}[f(e^{-t})e^{-at}; \omega]$,

where $\mathscr{L}$ is the two-sided Laplace transform and $\mathscr{F}$ is the Fourier transform without the factor $(2\pi)^{-\frac{1}{2}}$.

28. Prove the following properties of convolution:

(a) $f * g = g * f$,
(b) $(f * g) * h = f * (g * h)$,
(c) $f(x) * \delta(x-1) = f(x)$,
(d) $\delta(x-a) * f(x) = a^{-1} f\left(\frac{x}{a}\right)$,
(e) $\delta^n(n-1) * f(x) = \left(\frac{d}{dx}\right)^n (x^n f(x))$,
(f) $\left(x\frac{d}{dx}\right)^n (f * g) = \left[\left(x\frac{d}{dx}\right)^n f\right] * g = f * \left[\left(x\frac{d}{dx}\right)^n g\right].$

29. If $\mathscr{M}\{f(r,\theta)\} = \tilde{f}(p,\theta)$ and $\nabla^2 f(r,\theta) = f_{rr} + \frac{1}{r} f_r + \frac{1}{r^2} f_{\theta\theta}$, show that

$$\mathscr{M}\left\{\nabla^2 f(r,\theta)\right\} = \left[\frac{d^2}{d\theta^2} + (p-2)^2\right] \tilde{f}(p-2,\theta).$$

CHAPTER 28

Series Solutions Near a Regular Singular Point

REGULAR SINGULAR POINTS

The point x_0 is a *regular singular point* of the second-order homogeneous linear differential equation

$$y'' + P(x)y' + Q(x)y = 0 \tag{28.1}$$

if x_0 is not an ordinary point (see Chapter 27) but both $(x - x_0)P(x)$ and $(x - x_0)^2Q(x)$ are analytic at x_0. We only consider regular singular points at $x_0 = 0$; if this is not the case, then the change of variables $t = x - x_0$ will translate x_0 to the origin.

METHOD OF FROBENIUS

Theorem 28.1. If $x = 0$ is a regular singular point of *(28.1)*, then the equation has at least one solution of the form

$$y = x^\lambda \sum_{n=0}^{\infty} a_n x^n$$

where λ and a_n ($n = 0, 1, 2, \ldots$) are constants. This solution is valid in an interval $0 < x < R$ for some real number R.

To evaluate the coefficients a_n and λ in Theorem 28.1, one proceeds as in the power series method of Chapter 27. The infinite series

$$\begin{aligned} y &= x^\lambda \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{\lambda+n} \\ &= a_0 x^\lambda + a_1 x^{\lambda+1} + a_2 x^{\lambda+2} + \cdots + a_{n-1} x^{\lambda+n-1} + a_n x^{\lambda+n} + a_{n+1} x^{\lambda+n+1} + \cdots \end{aligned} \tag{28.2}$$

with its derivatives

$$\begin{aligned} y' &= \lambda a_0 x^{\lambda-1} + (\lambda+1)a_1 x^\lambda + (\lambda+2)a_2 x^{\lambda+1} + \cdots \\ &\quad + (\lambda+n-1)a_{n-1} x^{\lambda+n-2} + (\lambda+n)a_n x^{\lambda+n-1} + (\lambda+n+1)a_{n+1} x^{\lambda+n} + \cdots \end{aligned} \tag{28.3}$$

and
$$y'' = \lambda(\lambda-1)a_0x^{\lambda-2} + (\lambda+1)(\lambda)a_1x^{\lambda-1} + (\lambda+2)(\lambda+1)a_2x^{\lambda} + \cdots$$
$$+(\lambda+n-1)(\lambda+n-2)a_{n-1}x^{\lambda+n-3} + (\lambda+n)(\lambda+n-1)a_nx^{\lambda+n-2}$$
$$+ (\lambda+n+1)(\lambda+n)a_{n+1}x^{\lambda+n-1} + \cdots \tag{28.4}$$

are substituted into Eq. (*28.1*). Terms with like powers of x are collected together and set equal to zero. When this is done for x^n the resulting equation is a recurrence formula. A quadratic equation in λ, called the *indicial equation*, arises when the coefficient of x^0 is set to zero and a_0 is left arbitrary.

The two roots of the indicial equation can be real or complex. If complex they will occur in a conjugate pair and the complex solutions that they produce can be combined (by using Euler's relations and the identity $x^{a\pm ib} = x^a e^{\pm ib \ln x}$) to form real solutions. In this book we shall, for simplicity, suppose that both roots of the indicial equation are real. Then, if λ is taken as the *larger* indicial root, $\lambda = \lambda_1 \geq \lambda_2$, the method of Frobenius always yields a solution

$$y_1(x) = x^{\lambda_1} \sum_{n=0}^{\infty} a_n(\lambda_1)x^n \tag{28.5}$$

to Eq. (*28.1*). [We have written $a_n(\lambda_1)$ to indicate the coefficients produced by the method when $\lambda = \lambda_1$.]

If $P(x)$ and $Q(x)$ are quotients of polynomials, it is usually easier first to multiply (*28.1*) by their lowest common denominator and then to apply the method of Frobenius to the resulting equation.

GENERAL SOLUTION

The method of Frobenius always yields one solution to (*28.1*) of the form (*28.5*). The general solution (see Theorem 8.2) has the form $y = c_1y_1(x) + c_2y_2(x)$ where c_1 and c_2 are arbitrary constants and $y_2(x)$ is a second solution of (*28.1*) that is linearly independent from $y_1(x)$. The method for obtaining this second solution depends on the relationship between the two roots of the indicial equation.

Case 1. If $\lambda_1 - \lambda_2$ is not an integer, then

$$y_2(x) = x^{\lambda_2} \sum_{n=0}^{\infty} a_n(\lambda_2)x^n \tag{28.6}$$

where $y_2(x)$ is obtained in an identical manner as $y_1(x)$ by the method of Frobenius, using λ_2 in place of λ_1.

Case 2. If $\lambda_1 = \lambda_2$, then

$$y_2(x) = y_1(x)\ln x + x^{\lambda_1} \sum_{n=0}^{\infty} b_n(\lambda_1)x^n \tag{28.7}$$

To generate this solution, keep the recurrence formula in terms of λ and use it to find the coefficients a_n $(n \geq 1)$ in terms of both λ and a_0, where the coefficient a_0 remains arbitrary. Substitute these a_n into Eq. (*28.2*) to obtain a function $y(\lambda, x)$ which depends on the variables λ and x. Then

$$y_2(x) = \left.\frac{\partial y(\lambda, x)}{\partial \lambda}\right|_{\lambda=\lambda_1} \tag{28.8}$$

Case 3. If $\lambda_1 - \lambda_2 = N$, a positive integer, then

$$y_2(x) = d_{-1}y_1(x)\ln x + x^{\lambda_2} \sum_{n=0}^{\infty} d_n(\lambda_2)x^n \tag{28.9}$$

To generate this solution, first try the method of Frobenius. with λ_2. If it yields a second solution, then this solution is $y_2(x)$, having the form of (*28.9*) with $d_{-1} = 0$. Otherwise, proceed as in Case 2 to generate $y(\lambda, x)$, whence

$$y_2(x) = \frac{\partial}{\partial \lambda}[(\lambda - \lambda_2)y(\lambda, x)]\big|_{\lambda=\lambda_2} \tag{28.10}$$

Solved Problems

28.1. Determine whether $x = 0$ is a regular singular point of the differential equation

$$y'' - xy' + 2y = 0$$

As shown in Problem 27.1, $x = 0$ is an ordinary pont of this differential equation, so it cannot be a regular singular point.

28.2. Determine whether $x = 0$ is a regular singular point of the differential equation

$$2x^2y'' + 7x(x+1)y' - 3y = 0$$

Dividing by $2x^2$, we have

$$P(x) = \frac{7(x+1)}{2x} \quad \text{and} \quad Q(x) = \frac{-3}{2x^2}$$

As shown in Problem 27.7, $x = 0$ is a singular point. Furthermore, both

$$xP(x) = \frac{7}{2}(x+1) \quad \text{and} \quad x^2Q(x) = -\frac{3}{2}$$

are analytic everywhere: the first is a polynomial and the second a constant. Hence, both are analytic at $x = 0$, and this point is a regular singular point.

28.3. Determine whether $x = 0$ is a regular singular point of the differential equation

$$x^3y'' + 2x^2y' + y = 0$$

Dividing by x^3, we have

$$P(x) = \frac{2}{x} \quad \text{and} \quad Q(x) = \frac{1}{x^3}$$

Neither of these functions is defined at $x = 0$, so this point is a singular point. Here,

$$xP(x) = 2 \quad \text{and} \quad x^2Q(x) = \frac{1}{x}$$

The first of these terms is analytic everywhere, but the second is undefined at $x = 0$ and not analytic there. Therefore, $x = 0$ is *not* a regular singular point for the given differential equation.

28.4. Determine whether $x = 0$ is a regular singular point of the differential equation

$$8x^2y'' + 10xy' + (x-1)y = 0$$

Dividing by $8x^2$, we have

$$P(x) = \frac{5}{4x} \quad \text{and} \quad Q(x) = \frac{1}{8x} - \frac{1}{8x^2}$$

Neither of these functions is defined at $x = 0$, so this point is a singular point. Furthermore, both

$$xP(x) = \frac{5}{4} \quad \text{and} \quad x^2Q(x) = \frac{1}{8}(x-1)$$

are analytic everywhere: the first is a constant and the second a polynomial. Hence, both are analytic at $x = 0$, and this point is a regular singular point.

28.5. **Find a recurrence formula and the indicial equation for an infinite series solution around $x = 0$ for the differential equation given in Problem 28.4.**

It follows from Problem 28.4 that $x = 0$ is a regular singular point of the differential equation, so Theorem 24.1 holds. Substituting Eqs. (*28.2*) through (*28.4*) into the left side of the given differential equation and combining coefficients of like powers of x, we obtain

$$\begin{aligned} &x^{\lambda}[8\lambda(\lambda-1)a_0 + 10\lambda a_0 - a_0] + x^{\lambda+1}[8(\lambda+1)\lambda a_1 + 10(\lambda+1)a_1 + a_0 - a_1] + \cdots \\ &\quad + x^{\lambda+n}[8(\lambda+n)(\lambda+n-1)a_n + 10(\lambda+n)a_n + a_{n-1} - a_n] + \cdots = 0 \end{aligned}$$

Dividing by x^{λ} and simplifying, we have

$$\begin{aligned} &[8\lambda^2 + 2\lambda - 1]a_0 + x[(8\lambda^2 + 18\lambda + 9)a_1 + a_0] + \cdots \\ &\quad + x^n\{[8(\lambda+n)^2 + 2(\lambda+n) - 1]a_n + a_{n-1}\} + \cdots = 0 \end{aligned}$$

Factoring the coefficient of a_n and equating the coefficient of each power of x to zero, we find

$$(8\lambda^2 + 2\lambda - 1)a_0 = 0 \tag{1}$$

and, for $n \geq 1$,

$$[4(\lambda+n) - 1][2(\lambda+n) + 1]a_n + a_{n-1} = 0$$

or,

$$a_n = \frac{-1}{[4(\lambda+n)-1][2(\lambda+n)+1]}a_{n-1} \tag{2}$$

Equation (*2*) is a recurrence formula for this differential equation.

From (*1*), either $a_0 = 0$ or

$$8\lambda^2 + 2\lambda - 1 = 0 \tag{3}$$

It is convenient to keep a_0 arbitrary; therefore, we must choose λ to satisfy (3), which is the indicial equation.

28.6. **Find the general solution near $x = 0$ of $8x^2y'' + 10xy' + (x - 1)y = 0$.**

The roots of the indicial equation given by (*3*) of Problem 28.5 are $\lambda_1 = \frac{1}{4}$, and $\lambda_2 = -\frac{1}{2}$. Since $\lambda_1 - \lambda_2 = \frac{3}{4}$, the solution is given by Eqs. (*28.5*) and (*28.6*). Substituting $\lambda = \frac{1}{4}$ into the recurrence formula (*2*) of Problem 28.5 and simplifying, we obtain

$$a_n = \frac{-1}{2n(4n+3)}a_{n-1} \quad (n \geq 1)$$

Thus,

$$a_1 = \frac{-1}{14}a_0, \quad a_2 = \frac{-1}{44}a_1 = \frac{1}{616}a_0, \quad \ldots$$

and

$$y_1(x) = a_0x^{1/4}\left(1 - \frac{1}{14}x + \frac{1}{616}x^2 + \cdots\right)$$

Substituting $\lambda = -\frac{1}{2}$ into recurrence formula (*2*) of Problem 28.5 and simplifying, we obtain

$$a_n = \frac{-1}{2n(4n-3)}a_{n-1}$$

Thus,

$$a_1 = -\frac{1}{2}a_0, \quad a_2 = \frac{-1}{20}a_1 = \frac{1}{40}a_0, \quad \ldots$$

and

$$y_2(x) = a_0x^{-1/2}\left(1 - \frac{1}{2}x + \frac{1}{40}x^2 + \cdots\right)$$

The general solution is

$$y = c_1y_1(x) + c_2y_2(x)$$
$$= k_1x^{1/4}\left(1 - \frac{1}{14}x + \frac{1}{616}x^2 + \cdots\right) + k_2x^{-1/2}\left(1 - \frac{1}{2}x + \frac{1}{40}x^2 + \cdots\right)$$

where $k_1 = c_1a_0$ and $k_2 = c_2a_0$.

28.7. Find a recurrence formula and the indicial equation for an infinite series solution around $x = 0$ for the differential equation

$$2x^2y'' + 7x(x + 1)y' - 3y = 0$$

It follows from Problem 28.2 that $x = 0$ is a regular singular point of the differential equation, so Theorem 28.1 holds. Substituting Eqs. (*28.2*) through (*28.4*) into the left side of the given differential equation and combining coefficients of like powers of x, we obtain

$$x^{\lambda}[2\lambda(\lambda - 1)a_0 + 7\lambda a_0 - 3a_0] + x^{\lambda+1}[2(\lambda + 1)\lambda a_1 + 7\lambda a_0 + 7(\lambda + 1)a_1 - 3a_1] + \cdots$$
$$+ x^{\lambda+n}[2(\lambda + n)(\lambda + n - 1)a_n + 7(\lambda + n - 1)a_{n-1} + 7(\lambda + n)a_n - 3a_n] + \cdots 0$$

Dividing by x^{λ} and simplifying, we have

$$(2\lambda^2 + 5\lambda - 3)a_0 + x[(2\lambda^2 + 9\lambda + 4)a_1 + 7\lambda a_0] + \cdots$$
$$+ x^n\{[2(\lambda + n)^2 + 5(\lambda + n) - 3]a_n + 7(\lambda + n - 1)a_{n-1}\} + \cdots = 0$$

Factoring the coefficient of a_n and equating each coefficient to zero, we find

$$(2\lambda^2 + 5\lambda - 3)a_0 = 0 \tag{1}$$

and, for $n \geq 1$,

$$[2(\lambda + n) - 1][(\lambda + n) + 3]a_n + 7(\lambda + n - 1)a_{n-1} = 0$$

or,

$$a_n = \frac{-7(\lambda + n - 1)}{[2(\lambda + n) - 1][(\lambda + n) + 3]}a_{n-1} \tag{2}$$

Equation (*2*) is a recurrence formula for this differential equation.
From (*1*), either $a_0 = 0$ or

$$2\lambda^2 + 5\lambda - 3 = 0 \tag{3}$$

It is convenient to keep a_0 arbitrary; therefore, We require λ to satisfy the indicial equation (*3*).

28.8. Find the general solution near $x = 0$ of $2x^2y'' + 7x(x + 1)y' - 3y = 0$.

The roots of the indicial equation given by (*3*) of Problem 28.7 are $\lambda_1 = \frac{1}{2}$ and $\lambda_2 = -3$. Since $\lambda_1 - \lambda_2 = \frac{7}{2}$, the solution is given by Eqs. (*28.5*) and (*28.6*). Substituting $\lambda = \frac{1}{2}$ into (*2*) of Problem 28.7 and simplifying, we obtain

$$a_n = \frac{-7(2n - 1)}{2n(2n + 7)}a_{n-1} \quad (n \geq 1)$$

Thus,

$$a_1 = -\frac{7}{18}a_0, \quad a_2 = -\frac{21}{44}a_1 = \frac{147}{792}a_0, \quad \ldots$$

and

$$y_1(x) = a_0x^{1/2}\left(1 - \frac{7}{18}x + \frac{147}{792}x^2 + \cdots\right)$$

Substituting $\lambda = -3$ into (*2*) of Problem 28.7 and simplifying, we obtain

$$a_n = \frac{-7(n-4)}{n(2n-7)}a_{n-1} \quad (n \geq 1)$$

Thus, $$a_1 = -\frac{21}{5}a_0, \quad a_2 = -\frac{7}{3}a_1 = \frac{49}{5}a_0, \quad a_3 = -\frac{7}{3}a_2 = -\frac{343}{15}a_0, \quad a_4 = 0$$

and, since $a_4 = 0$, $a_n = 0$ for $n \geq 4$. Thus,

$$y_2(x) = a_0 x^{-3}\left(1 - \frac{21}{5}x + \frac{49}{5}x^2 - \frac{343}{15}x^3\right)$$

The general solution is

$$\begin{aligned} y &= c_1 y_1(x) + c_2 y_2(x) \\ &= k_1 x^{1/2}\left(1 - \frac{7}{18}x + \frac{147}{792}x^2 + \cdots\right) + k_2 x^{-3}\left(1 - \frac{21}{5}x + \frac{49}{5}x^2 - \frac{343}{15}x^3\right) \end{aligned}$$

where $k_1 = c_1 a_0$ and $k_2 = c_2 a_0$.

28.9. Find the general solution near $x = 0$ of $3x^2y'' - xy' + y = 0$.

Here $P(x) = -1/(3x)$ and $Q(x) = 1/(3x^2)$; hence, $x = 0$ is a regular singular point and the method of Frobenius is applicable. Substituting Eqs. (*28.2*) through (*28.4*) into the differential equation and simplifying, we have

$$x^{\lambda}[3\lambda^2 - 4\lambda + 1]a_0 + x^{\lambda+1}[3\lambda^2 + 2\lambda]a_1 + \cdots + x^{\lambda+n}[3(\lambda + \text{n})^2 - 4(\lambda + n) + 1]a_n + \cdots = 0$$

Dividing by x^{λ} and equating all coefficients to zero, we find

$$(3\lambda^2 - 4\lambda + 1)a_0 = 0 \tag{1}$$

and $$[3(\lambda + n)^2 - 4(\lambda + n) + 1]a_n = 0 \quad (n \geq 1) \tag{2}$$

From (*1*), we conclude that the indicial equation is $3\lambda^2 - 4\lambda + 1 = 0$, which has roots $\lambda_1 = 1$ and $\lambda_2 = \frac{1}{3}$.

Since $\lambda_1 - \lambda_2 = \frac{2}{3}$, the solution is given by Eqs. (*28.5*) and (*28.6*). Note that for either value of λ, (*2*) is satisfied by simply choosing $a_n = 0$, $n \geq 1$. Thus,

$$y_1(x) = x^1\sum_{n=0}^{\infty} a_n x^n = a_0 x \qquad y_2(x) = x^{1/3}\sum_{n=0}^{\infty} a_n x^n = a_0 x^{1/3}$$

and the general solution is

$$y = c_1 y_1(x) + c_2 y_2(x) = k_1 x + k_2 x^{1/3}$$

where $k_1 = c_1 a_0$ and $k_2 = c_2 a_0$.

28.10. Use the method of Frobenius to find one solution near $x = 0$ of $x^2y'' + xy' + x^2y = 0$.

Here $P(x) = 1/x$ and $Q(x) = 1$, so $x = 0$ is a regular singular point and the method of Frobenius is applicable. Substituting Eqs. (*28.2*) through (*28.4*) into the left side of the differential equation, as given, and combining coefficients of like powers of x, we obtain

$$x^{\lambda}[\lambda^2 a_0] + x^{\lambda+1}[(\lambda + 1)^2 a_1] + x^{\lambda+2}[(\lambda + 2)^2 a_2 + a_0] + \cdots + x^{\lambda+n}[(\lambda + n)^2 a_n + a_{n-2}] + \cdots = 0$$

Thus, $$\lambda^2 a_0 = 0 \tag{1}$$

$$(\lambda + 1)^2 a_1 = 0 \tag{2}$$

and, for $n \geq 2$, $(\lambda + n)^2 a_n + a_{n-2} = 0$, or,

$$a_n = \frac{-1}{(\lambda + n)^2}a_{n-2} \quad (n \geq 2) \tag{3}$$

The stipulation $n \geq 2$ is required in (*3*) because a_{n-2} is not defined for $n = 0$ or $n = 1$. From (*1*), the indicial equation is $\lambda^2 = 0$, which has roots, $\lambda_1 = \lambda_2 = 0$. Thus, we will obtain only *one* solution of the form of (*28.5*); the second solution, $y_2(x)$, will have the form of (*28.7*).

Substituting $\lambda = 0$ into (*2*) and (*3*), we find that $a_1 = 0$ and $a_n = -(1/n^2)a_{n-2}$. Since $a_1 = 0$, it follows that $0 = a_3 = a_5 = a_7 = \cdots$. Furthermore,

$$a_2 = -\frac{1}{4}a_0 = -\frac{1}{2^2(1!)^2}a_0 \qquad a_4 = -\frac{1}{16}a_2 = -\frac{1}{2^4(2!)^2}a_0$$

$$a_6 = -\frac{1}{36}a_4 = -\frac{1}{2^6(3!)^2}a_0 \qquad a_8 = -\frac{1}{64}a_6 = \frac{1}{2^8(4!)^2}a_0$$

and, in general, $a_{2k} = \dfrac{(-1)k}{2^{2k}(k!)^2}a_0$ $(k = 1, 2, 3, \ldots)$. Thus,

$$\begin{aligned} y_1(x) &= a_0x^0\left[1 - \frac{1}{2^2(1!)^2}x^2 + \frac{1}{2^4(2!)^2}x^4 + \cdots + \frac{(-1)^k}{2^{2k}(k!)^2}x^{2k} + \cdots\right] \\ &= a_0\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{2n}(n!)^2}x^{2n} \end{aligned} \tag{4}$$

28.11. Find the general solution near $x = 0$ to the differential equation given in Problem 28.10.

One solution is given by (*4*) in Problem 28.10. Because the roots of the indicial equation are equal, we use Eq. (*28.8*) to generate a second linearly independent solution. The recurrence formula is (*3*) of Problem 28.10, augmented by (*2*) of Problem 28.10 for the special case $n = 1$. From (*2*), $a_1 = 0$, which implies that $0 = a_3 = a_5 = a_7 = \cdots$. Then, from (*3*),

$$a_2 = \frac{-1}{(\lambda+2)^2}a_0, \quad a_4 = \frac{-1}{(\lambda+4)^2}a_2 = \frac{1}{(\lambda+4)^2(\lambda+2)^2}a_0, \quad \ldots$$

Substituting these values into Eq. (*28.2*), we have

$$y(\lambda, x) = a_0\left[x^\lambda - \frac{1}{(\lambda+2)^2}x^{\lambda+2} + \frac{1}{(\lambda+4)^2(\lambda+2)^2}x^{\lambda+4} + \cdots\right]$$

Recall that $\dfrac{\partial}{\partial\lambda}(x^{\lambda+k}) = x^{\lambda+k}\ln x$. (When differentiating with respect to λ, x can be thought of as a constant.) Thus,

$$\begin{aligned} \frac{\partial y(\lambda,x)}{\partial\lambda} = a_0\Bigg[x^\lambda \ln x &+ \frac{2}{(\lambda+2)^3}x^{\lambda+2} - \frac{1}{(\lambda+2)^2}x^{\lambda+2}\ln x \\ &- \frac{2}{(\lambda+4)^3(\lambda+2)^2}x^{\lambda+4} - \frac{2}{(\lambda+4)^2(\lambda+2)^3}x^{\lambda+4} \\ &+ \frac{1}{(\lambda+4)^2(\lambda+2)^2}x^{\lambda+4}\ln x + \cdots\Bigg] \end{aligned}$$

and

$$\begin{aligned} y_2(x) = \left.\frac{\partial y(\lambda,x)}{\partial\lambda}\right|_{\lambda=0} &= a_0\Bigg(\ln x + \frac{2}{2^3}x^2 - \frac{1}{2^2}x^2\ln x \\ &\qquad - \frac{2}{4^32^2}x^4 - \frac{2}{4^22^3}x^4 + \frac{1}{4^22^2}x^4\ln x + \cdots\Bigg) \\ &= (\ln x)a_0\left[1 - \frac{1}{2^2(1!)}x^2 + \frac{1}{2^4(2!)^2}x^4 + \cdots\right] \\ &\qquad + a_0\left[\frac{x^2}{2^2(1!)}(1) - \frac{x^4}{2^4(2!)^2}\left(\frac{1}{2}+1\right) + \cdots\right] \\ &= y_1(x)\ln x + a_0\left[\frac{x^2}{2^2(1!)^2}(1) - \frac{x^4}{2^4(2!)^2}\left(\frac{3}{2}\right) + \cdots\right] \end{aligned} \tag{1}$$

which is the form claimed in Eq. (*28.7*). The general solution is $y = c_1y_1(x) + c_2y_2(x)$.

28.12. Use the method of Frobenius to find one solution near $x = 0$ of $x^2y'' - xy' + y = 0$.

Here $P(x) = -1/x$ and $Q(x) = 1/x^2$, so $x = 0$ is a regular singular point and the method of Frobenius is applicable. Substituting Eqs. (*28.2*) through (*28.4*) into the left side of the differential equation, as given, and combining coefficients of like powers of x, we obtain

$$x^{\lambda}(\lambda-1)^2a_0 + x^{\lambda+1}[\lambda^2a_1] + \cdots + x^{\lambda+n}[(\lambda+n)^2 - 2(\lambda+n) + 1]a_n + \cdots = 0$$

Thus,
$$(\lambda-1)^2a_0 = 0 \tag{1}$$

and, in general,
$$[(\lambda+n)^2 - 2(\lambda+n) + 1]a_n = 0 \tag{2}$$

From (*1*), the indicial equation is $(\lambda-1)^2 = 0$, which has roots $\lambda_1 = \lambda_2 = 1$. Substituting $\lambda = 1$ into (*2*), we obtain $n^2a_n = 0$, which implies that $a_n = 0$, $n \geq 1$. Thus, $y_1(x) = a_0x$.

28.13. Find the general solution near $x = 0$ to the differential equation given in Problem 28.12.

One solution is given in Problem 28.12. Because the roots of the indicial equation are equal, we use Eq. (*28.8*) to generate a second linearly independent solution. The recurrence formula is (*2*) of Problem 28.12. Solving it for a_n, in terms of λ, we find that $a_n = 0$ $(n \geq 1)$, and when these values are substituted into Eq. (*28.2*), we have $y(\lambda, x) = a_0x^{\lambda}$. Thus,

$$\frac{\partial y(\lambda,x)}{\partial\lambda} = a_0x^{\lambda}\ln x$$

and
$$y_2(x) = \left.\frac{\partial y(\lambda,x)}{\partial\lambda}\right|_{\lambda=1} = a_0x\ln x = y_1(x)\ln x$$

which is precisely the form of Eq. (*28.7*), where, for this particular differential equation, $b_n(\lambda_1) = 0$ $(n = 0, 1, 2, \ldots)$. The general solution is

$$y = c_1y_1(x) + c_2y_2(x) = k_1(x) + k_2x\ln x$$

where $k_1 = c_1a_0$, and $k_2 = c_2a_0$.

28.14. Use the method of Frobenius to find one solution near $x = 0$ of $x^2y'' + (x^2 - 2x)y' + 2y = 0$.

Here
$$P(x) = 1 - \frac{2}{x} \quad \text{and} \quad Q(x) = \frac{2}{x^2}$$

so $x = 0$ is a regular singular point and the method of Frobenius is applicable. Substituting, Eqs. (*28.2*) through (*28.4*) into the left side of the differential equation, as given, and combining coefficients of like powers of x, we obtain

$$x^{\lambda}[(\lambda^2 - 3\lambda + 2)a_0] + x^{\lambda+1}[(\lambda^2-\lambda)a_1 + \lambda a_0] + \cdots$$
$$+ x^{\lambda+n}\{[(\lambda+n)^2 - 3(\lambda+n) + 2]a_n + (\lambda+n-1)a_{n-1}\} + \cdots = 0$$

Dividing by x^{λ}, factoring the coefficient of a_n, and equating the coefficient of each power of x to zero, we obtain

$$(\lambda^2 - 3\lambda + 2)a_0 = 0 \tag{1}$$

and, in general, $[(\lambda+n) - 2][(\lambda+n) - 1]a_n + (\lambda+n-1)a_{n-1} = 0$, or,

$$a_n = -\frac{1}{\lambda+n-2}a_{n-1} \quad (n \geq 1) \tag{2}$$

From (*1*), the indicial equation is $\lambda^2 - 3\lambda + 2 = 0$, which has roots $\lambda_1 = 2$ and $\lambda_2 = 1$. Since $\lambda_1 - \lambda_2 = 1$, a positive integer, the solution is given by Eqs. (*28.5*) and (*28.9*). Substituting $\lambda = 2$ into (*2*), we have $a_n = -(1/n)a_{n-1}$, from which we obtain

$$a_1 = -a_0$$
$$a_2 = -\frac{1}{2}a_1 = \frac{1}{2!}a_0$$
$$a_3 = -\frac{1}{3}a_2 = -\frac{1}{3}\frac{1}{2!}a_0 = -\frac{1}{3!}a_0$$

and, in general, $a_k = \frac{(-1)^k}{k!}a_0$. Thus,

$$y_1(x) = a_0x^2\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}x^n = a_0x^2e^{-x} \tag{3}$$

28.15. Find the general solution near $x = 0$ to the differential equation given in Problem 28.14.

One solution is given by (*3*) in Problem 28.14 for the indicial. root $\lambda_1 = 2$. If we try the method of Frobenius with the indicial root $\lambda_2 = 1$, recurrence formula (*2*) of Problem 28.14 becomes

$$a_n = -\frac{1}{n-1}a_{n-1}$$

which leaves a_1, undefined because the denominator is zero when $n = 1$. Instead, we must use (*28.10*) to generate a second linearly independent solution. Using the recurrence formula (*2*) of Problem 28.14 to solve sequentially for a_n ($n = 1, 2, 3, \ldots$) in terms of λ, we find

$$a_1 = -\frac{1}{\lambda-1}a_0, \quad a_2 = -\frac{1}{\lambda}a_1 = \frac{1}{\lambda(\lambda-1)}a_0, \quad a_3 = -\frac{1}{\lambda+1}a_2 = \frac{-1}{(\lambda+1)\lambda(\lambda-1)}a_0, \quad \ldots$$

Substituting these values into Eq. (*28.2*) we obtain

$$y(\lambda, x) = a_0\left[x^\lambda - \frac{1}{(\lambda-1)}x^{\lambda+1} + \frac{1}{\lambda(\lambda-1)}x^{\lambda+2} - \frac{1}{(\lambda+1)\lambda(\lambda-1)}x^{\lambda+3} + \cdots\right]$$

and, since $\lambda - \lambda_2 = \lambda - 1$,

$$(\lambda-\lambda_2)y(\lambda, x) = a_0\left[(\lambda-1)x^\lambda - x^{\lambda+1} + \frac{1}{\lambda}x^{\lambda+2} - \frac{1}{\lambda(\lambda+1)}x^{\lambda+3} + \cdots\right]$$

Then

$$\frac{\partial}{\partial\lambda}[(\lambda-\lambda_2)y(\lambda,x)] = a_0\left[x^\lambda + (\lambda-1)x^\lambda\ln x - x^{\lambda+1}\ln x - \frac{1}{\lambda^2}x^{\lambda+2} + \frac{1}{\lambda}x^{\lambda+2}\ln x\right.$$
$$\left. + \frac{1}{\lambda^2(\lambda+1)}x^{\lambda+3} + \frac{1}{\lambda(\lambda+1)^2}x^{\lambda+3} - \frac{1}{\lambda(\lambda+1)}x^{\lambda+3}\ln x + \cdots\right]$$

and

$$y_2(x) = \frac{\partial}{\partial\lambda}[(\lambda-\lambda_2)y(\lambda, x)]\Big|_{\lambda=\lambda_2=1}$$
$$= a_0\left(x + 0 - x^2\ln x - x^3 + x^3\ln x + \frac{1}{2}x^4 + \frac{1}{4}x^4 - \frac{1}{2}x^4\ln x + \cdots\right)$$
$$= (-\ln x)a_0\left(x^2 - x^3 + \frac{1}{2}x^4 + \cdots\right) + a_0\left(x - x^3 + \frac{3}{4}x^4 + \cdots\right)$$
$$= -y_1(x)\ln x + a_0x\left(1 - x^2 + \frac{3}{4}x^3 + \cdots\right)$$

This is the form claimed in Eq. (*28.9*), with $d_{-1} = -1$, $d_0 = a_0$, $d_1 = 0$, $d_3 = \frac{3}{4}a_0, \ldots$. The general solution is $y = c_1y_1(x) + c_2y_2(x)$.

28.16. Use the method of Frobenius to find one solution near $x = 0$ of $x^2y'' + xy' + (x^2 - 1)y = 0$.

Here

$$P(x) = \frac{1}{x} \quad \text{and} \quad Q(x) = 1 - \frac{1}{x^2}$$

so $x = 0$ is a regular singular point and the method of Frobenius is applicable. Substituting Eqs. (*28.2*) through (*28.4*) into the left side of the differential equation, as given, and combining coefficients of like powers of x, we obtain

$$x^{\lambda}[(\lambda^2 - 1)a_0] + x^{\lambda+1}[(\lambda+1)^2 - 1]a_1 + x^{\lambda+2}\{[(\lambda+2)^2 - 1]a_2 + a_0\} + \cdots$$
$$+ x^{\lambda+n}\{[(\lambda+n)^2 - 1]a_n + a_{n-2}\} + \cdots = 0$$

Thus,
$$(\lambda^2 - 1)a_0 = 0 \tag{1}$$
$$[(\lambda+1)^2 - 1]a_1 = 0 \tag{2}$$

and, for $n \geq 2$, $[(\lambda+n)^2 - 1]a_n + a_{n-2} = 0$, or,

$$a_n = \frac{-1}{(\lambda+n)^2 - 1}a_{n-2} \quad (n \geq 2) \tag{3}$$

From (*1*), the indicial equation is $\lambda^2 - 1 = 0$, which has roots $\lambda_1 = 1$ and $\lambda_2 = -1$. Since $\lambda_1 - \lambda_2 = 2$, a positive integer, the solution is given by (*28.5*) and (*28.9*). Substituting $\lambda = 1$ into (*2*) and (*3*), we obtain $a_1 = 0$ and

$$a_n = \frac{-1}{n(n+2)}a_{n-2} \quad (n \geq 2)$$

Since $a_1 = 0$, it follows that $0 = a_3 = a_5 = a_7 = \cdots$. Furthermore,

$$a_2 = \frac{-1}{2(4)}a_0 = \frac{-1}{2^2 1!2!}a_0, \quad a_4 = \frac{-1}{4(6)}a_2 = \frac{1}{2^4 2!3!}a_0, \quad a_6 = \frac{-1}{6(8)}a_4 = \frac{-1}{2^6 3!4!}a_0$$

and, in general,

$$a_{2k} = \frac{(-1)^k}{2^{2k}k!(k+1)!}a_0 \quad (k = 1, 2, 3, \ldots)$$

Thus,
$$y_1(x) = a_0 x \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n} n!\,(n+1)!} x^{2n} \tag{4}$$

28.17. Find the general solution near $x = 0$ to the differential equation given in Problem 28.16.

One solution is given by (*4*) in Problem 28.16 for the indicial root $\lambda_1 = 1$. If we try the method of Frobenius with the indicial root $\lambda_2 = -1$, recurrence formula (*3*) of Problem 28.16 becomes

$$a_n = -\frac{1}{n(n-2)}a_{n-2}$$

which fails to define a_2 because the denominator is zero when $n = 2$. Instead, we must use Eq. (*28.10*) to generate a second linearly independent solution. Using Eqs. (*2*) and (*3*) of Problem 28.16 to solve sequentially for $a_n (n = 1, 2, 3, \ldots)$ in terms of λ, we find $0 = a_1 = a_3 = a_5 = \cdots$ and

$$a_2 = \frac{-1}{(\lambda+3)(\lambda+1)}a_0, \quad a_4 = \frac{1}{(\lambda+5)(\lambda+3)^2(\lambda+1)}a_0, \quad \ldots$$

Thus,
$$y(\lambda, x) = a_0\left[x^{\lambda} - \frac{1}{(\lambda+3)(\lambda+1)}x^{\lambda+2} + \frac{1}{(\lambda+5)(\lambda+3)^2(\lambda+1)}x^{\lambda+4} + \cdots\right]$$

Since $\lambda - \lambda_2 = \lambda + 1$,

$$(\lambda - \lambda_2)y(\lambda, x) = a_0\left[(\lambda+1)x^{\lambda} - \frac{-1}{(\lambda+3)}x^{\lambda+2} + \frac{1}{(\lambda+5)(\lambda+3)^2}x^{\lambda+4} + \cdots\right]$$

and

$$\frac{\partial}{\partial\lambda}[(\lambda-\lambda_2)y(\lambda, x)] = a_0\left[x^\lambda + (\lambda+1)x^\lambda \ln x + \frac{1}{(\lambda+3)^2}x^{\lambda+2} - \frac{1}{(\lambda+3)}x^{\lambda+2}\ln x - \frac{1}{(\lambda+5)^2(\lambda+3)^2}x^{\lambda+4} - \frac{2}{(\lambda+5)(\lambda+3)^3}x^{\lambda+4} + \frac{1}{(\lambda+5)(\lambda+3)^2}x^{\lambda+4}\ln x + \cdots\right]$$

Then

$$\begin{aligned} y_2(x) &= \left.\frac{\partial}{\partial\lambda}[(\lambda-\lambda_2)y(\lambda,x)]\right|_{\lambda=\lambda_2=-1} \\ &= a_0\left(x^{-1} + 0 + \frac{1}{4}x - \frac{1}{2}x\ln x - \frac{1}{64}x^3 - \frac{2}{32}x^3 + \frac{1}{16}x^3\ln x + \cdots\right) \\ &= -\frac{1}{2}(\ln x)a_0 x\left(1 - \frac{1}{8}x^2 + \cdots\right) + a_0\left(x^{-1} + \frac{1}{4}x - \frac{5}{64}x^3 + \cdots\right) \\ &= -\frac{1}{2}(\ln x)y_1(x) + a_0x^{-1}\left(1 + \frac{1}{4}x^2 - \frac{5}{64}x^4 + \cdots\right) \end{aligned} \tag{1}$$

This is in the form of (*28.9*) with $d_{-1} = -\frac{1}{2}$, $d_0 = a_0$, $d_1 = 0$, $d_2 = \frac{1}{4}a_0$, $d_3 = 0$, $d_4 = \frac{-5}{64}a_0, \ldots$. The general solution is $y = c_1y_1(x) + c_2y_2(x)$.

28.18. Use the method of Frobenius to find one solution near $x = 0$ of $x^2y'' + (x^2 + 2x)y' - 2y = 0$.

Here

$$P(x) = 1 + \frac{2}{x} \quad \text{and} \quad Q(x) = -\frac{2}{x^2}$$

so $x = 0$ is a regular singular point and the method of Frobenius is applicable. Substituting Eqs. (*28.2*) through (*28.4*) into the left side of the differential equation, as given, and combining coefficients of like powers of x, we obtain

$$\begin{aligned} &x^\lambda[(\lambda^2 + \lambda - 2)a_0] + x^{\lambda+1}[(\lambda^2 + 3\lambda)a_1 + \lambda a_0] + \cdots \\ &\quad + x^{\lambda+n}\{[(\lambda+n)^2 + (\lambda+n) - 2]a_n + (\lambda+n-1)a_{n-1}\} + \cdots = 0 \end{aligned}$$

Dividing by x^λ, factoring the coefficient of a_n, and equating to zero the coefficient of each power of x, we obtain

$$(\lambda^2 + \lambda - 2)a_0 = 0 \tag{1}$$

and, for $n \geq 1$,

$$[(\lambda+n) + 2][(\lambda+n) - 1]a_n + (\lambda+n-1)a_{n-1} = 0$$

which is equivalent to

$$a_n = -\frac{1}{\lambda+n+2}a_{n-1} \quad (n \geq 1) \tag{2}$$

From (*1*), the indicial equation is $\lambda^2 + \lambda - 2 = 0$, which has roots $\lambda_1 = 1$ and $\lambda_2 = -2$. Since $\lambda_1 - \lambda_2 = 3$, a positive integer, the solution is given by Eqs. (*28.5*) and (*28.9*). Substituting $\lambda = 1$ into (*2*), we obtain $a_n = [-1/(n+3)]a_{n-1}$, which in turn yields

$$\begin{aligned} a_1 &= -\frac{1}{4}a_0 = -\frac{3!}{4!}a_0 \\ a_2 &= -\frac{1}{5}a_1 = \left(-\frac{1}{5}\right)\left(-\frac{3!}{4!}\right)a_0 = \frac{3!}{5!}a_0 \\ a_3 &= -\frac{1}{6}a_2 = -\frac{3!}{6!}a_0 \end{aligned}$$

and, in general,

$$a_k = \frac{(-1)^k 3!}{(k+3)!} a_0$$

Hence,
$$y_1(x) = a_0 x \left[1 + 3! \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{(n+3)!}\right] = a_0 x \sum_{n=0}^{\infty} \frac{(-1)^n 3!\, x^n}{(n+3)!}$$

which can be simplified to

$$y_1(x) = \frac{3a_0}{x^2}(2 - 2x + x^2 - 2e^{-x}) \tag{3}$$

28.19. Find the general solution near $x = 0$ to the differential equation given in Problem 28.18.

One solution is given by (*3*) in Problem 28.18 for the indicial root $\lambda_1 = 1$. If we try the method of Frobenius with the indicial root $\lambda_2 = -2$, recurrence formula (*2*) of Problem 28.18 becomes

$$a_n = -\frac{1}{n} a_{n-1} \tag{1}$$

which does define all $a_n (n \geq 1)$. Solving sequentially, we obtain

$$a_1 = -a_0 = -\frac{1}{1!} a_0 \qquad a_2 = -\frac{1}{2} a_2 = \frac{1}{2!} a_0$$

and, in general, $a_k = (-1)^k a_0 / k!$. Therefore,

$$y_2(x) = a_0 x^{-2} \left[1 - \frac{1}{1!} x + \frac{1}{2!} x^2 + \cdots + \frac{(-1)^k}{k!} x^k + \cdots\right]$$
$$= a_0 x^{-2} \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!} = a_0 x^{-2} e^{-x}$$

This is precisely in the form of (*28.9*), with $d_{-1} = 0$ and $d_n = (-1)^n a_0 / n!$. The general solution is

$$y = c_1 y_1(x) + c_2 y_2(x)$$

28.20. Find a general expression for the indicial equation of (*28.1*).

Since $x = 0$ is a regular singular point; $xP(x)$ and $x^2 Q(x)$ are analytic near the origin and can be expanded in Taylor series there. Thus,

$$xP(x) = \sum_{n=0}^{\infty} p_n x^n = p_0 + p_1 x + p_2 x^2 + \cdots$$
$$x^2 Q(x) = \sum_{n=0}^{\infty} q_n x^n = q_0 + q_1 x + q_2 x^2 + \cdots$$

Dividing by x and x^2, respectively, we have

$$P(x) = p_0 x^{-1} + p_1 + p_2 x + \cdots \qquad Q(x) = q_0 x^{-2} + q_1 x^{-1} + q_2 + \cdots$$

Substituting these two results with Eqs. (*28.2*) through (*28.4*) into (*28.1*) and combining, we obtain

$$x^{\lambda-2}[\lambda(\lambda-1)a_0 + \lambda a_0 p_0 + a_0 q_0] + \cdots = 0$$

which can hold only if

$$a_0[\lambda^2 + (p_0 - 1)\lambda + q_0] = 0$$

Since $a_0 \neq 0$ (a_0 is an arbitrary constant, hence can be chosen nonzero), the indicial equation is

$$\lambda^2 + (p_0 - 1)\lambda + q_0 = 0 \tag{1}$$

28.21. **Find the indicial equation of $x^2y'' + xe^xy' + (x^3 - 1)y = 0$ if the solution is required near $x = 0$.**

Here

$$P(x) = \frac{e^x}{x} \quad \text{and} \quad Q(x) = x - \frac{1}{x^2}$$

and we have

$$xP(x) = e^x = 1 + x + \frac{x^2}{2!} + \cdots$$
$$x^2Q(x) = x^3 - 1 = -1 + 0x + 0x^2 + 1x^3 + 0x^4 + \cdots$$

from which $p_0 = 1$ and $q_0 = -1$. Using (*1*) of Problem 28.20, we obtain the indicial equation as $\lambda^2 - 1 = 0$.

28.22. **Solve Problem 28.9 by an alternative method.**

The given differential equation, $3x^2y'' - xy' + y = 0$, is a special case of *Euler's equation*

$$b_nx^ny^{(n)} + b_{n-1}x^{n-1}y^{(n-1)} + \cdots + b_2x^2y'' + b_1xy' + b_0y = \phi(x) \tag{1}$$

where $b_j(j = 0, 1, \ldots, n)$ is a constant. Euler's equation can always be transformed into a linear differential equation with *constant coefficients* by the change of variables

$$z = \ln x \quad \text{or} \quad x = e^z \tag{2}$$

It follows from (*2*) and from the chain rule and the product rule of differentiation that

$$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = \frac{1}{x}\frac{dy}{dz} = e^{-z}\frac{dy}{dz} \tag{3}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(e^{-z}\frac{dy}{dz}\right) = \left[\frac{d}{dz}\left(e^{-z}\frac{dy}{dz}\right)\right]\frac{dz}{dx}$$
$$= \left[-e^{-z}\left(\frac{dy}{dz}\right) + e^{-z}\left(\frac{d^2y}{dz^2}\right)\right]e^{-z} = e^{-2z}\left(\frac{d^2y}{dz^2}\right) - e^{-2z}\left(\frac{dy}{dz}\right) \tag{4}$$

Substituting Eqs. (*2*), (*3*), and (*4*) into the given differential equation and simplifying, we obtain

$$\frac{d^2y}{dz^2} - \frac{4}{3}\frac{dy}{dz} + \frac{1}{3}y = 0$$

Using the method of Chapter 9 we find that the solution of this last equation is $y = c_1e^z + c_2e^{(1/3)z}$. Then using (*2*) and noting that $e^{(1/3)z} = (e^z)^{1/3}$, we have as before,

$$y = c_1x + c_2x^{1/3}$$

28.23. **Solve the differential equation given in Problem 28.12 by an alternative method.**

The given differential equation, $x^2y'' - xy' + y = 0$, is a special case of Euler's equation, (*1*) of Problem 28.22. Using the transformations (*2*), (*3*), and (*4*) of Problem 28.22, we reduce the given equation to

$$\frac{d^2y}{dz^2} - 2\frac{dy}{dz} + y = 0$$

The solution to this equation is (see Chapter 9) $y = c_1e^z + c_2ze^z$. Then, using (*2*) of Problem 28.22, we have for the solution of the original differential equation

$$y = c_1x + c_2x \ln x$$

as before.

28.24. Find the general solution near $x = 0$ of the *hypergeometric equation*

$$x(1-x)y'' + [C - (A+B+1)x]y' - ABy = 0$$

where A and B are any real numbers, and C is any real nonintegral number.

Since $x = 0$ is a regular singular point, the method of Frobenius is applicable. Substituting, Eqs. (*28.2*) through (*28.4*) into the differential equation, simplifying and equating the coefficient of each power of x to zero, we obtain

$$\lambda^2 + (C-1)\lambda = 0 \tag{1}$$

as the indicial equation and

$$a_{n+1} = \frac{(\lambda+n)(\lambda+n+A+B)+AB}{(\lambda+n+1)(\lambda+n+C)}a_n \tag{2}$$

as the recurrence formula. The roots of (*1*) are $\lambda_1 = 0$ and $\lambda_2 = 1 - C$; hence, $\lambda_1 - \lambda_2 = C - 1$. Since C is not an integer, the solution of the hypergeometric equation is given by Eqs. (*28.5*) and (*28.6*).

Substituting $\lambda = 0$ into (*2*), we have

$$a_{n+1} = \frac{n(n+A+B)+AB}{(n+1)(n+C)}a_n$$

which is equivalent to

$$a_{n+1} = \frac{(A+n)\,(B+n)}{(n+1)(n+C)}a_n$$

Thus

$$a_1 = \frac{AB}{C}a_0 = \frac{AB}{1!\,C}a_0$$
$$a_2 = \frac{(A+1)(B+1)}{2(C+1)}a_1 = \frac{A(A+1)B(B+1)}{2!\,C(C+1)}a_0$$
$$a_3 = \frac{(A+2)(B+2)}{3(C+2)}a_2 = \frac{A(A+1)(A+2)B(B+1)(B+2)}{3!\,C(C+1)(C+2)}a_0$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

and $y_1(x) = a_0F(A, B; C; x)$, where

$$F(A,B;C;x) = 1 + \frac{AB}{1!\,C}x + \frac{A(A+1)B(B+1)}{2!\,C(C+1)}x^2 + \frac{A(A+1)(A+2)B(B+1)(B+2)}{3!\,C(C+1)(C+2)}x^3 + \cdots$$

The series $F(A, B; C; x)$ is known as the hypergeometric series; it can be shown that this series converges for $-1 < x < 1$. It is customary to assign the arbitrary constant a_0 the value 1. Then $y_1(x) = F(A, B; C; x)$ and the hypergeometric series is a solution of the hypergeometric equation.

To find $y_2(x)$, we substitute $\lambda = 1 - C$ into (*2*) and obtain

$$a_{n+1} = \frac{(n+1-C)(n+1+A+B-C)+AB}{(n+2-C)(n+1)}a_n$$

or

$$a_{n+1} = \frac{(A-C+n+1)(B-C+n+1)}{(n+2-C)(n+1)}a_n$$

Solving for a_n in terms of a_0, and again setting $a_0 = 1$, it follows that

$$y_2(x) = x^{1-C}F(A-C+1,\, B-C+1;\, 2-C;\, x)$$

The general solution is $y = c_1y_1(x) + c_2y_2(x)$.

Supplementary Problems

In Problems 28.25 through 28.33, find two linearly independent solutions to the given differential equations.

28.25. $2x^2y'' - xy' + (1 - x)y = 0$

28.26. $2x^2y'' + (x^2 - x)y' + y = 0$

28.27. $3x^2y'' - 2xy' - (2 + x^2)y = 0$

28.28. $xy'' + y' - y = 0$

28.29. $x^2y'' + xy' + x^3y = 0$

28.30. $x^2y'' + (x - x^2)y' - y = 0$

28.31. $xy'' - (x + 1)y' - y = 0$

28.32. $4x^2y'' + (4x + 2x^2)y' + (3x - 1)y = 0$

28.33. $x^2y'' + (x^2 - 3x)y' - (x - 4)y = 0$

In Problem 28.34 through 28.38, find the general solution to the given equations using the method described in Problem 28.22.

28.34. $4x^2y'' + 4xy' - y = 0$

28.35. $x^2y'' - 3xy' + 4y = 0$

28.36. $2x^2y'' + 11xy' + 4y = 0$

28.37. $x^2y'' - 2y = 0$

28.38. $x^2y'' - 6xy' = 0$

CHAPTER 29

Some Classical Differential Equations

CLASSICAL DIFFERENTIAL EQUATIONS

Because some special differential equations have been studied for many years, both for the aesthetic beauty of their solutions and because they lend themselves to many physical applications, they may be considered *classical*. We have already seen an example of such an equation, the equation of *Legendre*, in Problem 27.13.

We will touch upon four classical equations: the *Chebyshev* differential equation, named in honor of Pafnuty Chebyshey (1821–1894); the *Hermite* differential equation, so named because of Charles Hermite (1822–1901); the *Laguerre* differential equation, labeled after Edmond Laguerre (1834–1886); and the *Legendre* differential equation, so titled because of Adrien Legendre (1752–1833). These equations are given in Table 29-1 below:

Table 29-1
(Note: $n = 0, 1, 2, 3, \ldots$)

Chebyshev Differential Equation	$(1-x^2)\,y'' - xy' + n^2y = 0$
Hermite Differential Equation	$y'' - 2xy' + 2ny = 0$
Laguerre Differential Equation	$xy'' + (1-x)y' + ny = 0$
Legendre Differential Equation	$(1-x^2)y'' - 2xy' + n(n+1)y = 0$

POLYNOMIAL SOLUTIONS AND ASSOCIATED CONCEPTS

One of the most important properties these four equations possess, is the fact that they have *polynomial* solutions, naturally called Chebyshev polynomials, Hermite polynomials, etc.

There are many ways to obtain these polynomial solutions. One way is to employ series techniques, as discussed in Chapters 27 and 28. An alternate way is by the use of *Rodrigues* formulas, so named in honor of O. Rodrigues (1794–1851), a French banker. This method makes use of repeated differentiations (see, for example, Problem 29.1).

These polynomial solutions can also be obtained by the use of *generating functions*. In this approach, infinite series expansions of the specific function "generates" the desired polynomials (see Problem 29.3). It should be noted, from a computational perspective, that this approach becomes more time-consuming the further along we go in the series.