طرق التكامل Methods of Integral:

أولاً: التكامل بالتعويض بالدوال المثلثية

$$Ex:\ \int \frac{dx}{\sqrt{3-2x^2}}\ ,\quad a^2=3 \implies a=\sqrt{3}\ ,\quad b^2=2 \implies b=\sqrt{2}$$

$$x=\frac{\sqrt{3}}{\sqrt{2}} sin\theta \implies dx=\frac{\sqrt{3}}{\sqrt{2}} cos\theta\ d\theta$$

$$sin\theta=\frac{\sqrt{2}}{\sqrt{3}}x \implies \theta=\sin^{-1}\left(\frac{\sqrt{2}}{\sqrt{3}}x\right)$$

$$3-2x^2 \implies 3-2\left(\frac{3}{2}sin^2\theta\right) \implies 3-3sin^2\theta \quad 3(1-sin^2\theta) \implies 3cos^2\theta$$

$$\int \frac{\frac{\sqrt{3}}{\sqrt{2}} cos\theta\ d\theta}{\sqrt{3cos^2\theta}} \implies \frac{\sqrt{3}}{\sqrt{2}}\int \frac{cos\theta\ d\theta}{\sqrt{3}\ cos\theta} \implies \frac{1}{\sqrt{2}}\int d\theta \implies \frac{1}{\sqrt{2}}\theta+c$$

$$\implies \frac{1}{\sqrt{2}}\sin^{-1}\left(\frac{\sqrt{2}}{\sqrt{3}}x\right)+c$$

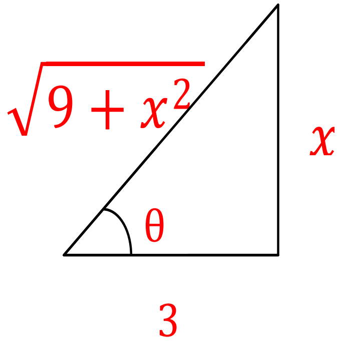
$$Ex: \int \frac{dx}{\sqrt{9+x^2}} \quad , \qquad a = 3\ , b = 1 \ \Rightarrow x = 3\ tan\theta \ \Rightarrow \mathrm{d}x = 3sec^2\theta\ d\theta$$

$$tan\theta = \frac{x}{3} \quad \Rightarrow \quad \theta = \tan^{-1}\left(\frac{x}{3}\right)$$

$$\int \frac{3sec^2\theta\ d\theta}{\sqrt{9+9tan^2\theta}} \ \Rightarrow \ \int \frac{3sec^2\theta\ d\theta}{3\sqrt{1+tan^2\theta}} \ \Rightarrow \ \int \frac{sec^2\theta\ d\theta}{\sqrt{sec^2\theta}}$$

$$\int sec\theta\ d\theta \qquad \Rightarrow \ \ln|(\sec\theta + \tan\theta)| + c$$

$$\Rightarrow \ \ln\left|\frac{\sqrt{9+x^2}}{3} + \frac{x}{3}\right| + c$$



ثانياَ:طريقة إكمال المربع

$$Ex: \int \frac{dx}{4x^2+4x+2}$$

$$4x^2+4x+2 \implies 4\left(x^2+x+\frac{1}{2}\right)$$

$$4\left[\left(x^2+x+\frac{1}{4}\right)+\frac{1}{2}-\frac{1}{4}\right] \implies 4\left[\left(x+\frac{1}{2}\right)^2+\frac{1}{4}\right]$$

$$4\left[\left(\frac{2x+1}{2}\right)^2+\frac{1}{4}\right] \implies (2x+1)^2+1$$

$$u = 2x + 1 \implies du = 2\,dx$$

$$(2x+1)^2 + 1 \implies u^2 + 1$$

$$\int \frac{\frac{1}{2}\mathrm{du}}{u^2+1} \implies \frac{1}{2}\tan^{-1} u + c \implies \frac{1}{2}\tan^{-1}(2x+1) + c$$

Ex: $\int \frac{x+1}{\sqrt{2x-x^2}}\,dx$

$$2x - x^2 \implies -(x^2 - 2x) \implies -(x^2 - 2x + 1 - 1)$$

$$\implies -[(x^2 - 2x + 1) - 1] \implies 1 - (x^2 - 2x + 1)$$

$$\implies 1 - (x-1)^2 \implies 1 - u^2$$

$$\mathrm{u} = x - 1 \;\Rightarrow\; \mathrm{du} = \mathrm{d}x \;, \qquad x + 1 \;\Rightarrow\; \mathrm{u} + 1 + 1 \;\Rightarrow\; \mathrm{u} + 2$$

$$\Rightarrow \int \frac{u+2}{\sqrt{1-u^2}}\mathrm{du} \;\Rightarrow\; \int \frac{u}{\sqrt{1-u^2}}\mathrm{du} + 2\int \frac{\mathrm{du}}{\sqrt{1-u^2}}$$

$$\Rightarrow -\frac{1}{2}\int -2u(1-\mathrm{u}^2)^{-\frac{1}{2}}\,\mathrm{du} \quad + 2\sin^{-1}\mathrm{u} + \mathrm{c}$$

$$\Rightarrow -\frac{1}{2}\frac{\sqrt{1-u^2}}{\frac{1}{2}} + 2\sin^{-1}\mathrm{u} + \mathrm{c}$$

$$\Rightarrow -\sqrt{1-u^2} + 2\sin^{-1}\mathrm{u} + \mathrm{c} \;\Rightarrow -\sqrt{1-(x-1)^2} + 2\sin^{-1}(x-1) + \mathrm{c}$$

ثالثاَ: طريقة تجزئة الكسور partial fractions

الطريقة الأولى: اذا كان المقام يتجزأ الى عوامل مختلفة بدون تكرار من الدرجة الأولى.

$$Ex:\ I=\int \frac{5x-3}{x^2-2x-3}dx$$

$$\frac{5x-3}{x^2-2x-3}=\frac{5x-3}{(x+1)(x-3)}=\frac{A}{(x+1)}+\frac{B}{(x-3)}$$

$$\Longrightarrow \frac{5x-3}{(x+1)(x-3)}=\frac{A(x+1)+B(x-3)}{(x+1)(x-3)}$$

$$5x-3=Ax-3A+Bx+B=x(A+B)-3A+B$$

$$5=A+B---(1),\qquad -3=-3A+B---(2)\ \Longrightarrow A=2\ ,B=3$$

$$I=\int \frac{A}{(x+1)}dx+\frac{B}{(x-3)}dx$$

$$\Longrightarrow \int \frac{2}{(x+1)}dx+\frac{3}{(x-3)}dx \Longrightarrow 2ln|x+1|+3ln|x-3|+c$$

$$Ex:\ I = \int \frac{3x+5}{x^3 - x^2 - x + 1}\,dx$$

$$\Longrightarrow x^3 - x^2 - x + 1 \Longrightarrow (x^3 - x^2) - (x - 1)$$

$$\Longrightarrow x^2(x-1) - (x-1) \Longrightarrow (x-1)(x^2-1)$$

$$\Longrightarrow (x-1)(x-1)(x+1) \Longrightarrow (x+1)(x-1)^2$$

$$\frac{3x+5}{x^3 - x^2 - x + 1} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$$

$$\Longrightarrow \frac{A(x-1)^2 + B(x+1)(x-1) + C(x+1)}{(x+1)(x-1)^2}$$

$$3x + 5 = Ax^2 - 2Ax + A + Bx^2 - B + Cx + C$$

$$\left.\begin{array}{r} 0 = \mathrm{A} + \mathrm{B} \\ 3 = -2\mathrm{A} + \mathrm{C} \\ 5 = \mathrm{A} - \mathrm{B} + \mathrm{C} \end{array}\right\} \Longrightarrow A = \frac{1}{2}\ , B = -\frac{1}{2}\ , C = 4$$

$$I = \int \frac{\frac{1}{2}}{x+1}\,dx + \int \frac{-\frac{1}{2}}{x-1}\,dx + \int \frac{4}{(x-1)^2}\,dx$$

$$\Longrightarrow \frac{1}{2} ln|x+1| - \frac{1}{2} ln|x-1| - \frac{4}{x-1} + c$$

الطريقة الثالثة: اذا تحلل المقام الى عوامل بعضها من الدرجة الثانية.

$$Ex:\ I=\int\frac{4-2x}{(x^2+1)(x-1)^2}dx$$

$$\frac{4-2x}{(x^2+1)(x-1)^2}=\frac{\mathrm{A}x+\mathrm{B}}{x^2+1}+\frac{\mathrm{C}}{x-1}+\frac{\mathrm{D}}{(x-1)^2}$$

$$4-2x=(\mathrm{A}x+\mathrm{B})(x-1)^2+C(x^2+1)(x-1)+D(x^2+1)$$

$$\Longrightarrow\ A=2\ ,B=1\ ,C=-2\ ,D=1$$

$$\Longrightarrow I=\int\frac{2x+1}{(x^2+1)}dx+\int\frac{-2}{x-1}dx+\int\frac{dx}{(x-1)^2}$$

$$\Longrightarrow I=\int\frac{2x}{(x^2+1)}dx+\int\frac{dx}{(x^2+1)}-2ln|x-1|-\frac{1}{x-1}+c$$

$$\Longrightarrow I=ln|x^2+1|+\tan^{-1}x-ln|x-1|-\frac{1}{x-1}+c$$

Integration by parts رابعاَ: طريقة التجزئة

$$\int u\, dv \;=\; u\, v - \int v\, du$$

$Ex:\ \int \sin^{-1} x\, dx$

$$u = \sin^{-1} x \;\Longrightarrow\; du = \frac{dx}{\sqrt{1-x^2}} \;,\; \mathrm{d}v = \mathrm{d}x \;\Longrightarrow v = x$$

$$\int \sin^{-1} x\, dx \;=\; x\ \sin^{-1} x - \int \frac{x}{\sqrt{1-x^2}} dx$$

$$\int \sin^{-1} x\, dx \;=\; x\ \sin^{-1} x + \frac{1}{2} \int -2x(1-x^2)^{-\frac{1}{2}} dx$$

$$\int \sin^{-1} x\, dx \;=\; x\ \sin^{-1} x + \frac{1}{2}\ \sqrt{1-x^2} + \mathrm{c}$$

$$Ex:\ I = \int \sec^{-1} x\, dx$$

$$u = \sec^{-1} x \ \Longrightarrow\ du = \frac{dx}{x\sqrt{x^2 - 1}}\ ,\ \ \mathrm{d}v = \mathrm{d}x \ \Longrightarrow v = x$$

$$\mathrm{I} = x\ \sec^{-1} x - \int \frac{x}{x\sqrt{1 - x^2}} dx = x\ \sec^{-1} x - \int \frac{dx}{\sqrt{1 - x^2}}$$

$$\mathrm{I} = x\ \sec^{-1} x - I_1$$

I_1 : يحل بطريقة التعويض بالدوال المثلثية حسب صيغة (١)

$$\mathrm{H.W:}\ \int \cos^{-1} x\, dx\ ,\int \tan^{-1} x\, dx\, ,\int \cot^{-1} x\, dx\, ,\int \csc^{-1} x\, dx$$