مثال : أوجد حاصل ضرب المصفوفتين $A \times B$

$$A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}_{2\times3} \quad , \quad B = \begin{bmatrix} 3 & 6 & 9 \\ 4 & 7 & 10 \\ 5 & 8 & 11 \end{bmatrix}_{3\times3}$$

$$A \times B = C$$

$$= \begin{bmatrix} \overset{c_{11}}{(1\times3+3\times4+5\times5)} & \overset{c_{12}}{(1\times6+3\times7+5\times8)} & \overset{c_{13}}{(1\times9+3\times10+5\times11)} \\ \overset{c_{21}}{(2\times3+4\times4+6\times5)} & \overset{c_{22}}{(2\times6+4\times7+6\times8)} & \overset{c_{23}}{(2\times9+4\times10+6\times11)} \end{bmatrix}$$

$$A \times B = C = \begin{bmatrix} 40 & 67 & 94 \\ 52 & 88 & 124 \end{bmatrix}$$

المحددة Determinate:

اذا كان A مصفوفة مربعة من الدرجة $(n \times n)$ فانه يوجد عدد يسمى محدد المصفوفة ويرمز له بالرمز $|A|$ أو $\det(A)$ أو D

$$for \;\; n = 1 \;\; \Longrightarrow \;\; |A| = |a_{11}| = D$$

$$for \;\; n = 2 \;\; \Longrightarrow \;\; |A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21} = D$$

$$for \;\; n = 3 \;\; \Longrightarrow \;\; |A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{matrix}$$

$$= (a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}) - (a_{31}a_{22}a_{13} + a_{32}a_{23}a_{11} + a_{33}a_{21}a_{12})$$

مثال: جـد قيمة المحددة $\begin{vmatrix} 1 & 2 & 3 \\ 4 & -1 & 5 \\ 6 & 7 & 0 \end{vmatrix}$ بطريقة فتح لابلاس على الصف الأول ، العمود الثالث ، الصف الثاني

الصف الأول

1. $|A| = +1 \begin{vmatrix} -1 & 5 \\ 7 & 0 \end{vmatrix} - 2 \begin{vmatrix} 4 & 5 \\ 6 & 0 \end{vmatrix} + 3 \begin{vmatrix} 4 & -1 \\ 6 & 7 \end{vmatrix} = 127$

العمود الثالث

2. $|A| = +3 \begin{vmatrix} 4 & -1 \\ 6 & 7 \end{vmatrix} - 5 \begin{vmatrix} 1 & 2 \\ 6 & 7 \end{vmatrix} + 0 = 127$

الصف الثاني

3. $|A| = -4 \begin{vmatrix} 2 & 3 \\ 7 & 0 \end{vmatrix} - 1 \begin{vmatrix} 1 & 3 \\ 6 & 0 \end{vmatrix} + 5 \begin{vmatrix} 1 & 2 \\ 6 & 7 \end{vmatrix} = 127$

مثال: حل منظومة المعادلات الخطية التالية بطريقة كريمر .

$$x + 2y + z = 0$$
$$3x - y - 2z = 9$$
$$4x + 3y - 3z = 3$$

$$D = \begin{vmatrix} 1 & 2 & 1 \\ 3 & -1 & -2 \\ 4 & 3 & -3 \end{vmatrix} \quad , \quad W = \begin{vmatrix} x \\ y \\ z \end{vmatrix} \quad , \quad b = \begin{vmatrix} 0 \\ 9 \\ 3 \end{vmatrix}$$

$$D = \begin{vmatrix} 1 & 2 & 1 \\ 3 & -1 & -2 \\ 4 & 3 & -3 \end{vmatrix} \begin{matrix} 1 & 2 \\ 3 & -1 \\ 4 & 3 \end{matrix} = (3 - 16 + 9) - (-4 - 6 - 18) = 24$$

$$x = \frac{D_x}{D} \quad , \quad y = \frac{D_y}{D} \quad , \quad z = \frac{D_z}{D}$$

$$D_x = \begin{vmatrix} 0 & 2 & 1 \\ 9 & -1 & -2 \\ 3 & 3 & -3 \end{vmatrix} \begin{matrix} 0 & 2 \\ 9 & -1 \\ 3 & 3 \end{matrix} = 72$$

$$D_y = \begin{vmatrix} 1 & 0 & 1 \\ 3 & 9 & -2 \\ 4 & 3 & -3 \end{vmatrix} \begin{matrix} 1 & 0 \\ 3 & 9 \\ 4 & 3 \end{matrix} = -48$$

$$D_z = \begin{vmatrix} 1 & 2 & 0 \\ 3 & -1 & 9 \\ 4 & 3 & 3 \end{vmatrix} \begin{matrix} 1 & 2 \\ 3 & -1 \\ 4 & 3 \end{matrix} = 24$$

$$x = \frac{D_x}{D} = \frac{72}{24} = 3 \quad , \quad y = \frac{D_y}{D} = \frac{-48}{24} = -2 \quad , \qquad z = \frac{D_z}{D} = \frac{24}{24} = 1$$

$$3 + (-2)2 + 1 = 0 \quad \Rightarrow 0 = 0$$

To cheak :

$$3(3) - (-2) - 2(1) = 9 \quad \Rightarrow 9 = 9$$

$$4(3) + 3(-2) - 3(1) = 3 \quad \Rightarrow \; 3 = 3$$

مثال: حل منظومة المعادلات الخطية التالية بطريقة كريمر .

$$x + 5y + 2z = 1$$
$$x + y + 7z = 0$$
$$-3y + 4z = 1$$

$$D = \begin{vmatrix} 1 & 5 & 2 \\ 1 & 1 & 7 \\ 0 & -3 & 4 \end{vmatrix} \;, W = \begin{vmatrix} x \\ y \\ z \end{vmatrix} \;, b = \begin{vmatrix} 1 \\ 0 \\ 1 \end{vmatrix} \;, \quad x = \frac{D_x}{D} \;, \quad y = \frac{D_y}{D} \;, \quad z = \frac{D_z}{D}$$

$$D = \begin{vmatrix} 1 & 5 & 2 \\ 1 & 1 & 7 \\ 0 & -3 & 4 \end{vmatrix} \begin{matrix} 1 & 5 \\ 1 & 1 \\ 0 & -3 \end{matrix} = -1 \;, \qquad D_x = \begin{vmatrix} 1 & 5 & 2 \\ 0 & 1 & 7 \\ 1 & -3 & 4 \end{vmatrix} \begin{matrix} 1 & 5 \\ 0 & 1 \\ 1 & -3 \end{matrix} = 58$$

$$D_y = \begin{vmatrix} 1 & 1 & 2 \\ 1 & 0 & 7 \\ 0 & 1 & 4 \end{vmatrix} \begin{matrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{matrix} = -9 \;, D_z = \begin{vmatrix} 1 & 5 & 1 \\ 1 & 1 & 0 \\ 0 & -3 & 1 \end{vmatrix} \begin{matrix} 1 & 5 \\ 1 & 1 \\ 0 & -3 \end{matrix} = -7$$

$$x = \frac{D_x}{D} = \frac{58}{-1} = -58 \;, \quad y = \frac{D_y}{D} = \frac{-9}{-1} = 9 \;, \qquad z = \frac{D_z}{D} = \frac{-7}{-1} = 7$$

$$-58 + 5(9) + 2(7) = 1 \quad \Rightarrow \quad 1 = 1$$

ثانياَ: حل المعادلات الخطية بطريقة معكوس المصفوفة:

$$[W] = [A^{-1}][B] \quad , \qquad A^{-1} = \frac{[A^C]^T}{D}$$ معكوس المصفوفة , $D \neq 0$

$$[A^C] = \begin{vmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{vmatrix} \Rightarrow [A^C]^T = \begin{vmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{23} \\ C_{13} & C_{23} & C_{33} \end{vmatrix}$$ مدور مصفوفة العوامل المرافقة: $[A^C]^T$

$$\Rightarrow A^{-1} = \frac{\begin{vmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{23} \\ C_{13} & C_{23} & C_{33} \end{vmatrix}}{D} = \begin{vmatrix} \frac{C_{11}}{D} & \frac{C_{21}}{D} & \frac{C_{31}}{D} \\ \frac{C_{12}}{D} & \frac{C_{22}}{D} & \frac{C_{23}}{D} \\ \frac{C_{13}}{D} & \frac{C_{23}}{D} & \frac{C_{33}}{D} \end{vmatrix} \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{vmatrix} \frac{C_{11}}{D} & \frac{C_{21}}{D} & \frac{C_{31}}{D} \\ \frac{C_{12}}{D} & \frac{C_{22}}{D} & \frac{C_{23}}{D} \\ \frac{C_{13}}{D} & \frac{C_{23}}{D} & \frac{C_{33}}{D} \end{vmatrix} \begin{vmatrix} b_1 \\ b_2 \\ b_3 \end{vmatrix}$$

$$x = \frac{C_{11}}{D} \times b_1 + \frac{C_{21}}{D} \times b_2 + \frac{C_{31}}{D} \times b_3$$

$$y = \frac{C_{12}}{D} \times b_1 + \frac{C_{22}}{D} \times b_2 + \frac{C_{23}}{D} \times b_3$$

$$z = \frac{C_{13}}{D} \times b_1 + \frac{C_{23}}{D} \times b_2 + \frac{C_{33}}{D} \times b_3$$

مثال: حل منظومة المعادلات الخطية التالية بطريقة معكوس المصفوفة .

$$2x + 3y - 4z = -3$$

$$x + 2y + 3z = 3$$

$$3x - y - z = 6$$

$$D = \begin{vmatrix} 2 & 3 & -4 \\ 1 & 2 & 3 \\ 3 & -1 & -1 \end{vmatrix} = 60\,, \qquad [A^C] = \begin{vmatrix} 1 & 10 & -7 \\ 7 & 10 & 11 \\ 17 & -10 & 1 \end{vmatrix} \Rightarrow [A^C]^T = \begin{vmatrix} 1 & 7 & 17 \\ 10 & 10 & -10 \\ -7 & 11 & 1 \end{vmatrix}$$

$$A^{-1} = \frac{\begin{vmatrix} 1 & 7 & 17 \\ 10 & 10 & -10 \\ -7 & 11 & 1 \end{vmatrix}}{60} = \begin{vmatrix} \frac{1}{60} & \frac{7}{60} & \frac{17}{60} \\ \frac{10}{60} & \frac{10}{60} & \frac{-10}{60} \\ \frac{-7}{60} & \frac{11}{60} & \frac{1}{60} \end{vmatrix} = \begin{vmatrix} \frac{1}{60} & \frac{7}{60} & \frac{17}{60} \\ \frac{1}{6} & \frac{1}{6} & \frac{-1}{6} \\ \frac{-7}{60} & \frac{11}{60} & \frac{1}{60} \end{vmatrix}$$

$$To\ cheak \quad A^{-1}A = 1$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{vmatrix} \frac{1}{60} & \frac{7}{60} & \frac{17}{60} \\ \frac{1}{6} & \frac{1}{6} & \frac{-1}{6} \\ \frac{-7}{60} & \frac{11}{60} & \frac{1}{60} \end{vmatrix} \begin{vmatrix} -3 \\ 3 \\ 6 \end{vmatrix} \Longrightarrow x = \left(\frac{1}{60}\right)(-3) + \left(\frac{7}{60}\right)(3) + \left(\frac{17}{60}\right)(6) = 2$$

$$y = \left(\frac{1}{6}\right)(-3) + \left(\frac{1}{6}\right)(3) + \left(-\frac{1}{6}\right)(6) = -1$$

$$z = \left(\frac{-7}{60}\right)(-3) + \left(\frac{11}{60}\right)(3) + \left(\frac{1}{60}\right)(6) = 1$$

مثال: حل منظومة المعادلات الخطية التالية بطريقة معكوس المصفوفة .

$$x + y + z = 6$$

$$2x + 3y + z = 11$$

$$3x + 2y + 2z = 13$$

$$D = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 1 \\ 3 & 2 & 2 \end{vmatrix} = -2, \qquad [A^C] = \begin{vmatrix} 4 & -1 & -5 \\ 0 & -1 & 1 \\ -2 & 1 & 1 \end{vmatrix} \Rightarrow [A^C]^T = \begin{vmatrix} 4 & 0 & -2 \\ -1 & -1 & 1 \\ -5 & 1 & 1 \end{vmatrix}$$

$$A^{-1} = \frac{\begin{vmatrix} 4 & 0 & -2 \\ -1 & -1 & 1 \\ -5 & 1 & 1 \end{vmatrix}}{-2} = \begin{vmatrix} \frac{4}{-2} & \frac{0}{-2} & \frac{-2}{-2} \\ \frac{-1}{-2} & \frac{-1}{-2} & \frac{1}{-2} \\ \frac{-5}{-2} & \frac{1}{-2} & \frac{1}{-2} \end{vmatrix} = \begin{vmatrix} -2 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{5}{2} & \frac{-1}{2} & \frac{-1}{2} \end{vmatrix}$$

$$To\ cheak \quad A^{-1}A = 1$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{vmatrix} -2 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{5}{2} & \frac{-1}{2} & \frac{-1}{2} \end{vmatrix} \begin{vmatrix} 6 \\ 11 \\ 13 \end{vmatrix} \implies x = (-2)(6) + (0)(11) + (1)(13) = 1$$

$$y = \left(\frac{1}{2}\right)(6) + \left(\frac{1}{2}\right)(11) + \left(-\frac{1}{2}\right)(13) = 2$$

$$z = \left(\frac{5}{2}\right)(6) + \left(\frac{-1}{2}\right)(11) + \left(\frac{-1}{2}\right)(13) = 3$$

مشتقة وتكامل الدوال الزائدية

	Derivative	Integral
1	$\frac{d\ \sinh u}{dx} = \cosh u \cdot \frac{du}{dx}$	$\int \cosh u\, du = \sinh u + c$
2	$\frac{d \cosh u}{dx} = \sinh u \cdot \frac{du}{dx}$	$\int \sinh u\ du = \cosh u + c$
3	$\frac{d\ \tanh u}{dx} = \mathrm{sech}^2 u. \frac{du}{dx}$	$\int \mathrm{sech}^2 u\, du = \tanh u + c$
4	$\frac{d\ \coth u}{dx} = -\mathrm{csch}^2 x. \frac{du}{dx}$	$\int \mathrm{csch}^2 x\, du = -\coth u + c$
5	$\frac{d\ \mathrm{sech}\, u}{dx} = -\mathrm{sech}\, u\ \tanh u \cdot \frac{du}{dx}$	$\int \mathrm{sech}\, u\ \tanh u\, du = -\mathrm{sech}\, u\, u + c$
6	$\frac{d\ \mathrm{csch}\, u}{dx} = -\mathrm{csch}\, u \coth u \cdot \frac{du}{dx}$	$\int \mathrm{csch}\, u \coth u\, du = -\mathrm{csch}\, u + c$

$$\boldsymbol{Find\ \frac{dy}{dx}\ for:}$$

1. $y = cosh^2 5\text{x}$

$$\frac{dy}{dx} = 2cosh\,5x.sinh\,5x.5$$

2. $y = ln(tanh\,2x)$

$$\frac{\text{dy}}{\text{dx}} = \frac{sech^2 2x.2}{tanh\,2x}$$

3. $y = tan^{-1}(\sinh x)$

$$\frac{\text{dy}}{\text{dx}} = \frac{\cosh x}{1 + sinh^2 x}$$

$$\int \frac{\cosh x}{\sinh x + \cosh x} dx$$

$$\int \frac{\frac{e^x + e^{-x}}{2}}{\frac{e^x - e^{-x}}{2} + \frac{e^x + e^{-x}}{2}} dx \quad \Rightarrow \quad \int \frac{e^x + e^{-x}}{e^x - e^{-x} + e^x + e^{-x}} dx$$

$$\int \frac{e^x + e^{-x}}{2e^x} dx \;\Rightarrow\; \frac{1}{2}\int \frac{e^x(1 + e^{-2x})}{e^x} dx \quad \Rightarrow \frac{1}{2}\int dx + \frac{1}{2}\int e^{-2x}\, dx$$

$$\Rightarrow \; \frac{1}{2}x - \frac{1}{4}e^{-2x} + \mathrm{c}$$