Differential Topology

PART(1)

Smooth manifold.

1-smooth structure and smooth manifold.

Definition:.

Let $U, V \subseteq \mathbb{R}^m$ are open subset. A map $f: U \longrightarrow V$ is called a differentiable of class c^r , if the functions $f_i = f_i(x^1, \dots x^m); i = 1, \dots n$ have the partial derivative up to order r.

i.e. $f(x^1,\dots,x^m)$=$f_1 1(x^1, \dots, x^m), \dots f_n(x^1, \dots x^m)$ have $\frac{\partial^r f_k}{\partial^{r_1} x_1, \dots \partial^{r_s} x_m}$ $, r_1 + \cdots + r_s \quad = r.$

Definition:

If f is differentiable and bijective and f^{-1} differentiable ,then we say that f is diffeomorphisim U on to V and then U and V are said to be diffeomorphic .

Remark:

We will assume that $r = \infty$ and in this case , we say that f is a smooth.

Definition:

Suppose that M is a housdorff space ,an open chart of dimension $n \ in \ M$

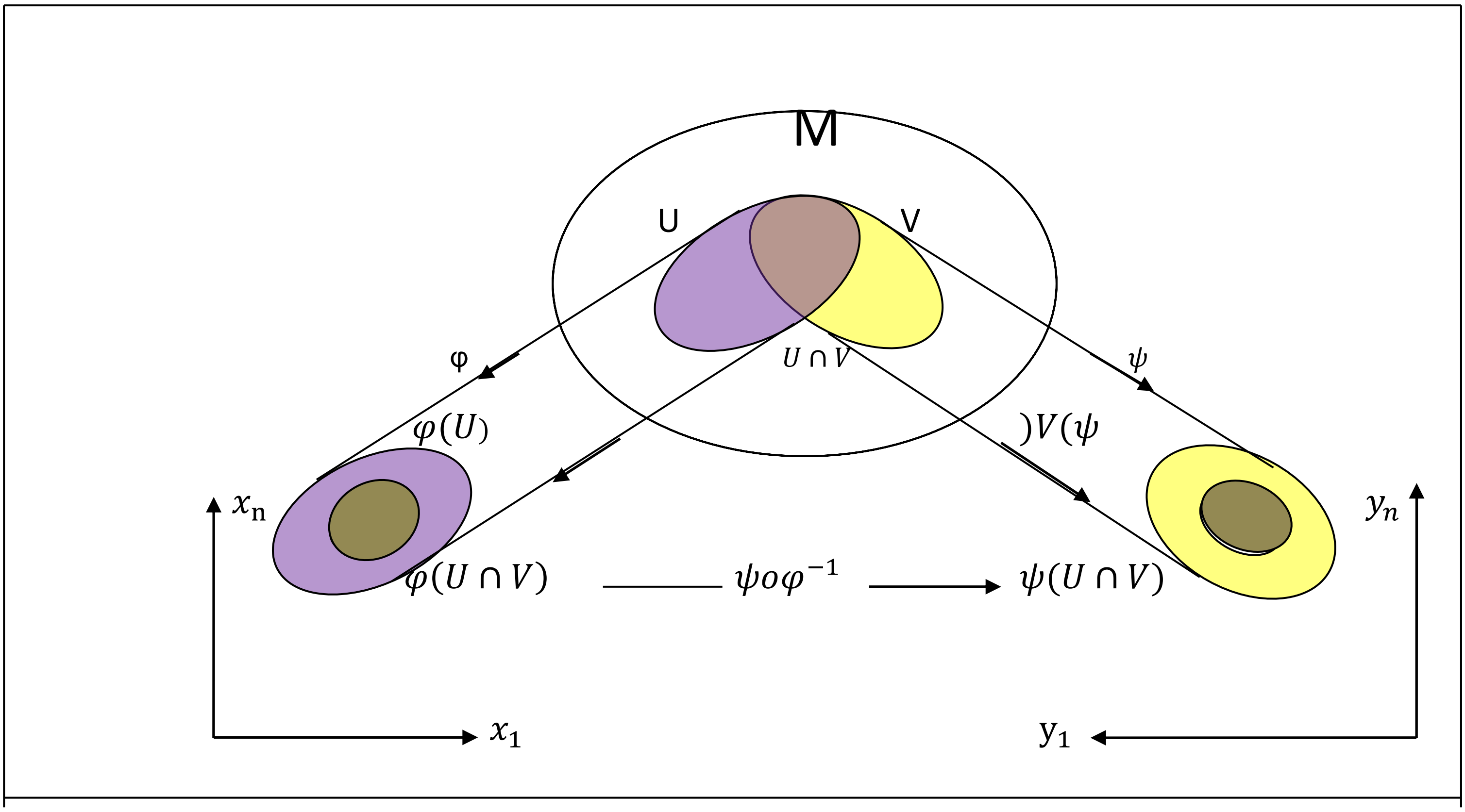
Is a pair (U, φ), where U is an open set in M and $\varphi: U \longrightarrow \varphi(U) \subseteq \mathbb{R}^n$ is a homeomorphisim on open sub set of $\mathbb{R}^n$ for $\in U$, $\varphi(p) = (x^1,\dots,x^n)$ are said to be a local coordinants of the point $p \in U$.

Definition:

Two charts $(U, \varphi), (V, \psi)$,are said to be a smooth connection, if the map :

$$\psi o \varphi^{-1}: \phi(U \cap V) \longrightarrow \psi(U \cap V)$$

is diffeomorphisim in Euclidean space.



Definition:

A family of n-dimensional charts on M $\{(U_i,\varphi_i)\}$ is called an atlas if

1) All charts are to be pairwise smooth connection;

2)$\cup_{i\in I} U_i = M.$

Definition:

An atlas on M is said to be a maximal , if every chart on M which is a smooth connection with each chart of this atlas ,then its belong to this atlas. i.e.atlas in M is maximal if its no contained in any other atlas.

Definition:

A maximal atlas is called a smooth structure .

Definition:

A space M with smooth structure is called a smooth manifold.

Definition:

Two smooth structure on M are said to be equivalent , if each two charts are smooth connection.

Example:(about non-equivelent smooth structure)

Suppose **M=**$\mathbb{R}$**,**

$$(\boldsymbol{U}, \boldsymbol{\varphi}); \boldsymbol{U} = \mathbb{R}\,, \boldsymbol{\varphi} = \boldsymbol{id};$$

$$(\boldsymbol{V}, \boldsymbol{\varphi}); \boldsymbol{V} = \mathbb{R}, \boldsymbol{\psi} = \boldsymbol{x^3}$$

Exampls:(about smooth manifolds)

1) M=$\mathbb{R}^n$ Euclidean space is an n dimensional smooth manifold.

A maximal atlas consist of one chart $(\mathbb{R}^n, 1_{\mathbb{R}^n})$; where $1_{\mathbb{R}^n}: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is the identity map.

2) Every finite dimensional linear space is a smooth manifold.

Suppose that V is an n dimensional linear space;

Let $\alpha = (e_1, \dots, e_n)$ be a basis for V. consider a map :

$$\varphi_\alpha: V \longrightarrow \mathbb{R}^n; \varphi_\alpha(x) = (x^1, \dots, x^n)$$

Where $(x^1, \dots, x^n)$ are the coordenentes of the vector $x \in V$ in the basis α. Clearly,φ_α is bijective and then is a homeomorphisim .

If $\beta = (\epsilon_1, \dots, \epsilon_n)$ is an another basis in V and $\varphi_\beta: V \longrightarrow \mathbb{R}^n$ given by $\varphi_\beta(x) = (x^1, \dots, x^n)$; $x^1, \dots, x^n$ are the coordenentes of the vector $x \in V$ in the basis β

Now,$\varphi_{\alpha\beta} = \varphi_\beta o \varphi_\alpha: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ givis the equation :

$$Y^i = C^i_j X^j; i = 1, \dots, n,$$

where (C^i_j) is the transition matrix from the basis β to the basis α .

Clearly that $\varphi_{\alpha\beta}$ is diffeomorphisim .Therefore, every basis α in V generats a chart (V, φ_α) in V will be atlas in V and then defind a smooth structure. Therefore, V is n dimensional smooth manifold .

Remark:

Suppose that M and N are smooth manifolds of dimensions m and n respectively with the smooth structures :

$\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in A}$ and $\{(V_\beta, \psi_\beta)\}_{\beta\in B}$.

Then, the set $M \times N$ has smooth structures which is generated by the family $\{(W_{\alpha\beta}, X_{\alpha\beta})\}_{\alpha\in A,\beta\in B}$.

Where $W_{\alpha\beta} = U_\alpha \times V_\beta$, $X_{\alpha\beta} = \varphi_\alpha \times \psi_\beta$.

The smooth manifold $M \times N$ will be of dimension n+m which is called a product manifold of M and N.

Clearly, this structure can be extended for any number of manifolds.

For example ,the product of any n copy of S^1 is n dimensional smooth manifold which is denoted by T^n and called n dimensional torus .

2-Algebra of smooth functions on smooth manifold .

Definition:

Let M^n be a smooth manifold. A map $f: M \longrightarrow \mathbb{R}$ is called a smooth function on M ,if for any chart (U, φ) on M , the map $fo\varphi^{-1}\varphi(U) \longrightarrow \mathbb{R}$ is smooth map of Euclidean space. Denote by $C^\infty(M)$ to the set of all smooth functions on M .

The set $C^\infty(M)$ will be algebra over field $\mathbb{R}$ with operations:

1) $(f + g)(p) = f(p) + g(p)$;
2) $(\lambda f)(p) = \lambda f(P)$;
3) $(f.g)(p) = f(p).g(p)$.

Where $p \in M,\ f, g \in C^\infty(M), \lambda \in \mathbb{R}$.

Algebra $C^\infty(M)$ is called an algebra of smooth functions on the manifold M .

3-Vector field on smooth manifold .

Definition:

Suppose that $C^\infty(M)$ is algebra of smooth functions on M . A linear operator $X: C^\infty(M) \longrightarrow C^\infty(M)$ is called a differentiation of algebra $C^\infty(M)$

If $X(f.g) = X(f).g + f.X(g)$; $f, g \in C^\infty(M)$ .By the another way the map $X: C^\infty(M) \longrightarrow C^\infty(M)$ is called a differentiation of algebra $C^\infty(M)$ if :

1) $X(f + g) = X(f) + X(g)$;

2) $X(\lambda f) = \lambda X(f)$;
3) $X(f.g) = X(f).g + f.X(g)$.

Where , $g \in C^{\infty}(M)$, $\lambda \in \mathbb{R}$.

Clearly that the set $X(M)$ of all differentiations of algebra $C^{\infty}(M)$ represent a module over a ring $C^{\infty}(M)$ with operations:

1) $(X + Y)(f) = X(f) + Y(f)$;
2) $(gX)(f) = g.X(f)$.
   Where $X, Y \in X(M)$, $f, g \in C^{\infty}(M)$.

Definition:

A differentiation $X(M)$ of algebra $C^{\infty}(M)$ is called a smooth vector field on a smooth manifold M.$C^{\infty}(M)$-module $X(M)$ is called a module of smooth vector fields on manifold M .

Proposition:

Let M be a smooth manifold , $X \in X(M)$,$X(C) = 0$; C is a constant.

Theorem:

Let (U, φ) be a local chart with coordinates $\{x^1, \dots, x^n\}$ on a smooth manifold M ,then the module $X(U)$ generated by $\{\frac{\partial}{x^1}, \dots, \frac{\partial}{\partial x^n}\}$, in particular,$\forall X \in X(M)$ then $X = \sum_{i=1}^{n} X^i \frac{\partial}{\partial x^i}$.

Definition:

The basis $\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}$ of the module $X(U)$ is called a canonical basis.

4- Tangent vectors and tangent space

Definition:

Let M be a smooth manifold and let $p \in M$. A tangent vector X_p at p is a map

$$X_p : C^{\infty}(U) \longrightarrow \mathbb{R}$$

Where U is an open neighborhood of p ,such that:

1) $X_p(\lambda f+\mu g)=\lambda X_p(f)+\mu X_p(g)$;
2) $X_p(f.g)=X_P(f).g(p)+f(p)X_p(g)$;

for each $\lambda,\mu\in\mathbb{R}$, $f,g\in C^{\infty}(U)$

The set of all tangent vectors at p is called the tangent space M at p and denoted by T_p(M)

Note:

The tangent space T_p(M) carries natural operations + and . turning it in to real vector space. i.e. For all X_p ,Y_p $\in$ T_p(M) $f\in C^{\infty}(M),\lambda\in\mathbb{R}$,

1) $(X_p+Y_p)(f)=X_p(f)+Y_p(f)$
2) $(\lambda X_p)(f)=\lambda X_p(f)$.

Theorem:

The giving of vector field $X\in X(M)$ on n – dimension of smooth manifold M equivalent to giving family of tangent vectors $\{X_p\in T_p(\mathrm{M})\}$ on M such that: in each local chart (U,φ) with coordinates $\{x^1,\dots,x^n\}$, the functions $X^i(p)=X_p(x^i)$ belong to algebra $C^{\infty}(U)$.

5- Lie algebra of vector fields on a smooth manifold .

Let M be a smooth manifold , $X(M)$ be module of it is vector fields,$X,Y\in X(M)$,its easy to check that $X\circ Y$ does not be vector field . if , $g\in C^{\infty}(M)$, then:

$$X\circ Y(f.g)=X(Y(f.g))$$
$$=X(Y(f).g+f.y(g))$$
$$=X(Y(f).g+Y(f).X(g)+X(f).Y(g)+f.X(Y(g))$$
$$=X\circ Y(f).g+Y(f).X(g)+X(f).Y(g)+f.X\circ Y(g)$$

In other hand if we change the vector fields X,Y we get :

$$Y\circ X(f.g)=Y\circ X(f).g+X(f).Y(g)+Y(f).X(g)+f.(Y\circ X)(g)$$

Now:

$$(X\circ Y-Y\circ X)(f.g)=(X\circ Y-Y\circ X)(f).g+f.(X\circ Y-Y\circ X)(g),$$

This mean that $(X \circ Y - Y \circ X) \in X(M)$. Denoted by $[X,Y] = (X \circ Y - Y \circ X)$

Definition:

The operation $\circ : X(M) \times X(M) \longrightarrow X(M)$ which is defined by $o(X,Y) = XoY - YoX = [X,Y]$ is called a commutator of X *and* Y and the symbole $[X,Y]$ is called a commutator or Lie bracket .

Proposition:

prove that :

1) $[X,Y] = -[Y,X]$;
2) $[[X,Y],Z] + \big[[Y,Z,]X\big] + \big[[Z,X],Y\big] = 0$.

Definition:

The pair $(X(M), o)$ is called a Lie algebra of smooth vector field of smooth manifold M .

Proposition:

The commutator operation is not$C^{\infty}(M) -$ linear map.

Remark:

We can write the commutator of vector fields X and Y in the local chart (U,φ) with coordinates $\{x^1, \dots, x^n\}$ as follows :

$$X = X^i \frac{\partial}{\partial x^i} \quad ; Y = Y^j \frac{\partial}{\partial x^j} \quad \text{, then,}$$

$$[X,Y]^i = [X,Y](x^i) = X\left(Y(x^i)\right) - Y\left(X(x^i)\right) = X(Y^i) - Y(X^i) = \frac{\partial Y^i}{\partial x^i} X^j - \frac{\partial X^i}{\partial x^j} Y^j$$

6-Tensor algebra of a smooth manifold .

Let V be a module over a commutative and associative ring K with identity or (K – module), V^* be dual module of K- linear functions on V with value in K . we have $V \cong V^{**} = (V^*)^*$ by the map $\tau: V \longrightarrow V^{**}$ which is defined by $\tau(x)(w) = w(x)\,; x \in V, w \in V^*$. The module V is called a reflexive if the map τ is isomorphisim .

Definition:

Let V be a reflexive K-module . consider a K-module τ_r^s(V) , the set of all maps $t: \underbrace{V \times V \times \dots \times V}_{r-times} \times \underbrace{V^* \times V^* \times \dots \times V^*}_{s-times} \longrightarrow K$,which are K-linear in every argument. It is element are called r-times covariante and s-times contravariants tensors of module V .For short are called tensors of type (r ,s).

Definition:

Define an operation $\otimes: \tau_{r_1}^{s_1}(V) \times \tau_{r_2}^{s_2}(V) \longrightarrow \tau_{r_{1+2}}^{s_{1+2}}(V)$ as follows:

If $t_1 \in \tau_{r_1}^{s_1}(V)$ and $t_2 \in \tau_{r_2}^{s_2}(V)$, then $t_1 \otimes t_2 \in \tau_{r_{1+2}}^{s_{1+2}}(V)$ such that :

$(t_1 \otimes t_2)(u_1, \dots, u_{r_1+r_2}, v^1, \dots, v^{s_1+s_2}) =$
$t_1(u_1, \dots u_{r_1}, v^1, \dots, v^{s_1}) t_2(u_{r_1+1}, \dots, u_{r_1+r_2}, v^{s_1+1}, \dots, v^{s_1+s_2})$, where $u_1, \dots, u_{r_1+r_2} \in V$ and $v^1, \dots, v^{s_1+s_2} \in V^*$

Theorem:

The operation $\otimes$ has the following properties :

1) $t_1 \otimes (t_2 + t_3) = t_1 \otimes t_2 + t_1 \otimes t_3$;
2) $(t_1 + t_2) \otimes t_3 = t_1 \otimes t_3 + t_2 \otimes t_3$;
3) $t_1 \otimes (t_2 \otimes t_3) = (t_1 \otimes t_2) \otimes t_3$.

The operation $\otimes$ which has the above properties , is called a tensor product .

The K-module $\tau(V) \cong \oplus_{r=0}^{\infty} \oplus_{s=0}^{\infty} \tau_r^s(V)$ with tensor product is called a tensor algebra.

Remark:

From the reflexivity of V ,we have:

$\tau_0^1(V) = \{t: V^* \longrightarrow K\} = V^{**} = V$, $\tau_1^0(V) = \{t: V \longrightarrow K\} = V^*$.

Definition:

From the above definition we get that the sub modules $\tau_*(V) = \oplus_{r=0}^{\infty} \tau_r^0(V)$ and $\tau^*(V) = \oplus_{s=0}^{\infty} \tau_0^s(V)$ represente sub algebras of the tensor algebra $\tau(V)$ and thy called covariant and contravariante tensor algebra of V respectively .

Remark:

If V is a finite linear space over a field K and $\{e_1, \dots, e_n\}$ is any basis of V , $\{e^1, \dots, e^n\}$ is a dual basis , then,

$$e^i(e_j) = \delta_j^i = \begin{cases} 1 & If \ \ i = j \\ 0 & If \ \ i \neq j \end{cases} \text{ , then,}$$

The tensor of the forms :

$$e_{i_1} \otimes \dots \otimes e_{i_r} \otimes e^{j_1} \otimes \dots \otimes e^{j_s} \,;$$

$i_1, \dots, i_r \;\; , j_1, \dots, j_r = 1, \dots, n$, are basis of linear space $\tau_r^s(V)$. In particular, we have :

$$\dim \tau_r^s(V) = n^{r+s}$$

Remark:

The coordinates $\{t_{i_1,\dots,i_r}^{j_1,\dots,j_s}\}$ of the tensor $t \in \tau_r^s(V)$ in this basis are equal to it is components . i.e. ,

$$t_{i_1,\dots,i_r}^{j_1,\dots,j_s} = t\left(e_{i_1}, \dots, e_{i_r}, e^{j_1}, \dots, e^{j_s}\right).$$

Definition:

Let M be n-dimensional smooth manifold , $p \in M$, then, the tensor algebra $\tau(T_P(M))$ denoted by $\tau_p(M)$ and is called a tensor algebra of the manifold M at the point p .In the other hand $\tau(X(M))$ denoted by $\tau(M)$ and is called a tensor algebra of manifold M .The element of the tensor algebra are called a tensor fields .

Definition:

A dual of the module $X(M)$ is called a module of differential 1-form on manifold M , and is denoted by $X^*(M)$;

$$X^*(M) = \{t: X(M) \longrightarrow \mathbb{R}\}.$$

Theorem:

The giving of tensor $t \in \tau_r^s(M)$ on smooth manifold M equivalent to the giving family of tensors$\{t_p \in \tau_r^s(M); p \in M\}$ such that, in each local chart (U, φ) with coordinates $\{x^1, \dots, x^n\}$ on M , the functions,

$$t_{i_1,\dots,i_r}^{j_1,\dots,j_s}(p) = t\left(\frac{\partial}{\partial x^{i_1}}\Big|_p, \dots, \frac{\partial}{\partial x^{i_r}}\Big|_p, w_p^{j_1}, \dots\, w_p^{j_s}\right) .$$

Where$\{w_p^1, \dots, w_p^n\}$ is the dual basis of the canonical basis of the space$T_p(M)$ at the point $\in M$.

7-Grassman algebra of smooth manifold.

Operator of exterior differentiation .

Let $T_*(V) = \oplus_{r=0}^{\infty} \tau_r^0(V)$ be the covariant tensor algebra of reflexive K-module V . In the module $\tau_r^0(V)$ acts symmetric group s_r of order r (permetation group) as the following:

If module $t \in \tau_r^0(V), \sigma \in s_r$, then$(\sigma t)(x_1,\ldots,x_r) = t(x_{\sigma(1)}, \ldots, x_{\sigma(r)})$.

Definition:

The tensor $t \in \tau_r^0(V)$ is called a symmetric, if $\forall \sigma \in s_r$,then $\sigma t = t$ and the tensor $t \in \tau_r^0(V)$ is called antisymmetric if for each $\sigma \in s_r$, then $\sigma t = \mathcal{E}(\sigma)$ where $\mathcal{E}(\sigma)$ is the sign of permetation which equal to 1 for even permetation and -1 for odd permetation:

$$\mathcal{E}(\sigma)=\begin{cases} 1 \; if \;\; \sigma \; even \\ -1 \; if \;\; \sigma \; odd \end{cases}$$

Note:

Clearly that the symmetric and antisymmetric tensors are submodules of the module $\tau_r^0(V)$ and we will denote them by $S_r(V)$ and $\wedge_r (V)$ respectivly .

Definition:

Define endomorphisims Sym and Alt of $\tau_r^0(V)$ as follows:

$$Sym(t) = \frac{1}{r!}\sum_{\sigma \in s_r} \sigma t \;\; ; Alt = \frac{1}{r!}\sum_{\sigma \in s_r} \varepsilon(\sigma) t \; .$$

Which are projections ;on modules $S_r(V)$ and $\wedge_r (V)$ respectivly , and are called symmetric and alterative operators .Define an operation as follows:

$$\wedge : \wedge_r (V) \times \wedge_S (V) \longrightarrow \wedge_{r+s} (V) \; .$$

If $w_1 \in \wedge_r (V)$, $w_2 \in \wedge_S (V)$, then $w_1 \wedge w_2 \in \wedge_{r+s} (V)$ which is defined by the form:

$$w_1 \wedge w_2 = \frac{(r+s)!}{r!\, s!} Alt(w_1 \otimes w_2) \; .$$

Proposition:

Prove that :

1) $(w_1 + w_2) \wedge w_3 = w_1 \wedge w_3 + w_2 \wedge w_3;$
2) $w_1 \wedge (w_2 + w_3) = w_1 \wedge w_2 + w_1 \wedge w_{3;}$

3) $w_1 \wedge (w_2 \wedge w_3) = (w_1 \wedge w_2) \wedge w_3$.

Definition:

The operator $\wedge$ is called an exterior product .

Let $\wedge (V) = \oplus_{r=0}^{\infty} \Lambda_r(V)$, where $\wedge_0 (V) = K$, and $\wedge_1 (V) = V^*$.

$\wedge (V)$ with operation $\wedge$ is called an exterior algebra .

Remark:

If $\wedge_r (V)$, then, $w: \underbrace{V \times \ldots \times V \longrightarrow K}_{r-times}$, which is called a form of degree r or r-form.

Definition:

Let V be an n-dimensional linear space over a field K ,$\{e_1, \ldots, e_n\}$ be a basis of V , then the r-forms:

$e_{i_1} \wedge \ldots \wedge e_{i_r}$ are basis of the module $\wedge_r (V)$. The coordinates $\{w_{i_1,\ldots,i_r}\}$ of r-form $w \in \wedge_r (V)$, in this basis, considens with it is components, i.e.

$$w_{i_1,\ldots,i_r} = w(e_{i_1}, \ldots, e_{i_r})$$

Clearly that $\dim \wedge_r (V) = \binom{n}{r} = \frac{n!}{r!(n-r)!}$.

Definition:

Suppose that M is smooth manifold . Exterior algebra $\wedge (X(M))$ denoted by $\wedge (M)$ which is called a Grassman algebra of smooth manifold M .It is elements are called differential form .

Theorem:

Suppose that M is a smooth manifold, then there exist a unique mapping:

$$d: \wedge (M) \longrightarrow \wedge (M)$$

With the following properties:

1) $d(\wedge_r (M) \subset \wedge_{r+1} (M)$;
2) $df(X) = X(f)$;
3) $d \circ d = 0$;

4) $d(w_1 \wedge w_2) = dw_1 \wedge w_2 + (-1)^r w_1 \wedge dw_2$,

Where $w_1 \in \wedge_r (M)$, $w_2 \in \wedge (M)$.

Definition:

The operator d which has the above properties is called the operator exterior differentiation .

Proposition:

Suppose that M is a smooth manifold , (U, φ) is a local chart with coordinates$\{x_1, \dots, x_n\}$ on M and $\{\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\}$ is the canonical basis for the module $X(U)$,then the differential 1-forms $\{dx_1, \dots, dx_n\}$ is the dual basis of the canonical basis of $X(U)$

8- Smooth map,differential of smooth map.

Proposition:

Suppose that M and N are smooth manifolds, a map $\phi: M \longrightarrow N$ is called a smooth , if$\forall f \in C^\infty(N)$, then$f \circ \phi \in C^\infty(M)$.

Remark:

The above definition equivalent to the following:

A map $\phi: M \longrightarrow N$ is called a smooth , if for each chart (U, φ) on M and (V, ψ) on N with coordinates $\{x_1, \dots, x_n\}$ and $\{y_1, \dots, y_n\}$ respectivly, then, the map :

$$\psi o \phi o \varphi^{-1}: \varphi(U) \longrightarrow \psi(V)$$

Is a smooth of Eucledian space .

Note:

If the smooth map ϕ is the bijective such that the map ϕ^{-1} is smooth, then the map ϕ is called a diffeomorphisim.

Definition:

Let $p \in M$, define a map:

$(\phi_*)_p: T_p(M) \longrightarrow T_{\phi(p)}(N)$ as follows:

Let $\in T_p(M)$, $(\phi_*)_p(\xi)(f) = \xi(fo\phi)$; $f \in C^\infty(N)$. The map $(\phi_*)_p$ is called a differential map of the smooth map ϕ .Note that $(\phi_*)_p(\xi) \in T_{\phi(p)}(N)$.

9- $\phi -$ connection of vector fields.

Definition:

Let $\phi: M \longrightarrow N$ be a smooth map, the vector fields $X \in X(M)$, $Y \in X(N)$ is called $\phi -$ connection ,if $\forall f \in C^\infty(N)$, $then\ X(fo\phi) = Y(f)o\phi$.

Theorem:

The vector field X and Y are $\phi -$ connection iff $\forall p \in M$, then $(\phi_*)_p X_p = Y_{\emptyset(P)}$.

Remark:

We will dente by $Y = \phi_* X$.

Definition:

The vector field $Y = \phi_* X$ is called a dragging of the vector field X with respect to the map .

H.W.:

If $Y_1 = \phi_* X_1$ and $Y_2 = \phi_* X_2$, then, Prove that: $[Y_1, Y_2] = \phi_*[X_1, X_2]$.

Remark:

By the same way for the vector field $\in X(M)$, we can define the dragging $\phi_* X$, where $\phi_*: X(M) \longrightarrow X(N)$, then $\phi_*^{-1} = \phi^*: X(N) \longrightarrow X(M)$ which is called an anti-dragging of the vector field.

10-Distribution and co distribution .

Definition:

A sub module D of the module $X(M)$ is called a distribution on M . The distribution D is called r-dimensional , if there exist atlas on M such that each chart (U, φ), then,

$$D|U = \{X|D : X \in D\}$$

Is a module of r-dimension .

Remark:

The giving of r-dimensional distribution on M is equivalent to the giving the family $\{D_P \subset T_P(M): dimD_p = r\}$.

Definition:

A sub module C of the module $X^*(M)$ is called a codistribution .

Definition:

Suppose that D is the distribution on M .The sub module :

$$C_D = \{w \in \wedge_1 (M): w(X) = 0 , \forall\, X \in D\}.$$

Is called a codistribution associated with the distribution .

Theorem:

If M is an n-dimensional smooth manifold , and $dimD = r$, then, $dimC_D = n - r$.

Proof:

Suppose that $\{X_1, \dots, X_r\}$ is a local basis for the distribution D . Compelet this basis to the basis $\{X_1, \dots, X_n\}$ for the module $X(M)$. Let $\{w^1, \dots, w^n\}$ be a dual basis . Let $w \in X^*(M)$, then,

$w = \sum_{i=1}^{n} a_i w^i$, where $a_i = w(X_i)$. We have :

$w \in C_D$ iff $w(X) = 0 , \forall\, X \in D$ iff $a_k = w(X_k) = 0 , k = 1, \dots, r$.

Then we get $= a_{r+1}w^{r+1} + \dots + a_n w^n$, since the form $\{w^{r+1}, \dots, w^n\}$ are linearly independent, then are will be basis of the module C_D.

Therefore, $dim\, C_D = n - r$.

11-Sub manifold of smooth manifold.

Definition:

Suppose that $\phi: N \longrightarrow M$ is a smooth function , the rank of ϕ at $p \in N$ is the rank of the$(\phi_*)_p: T_p(N) \longrightarrow T_{\phi(p)}(M)$. The dimension of range $(\phi_*)_p$ is called the rank of $(\phi_*)_p$.

Definition:

A smooth map $\phi: N \longrightarrow M$ is called an immersion if it is rank equal to the dimension of N .

Definition:

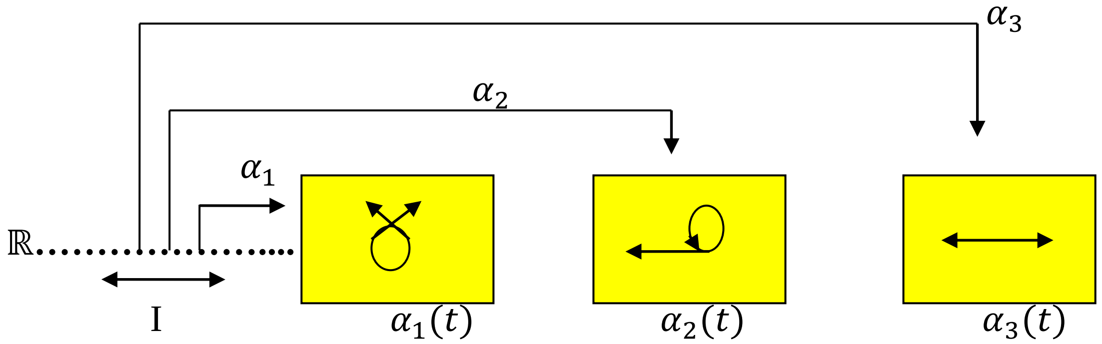
Suppose that $\phi: N \longrightarrow M$ is a smooth map, if ϕ is an immersion, then we say that the pair (N, ϕ) is an immbeding sub manifold. In this case, if ϕ is an injective, then the pair (N, ϕ)

Is called a sub manifold of M .

If (N, ϕ) is a sub manifold of M ,such that the map ϕ is an open , then we say that (N, ϕ) is an inclusion sub manifold of M and ϕ is called an inclusion map .

Example:

Let $N = I \subset \mathbb{R}$, $\alpha_i: I \longrightarrow M$ is a smooth curve ; i=1,2,3 , which are defined as following diagrams:



1) (I, α_1) is immbeding sub manifold , but not sub manifold ;
2) (I, α_2) is sub manifold , but not inclusion sub manifold ;
3) (I, α_3) is inclusion sub manifold .

PART(2) :

Lie group and Lie algebra .

1-Lie group:

Definition:

A Lie group is a group G which is al so smooth manifold such that, the map:

$$\phi: G \times G \longrightarrow G$$

Which is defined by:

$$\phi(x, y) = x . y^{-1}$$

Is a smooth $\forall\, x, y \in G$.

Proposition:

Suppose that G is a Lie group,then an operation $\alpha: G \longrightarrow G$ and $\alpha(x) = x^{-1}$ is a smooth .

Proof:

The map $\alpha: G \longrightarrow G$ can be written as the form:

$$x: \overset{i_e}{\rightrightarrows} (e, x) \overset{\varphi}{\rightrightarrows} e.x^{-1} = x^{-1}$$

Where e is the identity element of G. The map $\alpha = \varphi o i_e$ is a smooth , since $i_e$and φ are smooth .

Proposition:

The map $\mu: G \times G \longrightarrow G$, where $\mu(x, y) = x.y$ is a smooth .

Proof :(H.W).

Examples:

1) The space $\mathbb{R}^n = \{(x_1, \dots, x_n): x \in \mathbb{R}\}$ is a Lie group with respect to the operation $+$.

Solution:

Let$x = (x_i)$, $y = (y_i) \in \mathbb{R}^n$.

$(x_i).(y_i) = x_i + y_i$, and $(x_i)^{-1} = -x_i$, then $(x, y) = x.y^{-1} = x_i - y_i$.

Therefore, the map φ gives a smooth maps $u_i = x_i - y_i$, $i = 1, \dots, n$.

Hence, φ is a smooth map which means that $\mathbb{R}^n$ is a Lie group .

2) $\mathbb{C}^* = \{z \in \mathbb{C}\,; z \neq 0\}$, is a Lie group with respect to the complex product operation .

Solution:

Let $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$; $z = x + iy \in \mathbb{C}$, then,

$$z_1.z_2 = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)\ , z^{-1} = \frac{x - iy}{x^2 + y^2}\ .$$

$$\varphi(z_1, z_2) = x_1 + iy_1 \frac{x_2 - iy_2}{x_2^2 - y_2^2} = \frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2} + i\,\frac{-x_1y_2 + y_1x_2}{x_2^2 + y_2^2},$$

The map φ gives a smooth functions ,

$$u_1 = \frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2} \ , u_2 = \frac{-x_1y_2 + y_1x_2}{x_2^2 + y_2^2}.$$

3) Let G_1 and G_2 be a Lie groups, then the smooth manifold:

$$G_1 \times G_2 = \{(g_1, g_2) : g_1 \in G_1 \wedge g_2 \in G_2\}$$

Is a Lie group with respect to components of groups operation:

$$(g_1, g_2).(h_1, h_2) = (g_1h_1, g_2h_2);$$

$$(g_1, g_2)^{-1} = (g_1^{-1}, g_2^{-1}).$$

<u>Karatan's theorem :</u>

Suppose that G is a Lie group, $A \subset G$ is a closed sub group of , then A is a Lie group .

4) Let $S^1 = \{z \in \mathbb{C}^* : |z| = 1\}, (S^1 \subset \mathbb{C}^*)$

<u>Solution</u>:

If $z_1, z_2 \in S^1 \Rightarrow |z_1.z_2| = |z_1|.|z_2| = (1)(1) = 1$, thus $z_1.z_2 \in S^1$.

If $z \in S^1 \Rightarrow |z^{-1}| = \frac{1}{|z|} = \frac{1}{1} = 1$, thus $z^{-1} \in S^1$,

Therefore, S^1 is a sub group of $\mathbb{C}^*$.

Let $\{z_n\}$ be a sequence in S^1 and $\lim_{n\to\infty} z_n = z$,

So $|z| = \left|\lim_{n\to\infty} z_n\right| = \lim_{n\to\infty} |z_n| = \lim_{n\to\infty} 1 = 1$.

Thus, $z \in S^1$, therefore S^1 is closed .

Hence , by Karatan's theorem , we get that S^1 is a Lie group .

5) General linear group .

GL(n,ℝ) $= \{A = (a_{ij}) \in M_{n,n} : detA \neq 0\}$.

<u>Solution</u>:

Clearly that $GL(n, \mathbb{R})$ is open sub set in $M_{n,n} \cong (\mathbb{R}^n)^2$, then, $GL(n, \mathbb{R})$ is a smooth manifold and group .

$$\varphi: GL(n,\mathbb{R}) \times GL(n,\mathbb{R}) \longrightarrow GL(n,\mathbb{R})$$

$$\varphi(A,B) = A.B^{-1} = C = (c_{ij})$$

$$c_{ij} = \sum_{k=1}^{n} a_{ik}(B_{kj})^{-1} = \sum_{k=1}^{n} a_{ik}\frac{(-1)^{k+j}\Delta_{jk}}{detB} = \sum_{k=1}^{n}\frac{(-1)^{k+j}a_{ik}\Delta_{jk}}{detB}$$

Where Δ_{jk} is the complement of B_{kj} ,

Clearly that c_{ij} are smooth functions, therefore, φ is a smooth .

Hence, $GL(n,\mathbb{R})$ is a Lie group.

6) Orthogonal group of order n .

$GL(n,\mathbb{R}): A^{-1} = A^T\}O(n,\mathbb{R}) = \{A \in$

Then, by Karatan's theorem, $O(n,\mathbb{R})$ is a Lie group .

7) Unimodule group $SL(n,\mathbb{R}) = \{A \in GL(n,\mathbb{R}): detA = 1\}$;

8) Spicial orthogonal group $SoL(n,\mathbb{R}) = O(n,\mathbb{R}) \cap SL(n,\mathbb{R})$;

9) Complex general linear group $GL(n,\mathbb{C}) = \left\{C = \left(c_{ij}\right): c_{ij} \in \mathbb{C}\, ; detC \neq 0\right\}$;

10) Complex orthogonal group $(n,\mathbb{C}) = \{C \in GL(n,\mathbb{C}): C^{-1} = C^T\}$;

11) Complex unimodule group $SL(n,\mathbb{C}) = \{C \in GL(n,\mathbb{C}): detC = 1\}$;

12) Complex orthogonal unimodule group $SoL(n,\mathbb{C}) = O(n,\mathbb{C}) \cap SL(n,\mathbb{C})$;

13) Unitary group $(n) = \{C \in GL(n,\mathbb{C}): C^{-1} = C^{-T}\}$.

Realization of general complex group.

$\mathrm{GL}(\mathrm{n},\mathbb{C})^{\mathbb{R}} = \{A \in M_{2n,2n}: A \circ J = J \circ A\}$.

Where $J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$, it is easy to check that $J^2 = -I_{2n}$,

$$J^2 = J.J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}.\begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} = \begin{pmatrix} -I_{2n} & 0 \\ 0 & -I_{2n} \end{pmatrix}.$$

Let $\in \mathrm{GL}(\mathrm{n},\mathbb{C})^{\mathbb{R}}; A = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}$, then,

$A \circ J = J \circ A \Longrightarrow \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} \circ \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} \circ \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} \Longrightarrow A_2 = -A_3 , A_1 =$ A_4. Therefore, we get $= \begin{pmatrix} A_1 & A_2 \\ -A_2 & A_1 \end{pmatrix}$.

If $C = A + \sqrt{-1}\, B$ then, $= \begin{pmatrix} A & -B \\ B & A \end{pmatrix}$.

$\mathrm{GL}(\mathrm{n}, \mathbb{C})^{\mathbb{R}} \subset GL(2n, \mathbb{R})$,closed sub group, and then by Karatan's theorem it will be Lie group.

Proposition:

$GL(n, \mathbb{C}) \cong \mathrm{GL}(\mathrm{n}, \mathbb{C})^{\mathbb{R}}$.

Solution:

Define φ: $GL(n, \mathbb{C}) \longrightarrow \mathrm{GL}(\mathrm{n}, \mathbb{C})^{\mathbb{R}}$ as :

If $= (c_{ij}) \in GL(n, \mathbb{C})$, where $(c_{ij}) = \alpha_{ij} + \sqrt{-1}\, \beta_{ij}$,

Consider matrices $A = (\alpha_{IJ})$ and $B = (\beta_{ij}) \in \mathrm{GL}(\mathrm{n}, \mathbb{C})^{\mathbb{R}}$,

And $C = (A + \sqrt{-1}\, B) \in GL(n, \mathbb{C})$, then $\varphi\big(A + \sqrt{-1}\, B\big) = \begin{pmatrix} A & -B \\ B & A \end{pmatrix}) \in \mathrm{GL}(\mathrm{n}, \mathbb{C})^{\mathbb{R}}$.

Prove that φ is an isomorphisim .

Semi – direct product of Lie grops.

Let $G = GL(n, \mathbb{R})$ and $H = \mathbb{R}^{\mathrm{n}}$ are be Lie groups .we know that $M = G \times H$ has Lie group structure , this Lie group is the direct product of Lie groups.But there is another Lie group structure :

Let $(A, X), (B, Y) \in GL(n, \mathbb{R}) \times \mathbb{R}^{\mathrm{n}}$, define the operation $*$ by:

$(A, X) * (B, Y) = (AB, AY + X)$ and $(A, X)^{-1} = (A^{-1}, -A^{-1}X)$.

Directly, from this operation we can prove that $M = G \times H$ is a group (check).

Define $\varphi: M \times M \longrightarrow M$ by:

$\varphi((A, X), (B, Y\,))= (A, X) * (B, Y)^{-1} = (A, X) * (B^{-1}, -B^{-1}Y) = (AB^{-1},, -AB^{-1}Y + X)$.

Then we get that φ is a smooth map .Threfore, $GL(n, \mathbb{R}) \rtimes \mathbb{R}^{\mathrm{n}}$ is a Lie group , and is called a semi- direct product of Lie groups $GL(n, \mathbb{R})$ and $\mathbb{R}^{\mathrm{n}}$.

2- Lie algebra .

Definition:

A space $\mathcal{G}$ over a field $\mathbb{F}$ is called a Lie algebra if the binary operation , $[.,.]: \mathcal{G} \times \mathcal{G} \longrightarrow \mathcal{G}$

Satisfies the following properties :

1) $[X,Y] = -[Y,X]$;
2) $\big[[X,Y],Z\big] + \big[[Y,Z],X\big] + \big[[Z,X],Y\big] = 0$.

Note:

We will assume that $= \mathbb{R}$.

Examples:

1) suppose that M is a smooth manifold , then the module $X(M)$ is a Lie algebra under operation :

$$[X,Y] = XoY - YoX \; ; X,Y \in X(M) \; .$$

2) Every arthemetic linear space V is a Lie algebra with $[X,Y] = 0$, $X,Y \in V$.

3) Every associative algebra A is a Lie algebra with respect to the operation :

$$[X,Y] = X.Y - Y.X \; ; \; X,Y \in A \; .$$

In particular, the general matrix algebra $[A,B]A.B - B.A$; $A,B \in M_{n,n}$

Where . is the product matrix operation .

3- Lie algebra of Lie group .

Definition:

Let G be a Lie group , $g \in G$, define maps :

$L_g: G \longrightarrow G$ and $R_g: G \longrightarrow G$ by $L_g(h) = g.h$, $R_g(h) = h.g$

L_g is called a left shift to element g .

R_g is called a right shift to element g.

The maps R_g and L_g are a smooth maps have the following properties :

1) $L_g o L_h = L_{gh} : G \longrightarrow G \,; g, h \in G$.
   $(L_g o L_h)(p) = L_g(L_h(p)) = L_g(hp) = g(hp) = (gh)p = L_{gh}(p)$.

Therefore , $L_g o L_h = L_{gh}$.

2) $R_g o R_h = R_{hg}$, $(H.W)$.
3) The maps L_g and R_g are diffeomorphisim . This can be get directly from,
   $\forall g \in G$, $(L_g)^{-1} = L_{g^{-1}}$ and $(R_g)^{-1} = R_{g^{-1}}$,
   Then, $L_g o (L_g)^{-1} = L_g o L_{g^{-1}} = L_{gog^{-1}} = L_e$, where e is the identity element of G,
   i.e $L_e\,(g) = e.g = g$.

By the same way , we have:

$$(L_g)^{-1} o L_g = L_{g^{-1}} o L_g = L_{g^{-1}g} = L_e \,.$$

Therefore, we get that L_g is isomorphisim, and since $L_g, L_{g^{-1}}$ are differentiables (smooth), then we get that L_g is diffeomorphisim .

Definition:

A vector field $X \in X(G)$ is called a left invariant ,if $\forall g \in G$, then $(L_g)_* X = X$, where $(L_g)_*: X(G) \longrightarrow X(G)$ is the differential map of the smooth map $L_g: G \longrightarrow G$.

Theorem:

A set $\mathcal{G}$ of all left invariant vector fields on Lie group G is a linear space which is isomorphic to the tangent space $T_e(G)$ of Lie group G at the identity.

Inparticular, $im\, \mathcal{G} = dim\, G$.

Definition:

The Lie algebra $\mathcal{G}(G)$ of all left invariant vector fields of Lie group G is called a Lie algebra of Lie group.

Proposition:

The linear space $\mathcal{G}$ of all left invariant vector fields of Lie group G is a Lie algebra with respect to the commutator operation of vector fields.

Proof: (H.W) .

4-Homomorphisim of Lie groups and Lie algebras .

Definition:

A map $\varphi: G \longrightarrow H$ of Lie groups is called a homomorphisim of Lie groups if:

1) φ is smooth ;
2) $\varphi(x, y) = \varphi(x)\varphi(y)\ , x, y \in G$.

Definition:

A ma $\phi: \mathcal{G} \longrightarrow \hbar$ of Lie algebras is called a homomorhpisim of algebras if:

1) ϕ is a linear map ;
2) $\phi[X, Y] = [\phi X, \phi Y]$; $X, Y \in \mathcal{G}$.

5-The action of Lie group on a smooth manifold.

Definition:

Let G be a Lie group and M be a smooth manifold, we say that G act differentially on M of the left, if there exist a smooth map $\varphi: G \times M \longrightarrow M$ which satisfies the following conditions:

1) $\forall g \in G$, the map $\varphi_g: M \longrightarrow M$ which defined by $\varphi_g(m) = \varphi(g, m) = gm$ is diffeomorphisim.
2) $\varphi_{gh}(m) = \varphi_g o \varphi_h(m) = \varphi_g\left(\varphi_h(m)\right) = g(hm) . \forall g, h \in G, m \in M$.

Note that $\varphi_e(m) = em = m$ where e is the identity element of G.

Definition:

We say that G acts effectively if satisfies : If $\varphi_g(m) = m\ , \forall m \in M$ then $g = e$.

And we say that G acts freely , if $\varphi_g(m) = m$ for some $m \in M$, then $g = e$.

Definition:

The Lie group G act on M of the right , if there exist a smooth map $\varphi: M \times G \longrightarrow M$ which satisfies the following conditions :

1) $\forall g \in G$, the map $\varphi_g: M \longrightarrow M$ which defined by $\varphi_g(m) = \varphi(m, g) = mg$ is diffeomorphisim.
2) $\varphi_{gh}(m) = \varphi_h o \varphi_g(m) = \varphi_h\left(\varphi_g(m)\right) = \varphi_h(mg) = (mg)h\ , \forall g, h \in G, m \in M$.

Example:

Suppose that V is n-dimensional linear space , denoted by β to the set of all basis of V . The Lie group $GL(n,\mathbb{R})$ acts on β of the right as the follows:

Let $= (e_1, \dots, e_n) \in \beta$, $g = (g_j^i) \in GL(n,\mathbb{R})$.

Put $\varphi_g(b) = (g_1^{i_1} e_{i_1}, \dots, g_n^{i_n} e_{i_n})$.

We know that $\varepsilon_i = g_i^j e_j, (i = 1, \dots, n)$,

Where (g_i^j) is the transition matrix from the basis $\{ e_1, \dots, e_n\}$ to the basis $\{\varepsilon_1, \dots, \varepsilon_n\}$

$$\varphi_g(b) \in \beta$$

Clearly that φ_g is bijective and the diffeomorphisim.

Let , $h \in GL(n,\mathbb{R})$, then,

$$\varphi_h o \varphi_g(b) = \varphi_h o \varphi_g(e_1, \dots, e_n)$$

$$= \varphi_h(\varepsilon_1, \dots, \varepsilon_n) = \left(h_1^{i_1} \varepsilon_{i_1}, \dots, h_n^{i_n} \varepsilon_{i_n}\right)$$

$$= \left(h_1^{i_1} g_{i_1}^{j_1} e_{j_1}, \dots, h_n^{i_n} g_{i_n}^{j_n} e_{j_n}\right)$$

$$= \left((gh)_1^{j_1} e_{j_1}, \dots, (gh)_n^{j_n} e_{j_n}\right)$$

$$= \varphi_{gh}(b)$$

Therefore, $\varphi_{gh} = \varphi_h o \varphi_g$.

Then the Lie group $GL(n,\mathbb{R})$ acts on β on the right .

PART(3)

Principle fiber bundle space .

1- Principle fiber bundle.

Definition:

Suppose that the Lie group G acts on smooth manifold M then for each $m \in M$ generates a map $\delta_m: G \longrightarrow M$, such that , for each $g \in G$, $\delta_m(g) = \varphi_g(m)$.

The image of $\delta_m$called an orbit of the point $m \in M$ .The set of all orbit will be denoted by $Orb_G M$ which is a smooth manifold .

Definition:

A principle fiber bundle is a set of four (P, M, Π, G) , where P is a smooth manifold, and G is a Lie group which is acts freely on P of the right , M=Orb_G P is the space of the orbits.

Π: $P \longrightarrow M$, is a projection (which is smooth), such that the following are satisfies:

There is an open cover U of M, such that ,

$$\forall u \in U, \exists\, F_u: \Pi^{-1}\,(U) \longrightarrow G$$

Where (F_u is a smooth map) satisfies the conditions:

1) $F_u(pg) = F_u(p)g$; $(p \in P, g \in G)$;
2) The map ψ_u: F_u: $\Pi^{-1}\,(U) \longrightarrow U \times G$ satisfies:

$\psi_u(p) = (\Pi(p), F_u(p))$ is diffeomorphisim.

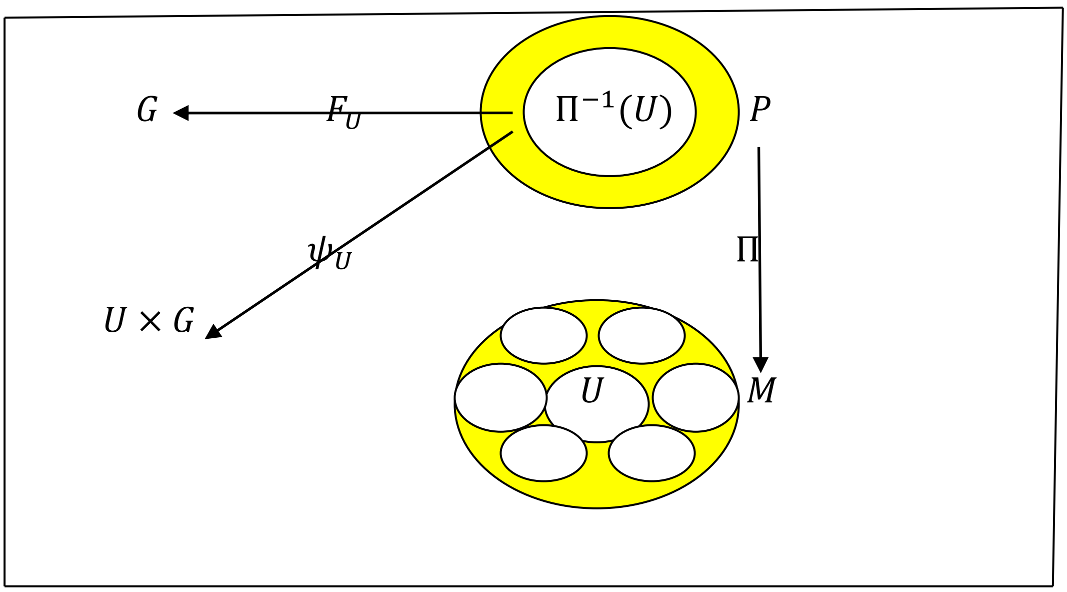
P: is called a fiber space (total space);

G: is called a structure group;

M: is called a basis of fiber bundle;

Π: is called a canonical projection;

$\forall m \in M$, Π^{-1}(m) is called a fiber over m .



Example:

Consider (P, M, Π_1, G), where M and G smooth manifold and Lie group respectivly.

$P = M \times G$; $\Pi_1: M \times G \longrightarrow M$ is the projection on the first factor ($\Pi_1(m, g) = m$). The Lie group G acts on P of the right as follows:

$$\varphi_h(m, g) = (m, g)h = (m, gh).$$

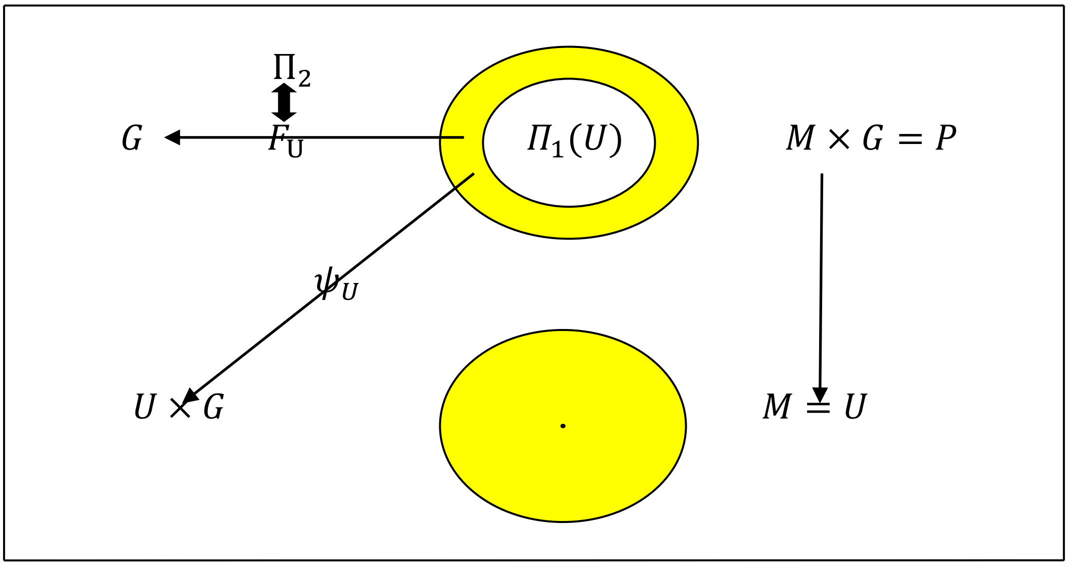
This action is freely because if $\varphi_h(m, g) = (m, g)$, so$(m, gh) = (m, g)$, then, $gh = g$ and thus $h = e$.

Now: suppose that the open cover $\mathcal{U}$ consist of element $U = M$.

1) $F_U(p) = F_U(m, g) = g = \Pi_2(\text{p}) \Rightarrow F_U = \Pi_2$;

$F_U(pg) = F_U\big((m, h)g\big) = F_U(m, hg) = \Pi_2(m, hg) = hg = \Pi_2(p)g = F_U(p)g$;

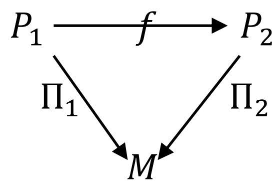
2) ψ_U is diffeomorphisim .



Definition:

Suppose that $\beta_1(P_1, M, \Pi_1, G_1)$ and $\beta_2(P_2, M, \Pi_2, G_2)$ are two fiber bundle spaces, a homomorphisim fiber bundle from β_1 to β_2 is a pair (f, ρ) , where $f: P_1 \longrightarrow P_2$ is a smooth map and $\rho: G_1 \longrightarrow G_2$ is a homomorphisim of Lie groups such that:

1) The following diagram is commutative,



2) $\forall p \in P_1$ and $\forall g \in G_1$,then $f(pg) = f(p)\rho(g)$.

In particular, if (P_1, f) is a sub manifold of P_2, and (G_1, ρ) is a Lie sub group of G_2, then, β_1 is called a sub fiber bundle of β_2 .

Note:

Another important case, if f is a diffeomorphisim, and ρ is an isomorphisim of Lie groups, then the pair (f, ρ) is called an isomorphisim of principle fiber bundles, or we say that β_1 and β_2 are equivalent principle fiber bundles .

Structure equation of principle fiber bundle .

a-Introduction:-

Definition:

A smooth map $\phi: M \longrightarrow N$ is called submersion if its rank equal to the dimension of N.

Theorem $\boxed{*}$:

Suppose that $\beta = (P, M, \Pi, G)$, is a principle fiber bundle, then the map $\Pi: P \longrightarrow M$ … … …

Definition:

Suppose that $\beta = (P, M, \Pi, G)$, is a principle fiber bundle ,denote by $X_\Pi(P)$ to the space of vector field of P , such that if its Π-connection with the vector fields on M , i.e.

$$X_\Pi(P) = \{X \in X(P): \exists Y \in X(M): \Pi_* X = Y\}.$$

Denote by $\tilde{\mathcal{V}} = ker\Pi_*$, then on P appear distribution $\mathcal{V} = C^\infty(P) \otimes \tilde{\mathcal{V}}$, i.e.

$$\mathcal{V} = \{\sum f_i X_i : f_i \in C^\infty(P), X_i \in \tilde{\mathcal{V}}\} .$$

The distribution $\mathcal{V}$ is called a vertical distribution on .

Note: According to theorem $\boxed{*}$, we have, if $p \in P$ any point, then,

$$dim\mathcal{V}_p = dim\tilde{\mathcal{V}}_p = \dim ker(\Pi_*)_p = dimT_p(P) - rank(\Pi_*)_p = dimP - dimM .$$

Fundamental Lie algebra.

Definition:

Suppose that a Lie group G acts on P (of the right), then defined a map :

$$\lambda : \mathcal{G} \longrightarrow X(P)$$

(since, if G acts on P, then $\delta_p : G \longrightarrow P$ is an orbit which is a smooth map).

The map λ generate vector field $X'=\lambda(X) \in X(P)$, λ is called a homomorphisim of Lie algebras, i.e :

$$\lambda([X,Y]) = [\lambda X, \lambda Y]$$

The image of λ is a Lie sub algebra $\notin \subset X(P)$, its elements are called fundamental vector fields on P. The Lie algebra $\notin$ is called a fundamental Lie algebra of vector fields on .

Remark:

The Lie algebra $\notin$ generates sub module $\mathcal{F} = C^{\infty}(P) \otimes \notin$ of the module $X(P)$, i.e:

$$\mathcal{F} = \{\sum f_i X_i : f_i \in C^{\infty}(P), X_i \in \notin\}.$$

Proposition:

The map $\lambda : \mathcal{G} \longrightarrow \notin$ is an isomorphisim .

Theorem:

The distribution $\mathcal{V}$ and $\mathcal{F}$ on the P are concides and $dimG = dimP - dimM = dim\mathcal{V}$

b- the structure equation:

Suppose that $\beta = (P, M, \Pi, G)$ is a principle fiber bundle, $\mathcal{V}$ its vertical distribution, and the indises:

$$i, j, k, \ldots = r+1, \ldots, r+n;$$

$$a, b, c = 1, \ldots, n \quad ; \quad n = dimM;$$

$$\alpha, \beta, \gamma = 1, \ldots, r+n \quad ; \quad r+n = dimP.$$

Suppose that $\{E_1, \ldots, E_r\}$ is a basis of the algebra . since, $\lambda : \mathcal{G} \longrightarrow \notin$ is isomorphisim, then the vector fields $\{E'_1, \ldots, E'_r\}$ is a basis of the linear space $\notin$, then, a basis of distribution $\mathcal{F} = \mathcal{V}$.

Lemma 1 :

Suppose that D is r-dimensional distribution on a smooth manifold M,then for each basis for the D,we can complete this basis to the basis for the module $X(M)$.

Lemma 2 :

Suppose that $\{E_1, \dots, E_n\}$ is a basis of the algebra $\mathcal{G}$, then, $[E_i, E_j] = C^i_{jk} E_k$, where C^i_{jk} are called the constant structure of Lie algebra.

Lemma 3 :

Suppose that $\{\omega^i\}$ is a basis of a codistribution , then,

$$d\omega^i = \omega^i_j \wedge \omega^j \ ; \ \omega^i_j \in \Lambda_1 (P).$$

Theorem:

The structure equation of principle fiber bundle $\beta = (P, M, \Pi, G)$ are :

1) $d\omega^i = \omega^i_j \wedge \omega^j;$
2) $d\omega^a = -\frac{1}{2} C^a_{bc} \omega^b \wedge \omega^c + \omega^a_j \wedge \omega^j$.

Connection on principle fiber bundle.

Definition:

A projection from the module $X(P)$ on the sub module $\mathcal{V}$ is called a vertical projection.

Definition:

We say that the endomorphisim f of the module $X(P)$ is invariant with respect to the action of the Lie group G if for each $g \in G$, then, $(\varphi_g)_* of = fo\ (\varphi_g)_*$; $\{\varphi_g : P \longrightarrow P\}$.

Since G acts on P of the right, then, φ_g can be written as R_g, and then we have,

$$(R_g)_* of = fo\ (R_g)_* \ ; R_g(p) = \varphi_g(p) = pg.$$

Definition:

A vertical projection which is invariant with respect to the structure group is called a connection on principle fiber bundle, this means,$\Pi_V \in End(X(P))$ is a connection if,

1) ${\Pi_V}^2 = \Pi_V$;
2) $Im\Pi_V = \mathcal{V}$;
3) $\forall g \in G,$ we have $(R_g)_* o\Pi_V = \Pi_V\ o\ (R_g)_*$.

Definition:

Suppose that Π_V is a vertical projection in , then, $\Pi_H = id - \Pi_V$ is the complement projection.

A distribution $\mathcal{H} = ker\Pi_V = Im\Pi_H$ is called a horizontal distribution, and the projection Π_H is called a horizontal projection.

Proposition: (H.W)

Suppose that Π_V is connection (i.e. Π_V is invariant w.r.t. action of the structure group G), then, Π_H also is invariant w.r.t. action of the structure group G.

Theorem:

The giving of the connection on a principle fiber bundle $\beta = (P, M, \Pi, G)$ is equivalent to the setting of distribution $\mathcal{H} \subset X(P)$, such that:

1) $X(P) = \mathcal{V} \oplus \mathcal{H}$;
2) $(R_g)_* o\Pi_H = \Pi_H o(R_g)_*$.

Definiti0n:

The isomorphisim $\lambda: \mathcal{G} \longrightarrow \not{f}$ generates an isomorphisim :

$$\wedge = id \otimes \lambda \otimes : C^\infty(P) \otimes \mathcal{G} \longrightarrow C^\infty(P)\not{f} = \mathcal{F} \cong \mathcal{V}.$$

Note that, $\wedge(1 \otimes X) = \lambda(X)$ and $\wedge(f \otimes X) = f \wedge (1 \otimes X) = f\, \lambda(X)$; $f \in C^\infty(P)$.

Define $\theta = \wedge^{-1} o\Pi_V$, where Π_V is a connection on P.

Since, $\lambda: \mathcal{G} \longrightarrow \not{f}$, $\wedge: C^\infty(P) \otimes \mathcal{G} \longrightarrow C^\infty(P)\not{f} = \mathcal{F} \cong \mathcal{V}$ and $\Pi_V: X(P) \longrightarrow \mathcal{V}$,

Then, $= \wedge^{-1} o\Pi_V: X(P) \longrightarrow C^\infty(P) \otimes \mathcal{G}$.

A homomorphisim θ is called a connection form which its value in Lie algebra $\mathcal{G}$.

Theorem:

The giving of the connection on principle fiber bundle $\beta = (P, M, \Pi, G)$ is equivalent to the giving the 1- form θ on a distribution with value in Lie algebra of structure Lie group which has the following properties:

1) $\theta o \wedge = id$;
2) $\theta(fX') = f \otimes X$; $X' \in \not{f} \subset X(P)$.

Structure equation of connection.

Theorem:

The principle fiber bundle $\beta = (P, M\ \Pi, G)$ has connection iff the system $\{\omega^a\}$ satisfies the following relation:

$$d\omega^a = -\frac{1}{2} C^a_{bc} \omega^b \wedge \omega^c + \frac{1}{2} R^a_{ij} \omega^i \wedge \omega^j \qquad \dots\dots\dots (*)$$

Definition:

The relations:

$$d\omega^i = \omega^i_j \wedge \omega^j \ ;$$

$$d\omega^a = -\frac{1}{2}C^a_{bc}\omega^b \wedge \omega^c + \frac{1}{2}R^a_{ij}\omega^i \wedge \omega^j \ .$$

Are called the structure equation of connection(the first and second group respectively).

Remark:

Let $X \in X(P)$, then, $X = X^a E'_a + X^i E_i$.

Remark:

Let θ be a connection form,

$\theta(X) = \theta(X^a E'_a) + \theta\big(X^i E_i\big) = X^a \otimes E_a = \omega^a(X)\otimes E_a = \omega^a \otimes E_a(X) \Longrightarrow \theta = \omega^a \otimes E_a$,

Then, $\theta = d\omega^a \otimes E_a$.

Denoted by $[\theta_1, \theta_2]$ $=\omega^b \wedge \omega^c \otimes [E_b, E_c]$,2 which is called the interior commutator of the forms $\theta_1$and θ_2 .then the relation $(*)$ can be written as the following form:

$$d\theta = -\frac{1}{2}[\theta_1, \theta_2] + \phi$$

Where, $\phi = \frac{1}{2}R^a_{ij}\omega^i \wedge \omega^j \otimes E_a$ is 2-form on P with value in the Lie algebra $\mathcal{G}$ which is called curvature.

Principle fiber bundle of frames.

Definition:

Let M be n-dimensional smooth manifold, $m \in M$ .Consider the space $T_m(M)$,Let $\{e_1, \dots, e_n\}$be a basis of $T_m(M)$, the set(m; $e_1, \dots, e_n$) is called a frame.

Denoted by BM =\{(m; $e_1, \dots, e_n$) : $m \in M$\} the set of all frames, then there exist a surjective map $\Pi: BM \longrightarrow M$.The subset $\Pi^{-1}(m)$ =\{all frames which based at the point m\}, which is called a fiber over M.

Remark:

The Lie group $GL(n, \mathbb{R})$ acts freely on BM on the right by the form:

(m; $e_1, \dots, e_n)g = \big(m; g^{i_1}_1 e_{i_1}, \dots, g^{e_n}_n e_{i_n}\big)$; $g = g^i_j$.

This action is freely, because,$\exists p$ and$pg = p \ \Longrightarrow g = I_n$;

(m; $e_1, \dots, e_n)g = \left(m; g_1^{i_1} e_{i_1}, \dots, g_n^{e_n} e_{i_n}\right) \Longrightarrow\ e_k = g_k^j e_j\ \Longrightarrow\ g = I_n$.

If $P_1$and P_2 are two frames then, $\Pi(P_1) = \Pi(P_2\ ) \Leftrightarrow\ \exists g \in GL(n, \mathbb{R})$ such that: $P_1 g = P_2$, Where g is the transition matrix from the frame $P_1 =$ (m; $e_1, \dots, e_n)$ to the frame $P_2 = (m; e_1', \dots, e_n')$.

Definition:

Let (U, φ) be a local chart in M with coordinates $(x^1, \dots, x^n)$. We define a map $F_U: \Pi^{-1}(U) \longrightarrow\ GL(n, \mathbb{R})$ by $F_U(p) = g$, where g is the transition matrix from the canonical frame $(m; \frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n})$ to the frame (m; $e_1, \dots, e_n$).

Define a map $\psi_U: \Pi^{-1}(U) \longrightarrow\ U \times GL(n, \mathbb{R})$ by the form:

$$\psi_U(p) = \left(\ \Pi(p), F_U(p)\right).$$

We have now $B(M) = (BM, M, \Pi, GL(n, \mathbb{R}))$ is a principle fiber bundle with base space M and canonical projection Π and structure group $GL(n, \mathbb{R})$. This principle fiber bundle is called is a principle fiber bundle of frames.

Remark:

Let M be an n-dimension smooth manifold,$m \in M$ be any point,P = (m; $e_1, \dots, e_n$) be any frame with based at m, then P can be identify with the linear isomorphisim $\rho: \mathbb{R}^n \longrightarrow T_m(M)$

Which defined by the form:

$$\rho(x^1, \dots, x^n) = X^i e_i.$$

Definition:

Let $B(M) = (BM, M, \Pi, GL(n, \mathbb{R}))$ be a principle fiber bundle of frames and $\rho: \mathbb{R}^n \longrightarrow T_m(M)$ be a linear isomorphisim on BM, defined 1-form ω with value in the space $\mathbb{R}^n$ by the form:$\omega_p(X) = \rho^{-1} o (\Pi_*)_p(X)\ ; X \in T_p(BM)$.

$\Pi: BM \longrightarrow M$ generates $(\Pi_*)_p: T_p(BM) \longrightarrow T_{\Pi(p)}(M) = T_m(M)$.

$\rho: \mathbb{R}^n \longrightarrow T_m(M)$ and $\omega_p: T_p(BM) \longrightarrow \mathbb{R}^n$.

The 1-form ω which is defined above is called mixture form.

Definition:

The r-form $\omega\ \in \wedge_r\ (P)$ is called a horizontal form if $\omega(X) = 0\ , \forall X \in \mathcal{V}$.

Theorem:

In the first and the second group of the structure equation of the principle fiber bundle of frames are given by the forms:

$$d\omega^i = -\omega^i_j \wedge \omega^j \ ;$$

$$d\omega^i_j = -\omega^i_k \wedge \omega^k_j + \omega^i_{jk} \wedge \omega^k .$$

Fundamental theorem of tensors analysis.

The setting of tensor field t of type (r,s) on smooth manifold M equivalent to the setting smooth functions $\{t^{j_1\ldots j_s}_{i_1\ldots i_r}\}$ on the principle fiber bundle of frames, which are satisfies:

$$dt^{j_1\ldots j_s}_{i_1\ldots i_r} - t^{j_1\ldots j_s}_{k i_2\ldots i_r}\omega^k_{i_1} - \cdots - t^{j_1\ldots j_s}_{i_1\ldots i_{r-1}k}\omega^k_{i_r} + t^{k j_2\ldots j_s}_{i_1\ldots i_r}\omega^{j_1}_k + \cdots + t^{j_1\ldots j_{s-1}k}_{i_1\ldots i_r}\omega^{j_s}_k$$

$$= t^{j_1\ldots j_s}_{i_1\ldots i_r k}\omega^k .$$

where$\{ t^{j_1\ldots j_s}_{i_1\ldots i_r k}\}$ are the system of smooth functions equal to the coorsponding components of the tensor t.

Structure equation of connection in principle fiber bundle of frames.

Lemma:

Let (P, M, Π, G) be a principle fiber bundle. Suppose that $\theta_1$and $\theta_2$are two connection forms on (P, M, Π, G), then, $\xi = \theta_1 - \theta_2$ is a horizontal form, this mean:

$\xi(X) = 0 \forall X \in \mathcal{V}$.

Proof:

Since $\theta_1$and θ_2 are two connection forms, then, $\theta_1 o \wedge = \theta_2 o \wedge = id$ .If G acts on P on the right, $: \mathcal{G} \longrightarrow X(P)$, $\lambda(X) = X'$, then,$\wedge = id \otimes \lambda : C^\infty(P) \otimes \mathcal{G} \longrightarrow C^\infty(P) \otimes X(P)$,

$\theta_1 o \wedge = \theta_2 o \wedge = id$ means $(\theta_1 - \theta_2)\ o \wedge = 0$

But we know that $\mathcal{F} = C^\infty(P) \otimes \notin = \mathcal{V}$

Then, $\forall X \in \mathcal{V}$, $\exists Y \in C^\infty(P) \otimes \mathcal{G}$ such that $\wedge\, Y = X$

$(\theta_1 - \theta_2)(X) = (\theta_1 - \theta_2)(\wedge\, Y) = (\theta_1 - \theta_2) o \wedge (Y) = 0$, $\forall X \in \mathcal{V}$.

The structure equation.

Suppose that $(B(M), M, \Pi, G)$ is a principle fiber bundle of frames, and suppose that θ is a connection:

$$\theta_j^i - \omega_j^i = \gamma_{jk}^i \omega^k \; ; \; \left\{\gamma_{jk}^i\right\} \in C^\infty\big(B(M)\big)$$

$$\Rightarrow \theta_j^i - \omega_j^i = \gamma_{jk}^i \omega^k$$

According to the first group of structure equation of principle fiber bundle of frames we have

$$d\omega^i = -\omega_j^i \wedge \omega^j = -\theta_j^i \wedge \omega^j + \gamma_{jk}^i \omega^k \wedge \omega^j$$

$$= -\theta_j^i \wedge \omega^j + \gamma_{[jk]}^i \omega^k \wedge \omega^j = -\theta_j^i \wedge \omega^j - \gamma_{[jk]}^i \omega^j \wedge \omega^k$$

Where the bracket [ ] refer to alternative of the indexes $i$and j.

$$d\omega^i = -\theta_j^i \wedge \omega^j + \frac{1}{2}\delta_{jk}^i \omega^j \wedge \omega^k \quad \ldots (1)$$

Where, $\delta_{jk}^i = -2\gamma_{[jk]}^i$.The equation (1) is called the first group of structure equation of connection.

Similar to the principle fiber bundle, we can write

$\omega = \omega^i \otimes \varepsilon_i$ (mixture form with respect to the canonical basis).

$$d\omega = d\omega^i \otimes \varepsilon_i = -\theta_j^i \wedge \omega^j \otimes \varepsilon_i + \frac{1}{2}\delta_{jk}^i \omega^j \wedge \omega^k \otimes \varepsilon_i$$

$$d\omega = -\theta \wedge \omega + \Omega.$$

Where $\Omega = +\frac{1}{2}\delta_{jk}^i \omega^j \wedge \omega^k \otimes \varepsilon_i$ is 2-form in $BM$with value in $\mathbb{R}^n$ which is called the torsion form of connection. on the other hand, remember the second group of structure equation of connection in principle fiber bundle , which has the form:

$$d\omega^a = -\frac{1}{2}C_{bc}^a \omega^b \wedge \omega^c + \frac{1}{2}\mathcal{R}_{k\ell}^a \omega^k \wedge \omega^\ell$$

In the case of principle fiber bundle of frames θ_j^i play the role of ω^a , then,

$$d\theta_j^i = -\theta_k^i \wedge \theta_j^k + \frac{1}{2}\mathcal{R}_{jk\ell}^i \omega^k \wedge \omega^\ell \quad \ldots (2)$$

$$\text{Or } d\theta = -\frac{1}{2}[\theta,\theta] + \phi$$

The equation (2) is called the second group of structure equation of connection in principle fiber bundle of frames, where,

$$\phi = \frac{1}{2}\mathcal{R}^i_{jk\ell}\omega^k \wedge \omega^\ell \text{ and } \frac{1}{2}[\theta,\theta] = \theta^i_k \wedge \theta^k_j.$$

From the above disscusion, we get the following theorem:

Theorem:

The complete group of the structure equations of connection in the principle fiber bundle of frames has the form:

1) $d\omega = -\theta \wedge \omega + \Omega$;
2) $d\theta = -\frac{1}{2}[\theta,\theta] + \phi$.

Where, .$\Omega = +\frac{1}{2}\delta^i_{jk}\omega^j \wedge \omega^k \otimes \varepsilon_i$, $\Phi = \frac{1}{2}\mathcal{R}^i_{jk\ell}\omega^k \wedge \omega^\ell \otimes E^j_i$ are the torsion and curvature forms of connection respectively.

Theorem:

The connection in the principle fiber bundle of frames induce two tensor lields, the first tensor of type (2,1) which is called a torsion tensor of connection, and the second tensor of type (3,1) which is called a curvature tensor of connection.

Problem:

Find $\nabla\delta^i_{jk}$ and $\nabla\mathcal{R}^i_{jk\ell}$.

Definition:

A smooth manifold which fixed connection on its principle fiber bundle of frames is called affine connection space.

Remark:

Let M be an n-dimensional affine connection space, θ be a connection form. Let t be a tensor of type (r,s) on M, according to the fundamental theorem of tensor analysis, the setting of tensor t on M equivalent to the setting a system of functions $t^\wedge$=$\{t^{j_1\dots j_s}_{i_1\dots i_r}\}$ on BM which satisfies the equation:

$$\nabla t^{j_1\dots j_s}_{i_1\dots i_r} = t^{j_1\dots j_s}_{i_1\dots i_r k}\omega^k.$$

Where, $\{ t^{j_1\dots j_s}_{i_1\dots i_r k}\}$ are smooth fuctions which are given on BM:

$$\nabla t^{j_1\dots j_s}_{i_1\dots i_r} = dt^{j_1\dots j_s}_{i_1\dots i_r} - t^{j_1\dots j_s}_{k i_2\dots i_r}\theta^k_{i_1} - \dots - t^{j_1\dots j_s}_{i_1\dots i_{r-1}k}\theta^k_{i_r} + t^{k j_2\dots j_s}_{i_1\dots i_r}\theta^{j_1}_k + \dots + t^{j_1\dots j_{s-1}k}_{i_1\dots i_r}\theta^{j_s}_k$$

$$= t^{j_1\dots j_s}_{i_1\dots i_r k}\theta^k.$$

The functions $\{t^{j_1\dots j_s}_{i_1\dots i_r k}\}$ are tensors of type $(r+1,s)$ this mean $\nabla t^{j_1\dots j_s}_{i_1\dots i_r}$ is a tensor of type $(r+1,s)$ which is called a covariant differential in the given connection and will be defined by ∇t.

Definition:

A tensor field $\nabla_X t$ is called a covariant derivative of the tensor field t in the direction of the vector field X, and the vector field $\nabla_X: \tau(M) \longrightarrow \tau(M)$ is called an operator of covariant derivative in the direction of the vector field X.

Theorem:

The operator ∇_X has the following properties:

1) $\nabla_X f = Xf$;
2) $\nabla_{fX+gY} t = f\nabla_X t + g\nabla_Y t$;
3) $\nabla_X(t_1 + t_2) = \nabla_X(t_1) + \nabla_X(t_2)$;
4) $\nabla_X(t_1 \otimes t_2) = \nabla_X(t_1)\otimes t_2 + t_1 \otimes \nabla_X(t_2)$.

   Where $X, Y \in X(M)$, $f, g \in C^\infty(M)$, $t_1, t_2, t \in \tau(M)$.

Corollary:

In the space M of affine connection defined operator $\nabla: X(M) \times X(M) \longrightarrow X(M)$ which has the following properties:

1) $\nabla(fX + gY, Z) = f\nabla(X, Z) + g\nabla(Y, Z)$;
2) $\nabla(X, Y + Z) = \nabla(X, Y) + \nabla(X, Z)$;
3) $\nabla(X, fY) = X(f)Y + f\nabla(X, Y)$.

   Where $X, Y, Z \in X(M)$ and $f, g \in C^\infty(M)$.

Definition:

The operator ∇ which has the above properties is called Kozel's operator, and we have $\nabla(X, Y) = \nabla_X Y$.

Remark:

The connection which identify with the Kozel's operator is called affine connectionor linear connection of the manifold M .

Theorem:

The setting of affine connection on smooth manifold is equivalent to the setting of Kozel's operator $\nabla: X(M) \times X(M) \longrightarrow X(M)$which has the following properties:

1) $\nabla_{fX+gY}Z = f\nabla_X t + g\nabla_Y Z$;
2) $\nabla_X(Y+Z) = \nabla_X(Y) + \nabla_X(Z)$;
3) $\nabla_X(fY) = X(f)Y + f\nabla_X$.
   Where $\mathrm{X}, \mathrm{Y}, Z \in X(M)$ and $f, g \in C^\infty(M)$.

Theorem:

Let M be the space of affine connection ∇, and $S, \mathcal{R}$ are torsion and curvature tensors respectively of this connection, then:

1) $S(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$;
2) $\mathcal{R}(X,Y)Z = \left([\nabla_X, \nabla_Y] - \nabla_{[X,Y]}\right)Z$.

Where $\mathrm{X}, \mathrm{Y}, Z \in X(M)$.

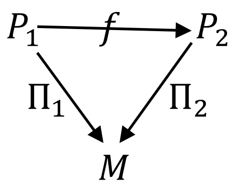
Remark:

This theorem (above) explain that torsion and curvature tensors can be written in the terms of Kozel's operator.

G- Structure of the first order on smooth manifold.

We call to, $\beta_1 = (P_1, M, \Pi_1, G_1)$ and $\beta_2 = (P_2, M, \Pi_2, G_2)$ are two principle fiber bundles on a smooth manifold M.A homomorphisim from β_1 in to β_2 is a pair (f, ρ), where $f: P_1 \longrightarrow P_2$ is a smooth map and $\rho: G_1 \longrightarrow G_2$ is a homomorphisim of Lie groups such that:

1) The following diagram is commutative :



2) $f(pg) = f(p)\rho(g)$

Remark:

In particular, if (P_1, f) is a sub manifold of P_2 and (G_1, ρ) is a Lie sub group of Lie group G_2then, $\beta_1 = (P_1, M, \Pi_1, G_1)$ is called a sub fiber bundle of $\beta_2 = (P_2, M, \Pi_2, G_2)$.

Definition:

The sub fiber bundle $\beta_1 = (P_1, M, \Pi_1, G_1)$ is called a reduction of the fiber bundle $\beta_2 = (P_2, M, \Pi_2, G_2)$ by a sub group (G_1, ρ).

Remark:

For us, the most interest is the case, where $\beta_2 = (BM, M, \Pi, GL(n, \mathbb{R}))$ is a principle fiber bundle of frames and $\beta_1 = (P, M, \widetilde{\Pi}, G)$ its sub fiber bundle , such that:$f: P \subset BM$ is the inclusion map, $\widetilde{\Pi} = \Pi|_P$, and G is a linear group. This means Lie sub group of the general linear group with respect to the inclusion $\rho: G \subset GL(n, \mathbb{R})$.

Example:

If $G = O(n, \mathbb{R}) \subset GL(n, \mathbb{R})$,and

$P =$ {all orthogonal frames of smooth manifold M} $\subseteq BM$={all frames of M}.

In this case $\beta_1 = (P, M, \widetilde{\Pi}, G)$ will be sub fiber bundle.

Definition:

The sub fiber bundle $(P, M, \widetilde{\Pi}, G)$ which is defined as above (this means reduction of the principle fiber bundle of frames over the smooth manifold M by the given subgroup) is called G-structure of the first order over M.