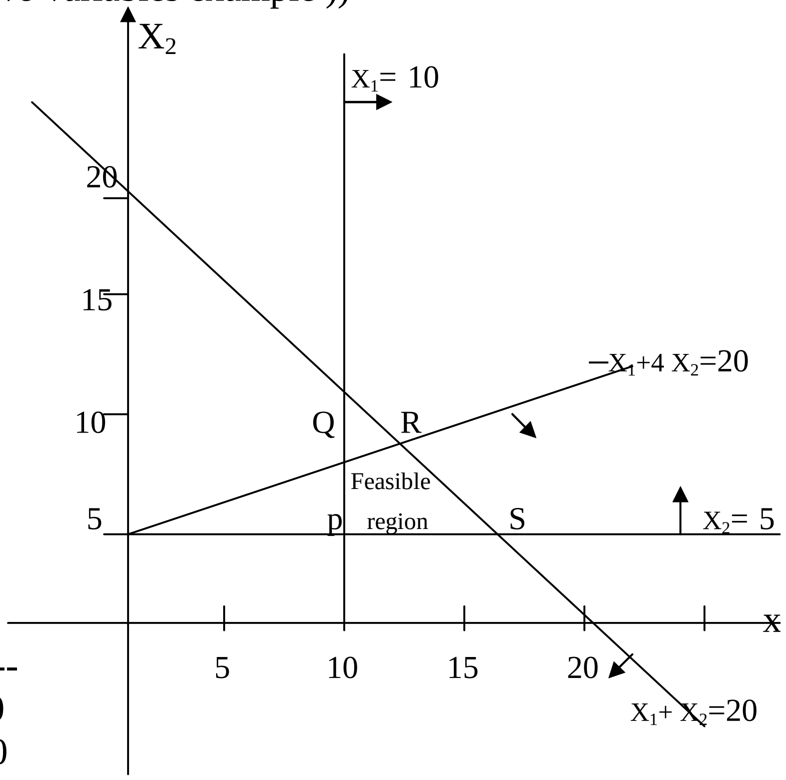
1- linear programming

((two variables example))

a ♣ Graphical method



Example:- (1)

Min. $\mathbf{Z} = -3\,X_1 - 4X_2$

Subject to :- $X_1 + X_2 \leq 20$

$-X_1 + 4\,X_2 \leq 20$

$X_1 \geq 10$

$X_2 \geq 5$

$X \geq 0$

Sol: -

Point	X_1, X_2	Z
--------	---------	------
Q	10 , 7.5	- 60
P	10 , 5	- 50
R	12 , 8	- 68
S	15, 5	- 65

← The min. value of z = – 68 at point R (12 , 8)

∴ P Q R S is the feasible region

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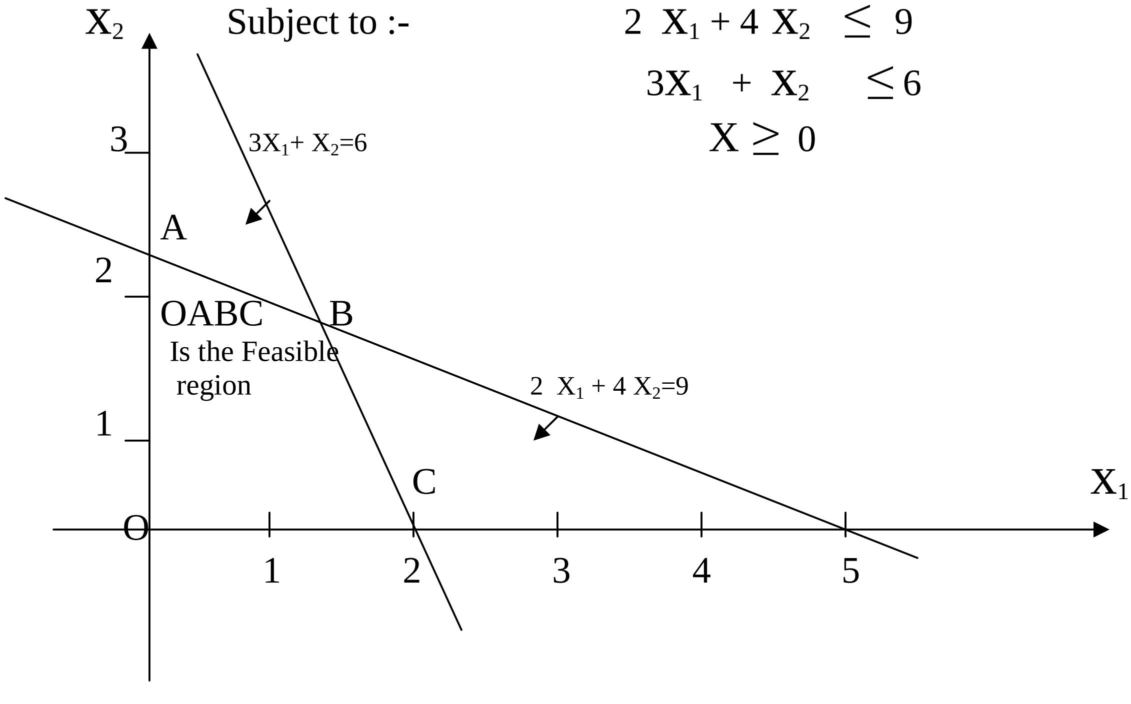
Example:- (2)

Min. $\mathbf{Z} = -6\,X_1 - 2X_2$

Subject to :- $2\,X_1 + 4\,X_2 \leq 9$

$3X_1 + X_2 \leq 6$

$X \geq 0$



Sol: -

Some times there is more than one optimal solution

Point	X_1, X_2	z
O	0 , 0	0
A	0 , 2.25	-4.5
B	1.5 , 1.5	-12
C	2 , 0	-12

← The min. value of z = – 12 at

← points B (1.5 , 1.5) & C(2,0)

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Example:- (3) Some times the solution is unbounded

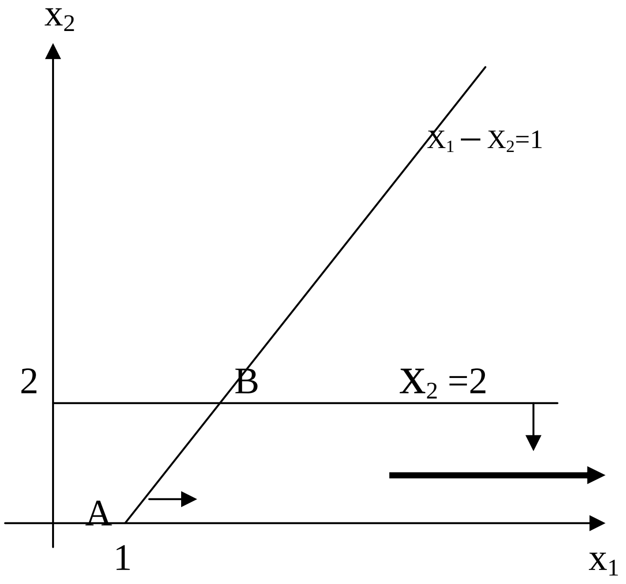
$$\text{Max. } Z = X_1 + X_2$$

Subject to :-

$$X_1 - X_2 \geq 1$$

$$X_2 \leq 2$$

$$X \geq 0$$



Sol: -

the feasible region is unbounded in the direction in which z increase

there is no finite point in the feasible region at which z attamins a maximum , the solution is unbounded & so the maximum value z

for minimize $Z = X_1 + X_2$, subject to the above constraints there is one finite minimize of Z (min) =1 at A, where $X_1 = 1$, $X_2 = 0$

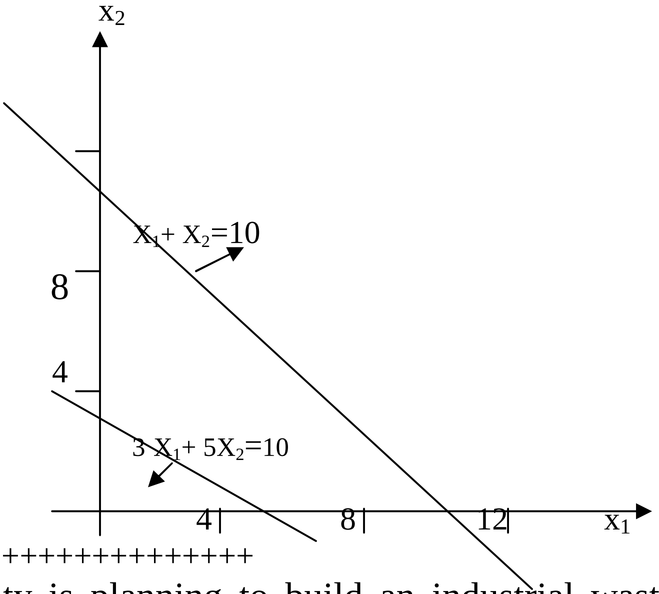
Example:- (4) Some times ,there is no solution at all because a feasible region doesn't exist .

$$\text{Min . } \mathbf{Z} = 2X_1 + 3X_2$$

$$\text{Subject to :-} \quad X_1 + X_2 \geq 10$$

$$3X_1 + 5X_2 \leq 15$$

$$X \geq 0$$



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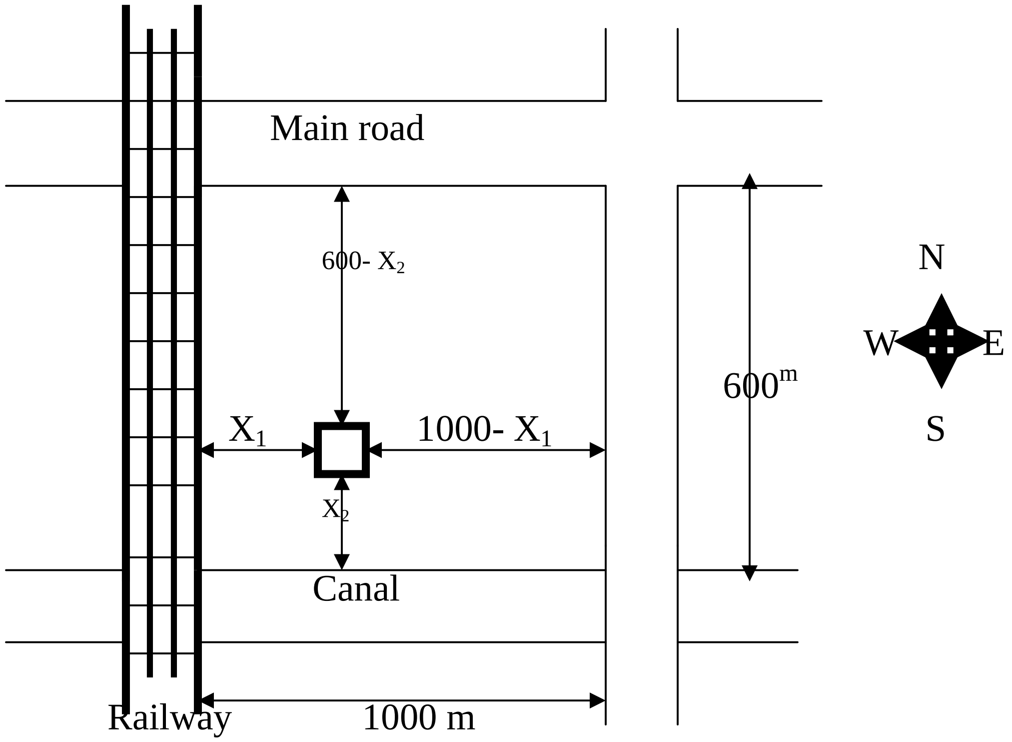
Example:- (5) A locale authority is planning to build an industrial waste processing plant at minimum expenditure on land adjacent to two main roads , a canal & a rail way track as shown below . the unprocessed & processed waste will be transported to & from the plant via the main roads & therefore two access roads will be required the plant location is given by the coordinates (X_1 , X_2) measured from an origin at the junction between railway & the canal as shown in the fig.



□ Plant location defined variable X_1 , X_2

The zone of possible plant locations is limited by several considerations :

1) the railway track on the western side of the land limits the zone of possible plant location to :

$$X_1 \geq 0 \qquad \text{------------} \quad (1)$$

2) the canal of southern side of the land limits the zone to :

$$X_2 \geq 0 \qquad \text{------------} \quad (2)$$

3) the local authority bounded line impose another restriction on the zone of possible plant locations . this boundary line represented by the equation

$3X_1 + 5X_2 = 3000$, thus limiting _the zone of possible plant locations to :

$$3X_1 + 5X_2 \leq 3000 \quad \text{------------} \quad (3)$$

4) precast pile foundations are required for the plant as the soil conditions are very poor on the site , & because of the transportation difficulties it has been decided to limit the average pile length to 16 m . the depth (d) to bed rock can thus reasonable by represented by the equation

d $= \left[\frac{X1}{150} + \frac{X2}{50} + 6 \right]$ since d $\leq$ 16 , a further limit is placed on the zone of possible plant locations, i.e;

$$\left[\frac{X1}{150} + \frac{X2}{50} + 6 \right] \leq 16$$

which can be re- written in amore convenient form as:

$$X_1 + 3X_2 \leq 1500 \qquad \text{----------} \quad (4)$$

5) the final restriction on the zone of possible plant locations is imposed by the presence of a green belt zone in which the construction of any industrial building is for bidden. The green belt zoning line cuts across the land under consideration & can be represented by the equation

$3X_1 + 2X_2 = 2100$, thus limiting the zone of possible plant locations to :

$$3X_1 + 2X_2 \leq 2100 \text{----------} \quad (4)$$

The break down of the total cost of the industrial waste processing plant is as follows:-

405000 \$ for the plant structure

$+ \quad 1500 \left[\frac{X1}{150} + \frac{X2}{50} + 6 \right]$ for the pile foundation

$+ \quad 210\ (1000 - X_1)$ for the east ward access

$+ \quad 210\ (600 - X_2)$ for the north ward access

There fore the objective function for the problem can expressed as the total cost :

$$Z = 405000 + 1500 \left[\frac{X1}{150} + \frac{X2}{50} + 6 \right] + 210\ (1000 - X_1) + 210\ (600 - X_2)$$

$$= 750000 - 200\ X_1 - 180\ X_2$$

∴ The problem can thus be written in the general form of linear programming as :

Minimize $Z = 750000 - 200\ X_1 - 180\ X_2$

Subject to :

$3X_1 + 5X_2 \leq 3000$

$X_1 + 3X_2 \leq 1500$

$3X_1 + 2X_2 \leq 2100$

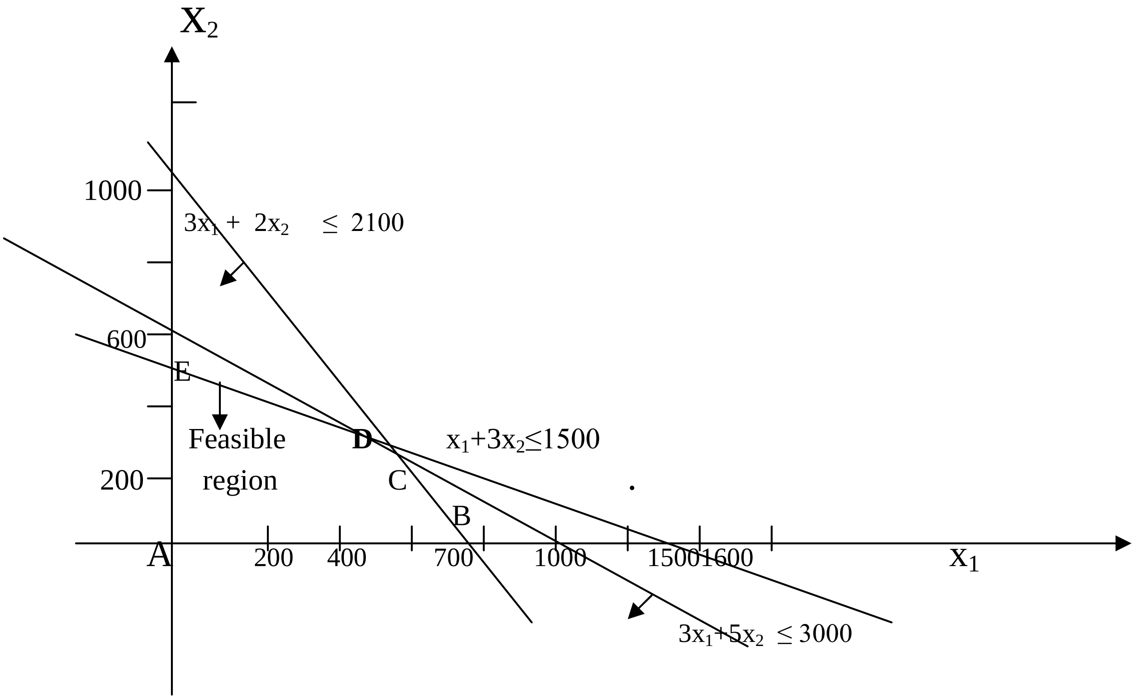
$X_1 \geq 0$

$X_2 \geq 0$

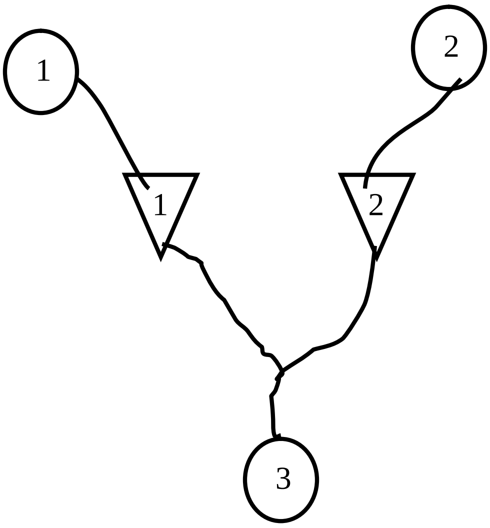
$X_1 \leq 1000 \ , \quad X_2 \leq 600$

Point	X_1 , X_2	Z
--------	---------	------
A	0 , 0	750000
B	700 , 0	610000
C	500 , 300	596000
D	375, 375	607500
E	0 , 500	660000

∴ the min . cost Z is 596000\$ at C ($X_1 = 500$, $X_2 = 300$)



Example:- (6) stream 1 & 2 possessed of a reservoir , join to form a common stream 3 as shown below:

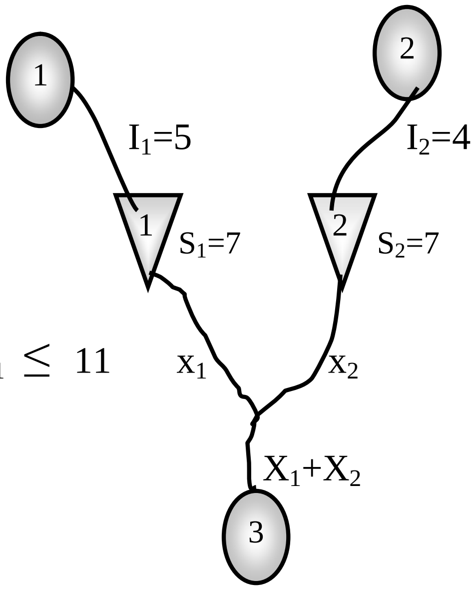


The total benefits derived from annual releases X_1 & X_2 from each reservoir are:

$$B = 5\ X_1 + 3\ X_2$$

The maximum capacity of reservoir (1) is (11) MAF & the maximum capacity of reservoir (2) is (10) MAF . Initial storage in each of the reservoirs at the beginning of the year is (7) MAF ; annual inflows from streams 1 & 2 are 5 MAF & 4 MAF respectively ; & the maximum capacities of channels 1 , 2 & 3 are 6 MAF , 5 MAF & 9 MAF respectively . formulate the optimization problem for maximizing annual benefit.

Solution : -



Max. benefit:

Objective function $Z = 5\ X_1 + 3\ X_2$

Res. 1

$$S_1 + I_1 - X_1 \leq C_1$$

$$7 + 5 - X_1 \leq 11 \longrightarrow 12 - X_1 \leq 11$$

$$X_1 \geq 1 \text{ ----- } (1)$$

Res. 2

$$S_2 + I_2 - X_2 \leq C_2$$

$$7 + 4 - X_2 \leq 10 \longrightarrow 11 - X_2 \leq 10$$

$$X_2 \geq 1 \text{ ------- } (2)$$

<u>River 1</u> : $X_1 \leq 6$ ------ (3)

<u>River 2</u> : $X_2 \leq 5$ ------ (4)

<u>River 3</u> : $X_1 + X_2 \leq 9$ ------ (5)

<u>Capacity</u>

Res. 1 C1 = 11 MAF

Res. 2 C2 = 10 MAF

Riv. 1 $q_1 = 6$

Riv. 2 $q_2 = 5$

Riv. 3 $q_3 = 9$

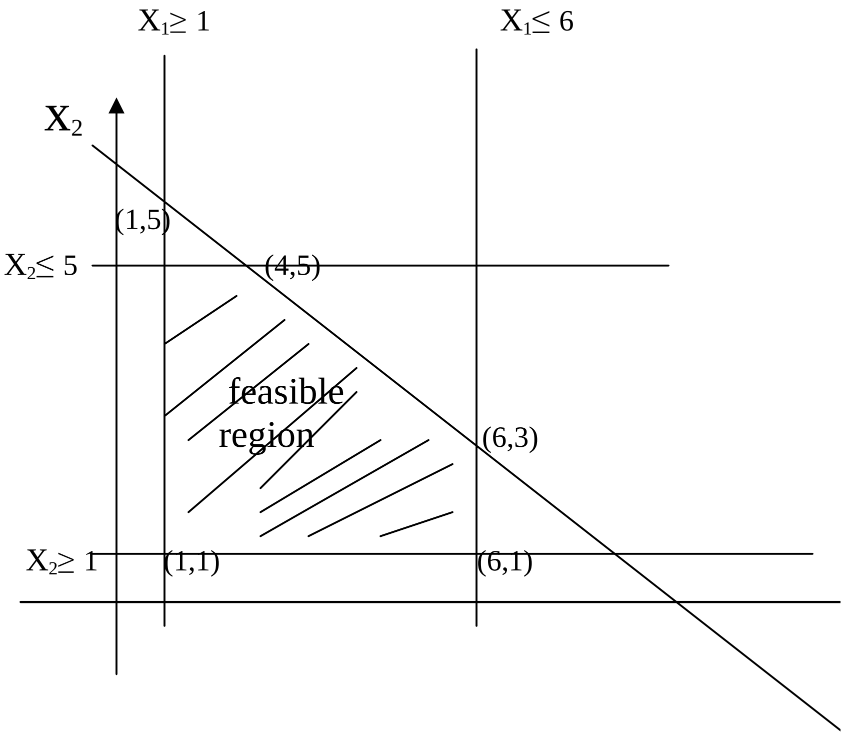
$$6 \geq X_1 \geq 1$$

$$5 \geq X_2 \geq 1$$

$$X_1 + X_2 \leq 9$$

Max. $Z = 5 X_1 + 3 X_2$

Point	Max.Z
(1,1)	8
(6,1)	33
(6,3)	39 ←
(4,5)	35
(1,5)	20



Example:- (7) A farm is assumed to consist of one type of soil. Two general classes of crops are grown on the farm (i.e , corn & wheat) with two varieties (i.e , high yield variety & normal yield variety) . the two crops will be treated with three different practices of irrigation water supply .

(i.e ,full irrigation , 3/4 full irrigation , & 1/2 full irrigation) . the available resources are land , water fertilizer & man power . the farmer would like to know the optimal amount of land should allocated to each crop to maximize his net return.

A standard form for linear programming problems

((two or more variables example))

b ♣ simplex method

a) 1- the objective function is to be either minimizer or maximized

2- the constrained is a mixture of " $\geq$ " , " = " , " $\leq$ " .

3- the all variables in the constraints are non- negative variables

b) maximizing the objective function is :

$$Z= c_1.x_1 + c_2.\ x_2 + c_3.x_3+ \dots\dots\dots+ c_n\ .x_n$$

which is equivalent to minimizing the objective function:

$$Z= - c_1.x_1 - c_2.\ x_2 - c_3.x_3- \dots\dots\dots- c_n\ .x_n$$

c) Inequality constraints:

1- A linear inequality constraint for the xj is of the form :

$$a_1.x_1 + a_2.\ x_2 + a_3.x_3+ \dots\dots\dots+ a_n\ .x_n \leq b$$

To covert the inequality to an equality constraints, we introduce a slack variable $x_{\ n+1} \geq 0$; thus

$$a_1.x_1 + a_2.\ x_2 + a_3.x_3+ \dots\dots\dots+ a_n\ .x_n + x_{\ n+1} = b$$

and we consider that $c_{n+1} = 0$ in the objective function z .

2- If we have the inequality form

$$a_1.x_1 + a_2.\ x_2 + a_3.x_3+ \dots\dots\dots+ a_n\ .x_n \geq b$$

we introduce slack $x_{n+1} > 0$ and artificial variables $x_{\ n+2} \geq 0$;
thus :

$$a_1.x_1 + a_2.\ x_2 + a_3.x_3+ \dots\dots\dots+ a_n\ .x_n = b$$

we introduce artificial variables variable $x_{\ n+1} \geq 0$; thus :

$$a_1.x_1 + a_2.\ x_2 + a_3.x_3+ \dots\dots\dots+ a_n\ .x_n + x_{\ n+1} = b$$

with $c_{n+1} =0$ in the objective function z

4- A different slack or Artificial variable is of course used for each inequality constraint s

d) Non- negative variable

if a particular variable x_k can take any value , then we write

$x_k = x_{k\,1} – x_{k2}$ where $x_{k\,1} \geq 0$ and $x_{k\,2} \geq 0$

e) Arrange the system of equations in (c) so that all (b) are non- negative by multiplying , where necessary , any equation through by minus one.

Example:- (1)

$$\text{Max. } \mathbf{Z} = 2X_1 + 4X_2$$

$$5\ X_1 + 7\ X_2 \leq 5$$

$$3\ X_1 + 2\ X_2 \leq 6$$

$$X \geq 0$$

Solution :-

$$5\ X_1 + 7\ X_2 + X_3 = 5$$

$$3\ X_1 + 2\ X_2 + X_4 = 6$$

$$-\mathbf{Z} + 2X_1 + 4X_2 = 0$$

First table

	X_1	X_2	X_3	X_4	b
	5	7	1	0	5
	3	2	0	1	6
–Z	2	4	0	0	0

Pivoting column

$X_1 = 0$, $X_2 = 0$ $X_3 = 5$, $X_4 = 6$ because they are in

a canonical form $\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

Second table

	X_1	X_2	X_3	X_4	b
	5/7	1	1/7	0	5/7
	$3-5\times\frac{2}{7}$	0	$0-1\times\frac{2}{7}$	$1-0\times\frac{2}{7}$	$6-5\times\frac{2}{7}$
–Z	$2-5\times\frac{4}{7}$	0	$0-1\times\frac{4}{7}$	$0-0\times\frac{4}{7}$	$0-5\times\frac{4}{7}$

	X_1	X_2	X_3	X_4	b
	5/7	1	1/7	0	5/7
	11/7	0	–2/7	1	32/7
–Z	–6/7	0	–4/7	0	–20/7

Note :- if the value of (X) in the objective function (z) raw are zero or negative , you will get the max. objective function **Z**= 20/7 . X_2= 5/7 , X_4= 32/7

X_1 & X_3 = 0

Example:- (2)

$$\text{Min. } Z = -2X_1 - 4X_2$$

$$3X_1 + 4X_2 \leq 1700$$

$$2X_1 + 5X_2 \leq 1600$$

$$X \geq 0$$

Solution :-

$$3X_1 + 4X_2 + X_3 = 1700$$

$$2X_1 + 5X_2 + X_4 = 1600$$

$$-Z - 2X_1 - 4X_2 = 0$$

X_3 & X_4 are slack variable ≥ 0

First table

	X_1	X_2	X_3	X_4	b
	3	4	1	0	1700
	2	5	0	1	1600
$-Z$	-2	-4	0	0	0

X_1 & $X_2 = 0$ $X_3 = 1700$, $X_4 = 1600$

Second table

	X_1	X_2	X_3	X_4	b
	$3-2\times\frac{4}{5}$	0	1–0	$0-1\times\frac{4}{5}$	$1700-1600\times\frac{4}{5}$
	2/5	1	0	1/5	320
–Z	$-2+2\times\frac{4}{5}$	0	0	$0+1\times\frac{4}{5}$	$0+1600\times\frac{4}{5}$

	X_1	X_2	X_3	X_4	b
	7/5	0	1	- 4/5	420
	2/5	1	0	1/5	320
–Z	–2/5	0	0	4/5	1280

Min.

$X_2 = 320$ $X_3 = 420$, X_4 & $X_1 = 0$ canonical form

table (3)

	X_1	X_2	X_3	X_4	b
	1	0	$\frac{5}{7}$	$-\frac{4}{7}$	300
	0	1	$-\frac{2}{7}$	$1/5+(\frac{2}{7})\times 4/5$	$320-(\frac{2}{7})\times 420$
-Z	0	0	$0+1\times\frac{2}{7}$	$(\frac{4}{5})-(\frac{2}{7})\times\frac{4}{5}$	$1280+(\frac{2}{7})\times 420$

	X_1	X_2	X_3	X_4	b
	1	0	5/7	–4/7	300
	0	1	–2/7	3/7	200
-Z	0	0	2/7	4/7	1400

Because all the coefficients of the objective functions $X \geq 0$

$\therefore$ you will get the min **Z** = - 1400

$X_1 = 300$, $X_2 = 200$ X_3 & $X_4 = 0$

From the canonical form $\begin{bmatrix}0\\1\\0\end{bmatrix}$ or $\begin{bmatrix}1\\0\\0\end{bmatrix}$

Check:-

$Z = -2X_1 - 4X_2 = -2 \times 300 - 4 \times 200 = -1400$

Subjects:-

$3X_1 + 4X_2 = 3\times 300 + 4\times 200 = 1700 \leq 1700$

$2X_1 + 5X_2 = 2\times 300 + 5 \times 200 = 1600 \leq 1600$

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<u>Example:- (3)</u>

$$\text{Max. } Z = 5X_1 + 10X_2 + 15X_3$$

Subject:

$$2X_1 + 3X_2 + 4X_3 \leq 10$$

$$3X_1 + 2X_2 + 5X_3 \leq 11$$

$$5X_1 + 10X_2 - 5X_3 \leq 12$$

$$X \geq 0$$

<u>Solution :-</u>

$$2X_1 + 3X_2 + 4X_3 + X_4 = 10$$

$$3X_1 + 2X_2 + 5X_3 + X_5 = 11$$

$$5X_1 + 10X_2 - 5X_3 + X_6 = 12$$

$$-Z + 5X_1 + 10X_2 + 15X_3 = 0$$

X_4 & X_5 & X_6 are slack variable ≥ 0

<u>table (1)</u>

	X_1	X_2	X_3	X_4	X_5	X_6	b	
	2	3	4	1	0	0	10	(10/4)
	3	2	$\boxed{5}$	0	1	0	11	(11/5)
	5	10	-5	0	0	1	12	(12/-5)
-Z	5	10	15	0	0	0	0	

Max, positive

Don't use – ve value because $b \geq 0$ (12/-5)

table (2)

	X_1	X_2	X_3	X_4	X_5	X_6	b	
eq .5	-2/5	7/5	0	1	- 4/5	0	6/5	(6/7)
eq .6	3/5	2/5	1	0	1/5	0	11/5	(11/2)
eq .7	8	12	0	0	1	1	23	(23/12)
eq .8 **-Z**	-4	4	0	0	-3	0	-33	

Max, positive

eq .5 = eq .1 – 4 × eq .6
eq .7 = eq .3 – (-5) × eq .6
eq .8 = eq .4 –(15) × eq .6

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Example:- (4)

$$\text{Min. } \mathbf{Z} = 2X_1 + 3X_2 + 8\ X_3$$

Subject: $X_1 - 5\ X_2 - 10\ X_3 \geq 10$

$$-X_1 + 3\ X_2 + 4\ X_3 \leq 15$$

$$X \geq 0$$

Solution :-

$$X_1 - 5\ X_2 - 10\ X_3 - X_4 = 10$$

$$-\ X_1 + 3\ X_2 + 4\ X_3 + X_5 = 15$$

where X_4 & X_5 are slack variable ≥ 0

& $X_1 = X_2 = X_3 = 0$ from canonical form

In this case the slack variable $X_5 = 15$ & $X_4 = -10$

(i .e $X \geq 0$)

$\therefore$ Add artificial variable $X_6 \geq 0$ in order to

X_4 & $X_6 \geq 0$ & $(-X_4 + X_6) \geq 0$

$\therefore\ X_1 - 5\ X_2 - 10\ X_3 - X_4 + X_6 = 10$

phase (1) :-

min $w = X_6 = -X_1 + 5\ X_2 + 10\ X_3 + X_4 + 10$

$-w - X_1 + 5\ X_2 + 10\ X_3 + X_4 + 10$

$-\mathbf{Z} + 2X_1 + 3X_2 + 8\ X_3$

table (1)

	X_1	X_2	X_3	X_4	X_5	X_6	b
	1	-5	-10	-1	0	1	10
	-1	3	4	0	1	0	15
-Z	2	3	8	0	0	0	0
-W	-1	5	10	1	0	0	-10

Is the bigger negative value which is choosing for min.

$X_5 = 15$; $X_6 = 10$; $X_1 = X_2 = X_{3=} X_4 = 0$

table (2)

	X_1	X_2	X_3	X_4	X_5	X_6	b
	1	-5	-10	-1	0	1	10
	0	-2	-6	-1	1	1	25
-Z	0	13	28	2	0	-2	-20
-W	0	0	0	0	0	1	0

Because W is zero and all coefficients $X \geq 0$ is

–w & X_6 =0

Drop X_6 & w from table (2)
∴ Phase (1) is ended

Phase (2):- continue to minimize (z)

table (3)

	X_1	X_2	X_3	X_4	X_5	b
	-----	-----	-----	-----	-----	-----
	1	-5	-10	-1	0	10
	0	-2	-6	-1	1	25
-Z	0	13	28	2	0	-20

Because all coefficients $X \geq 0$ in the min. obj. fnc. (-z)

$\therefore$ The min. z = 20

& $X_1 = 10$ $X_2 = 0$ $X_3 = 0$ $X_4 = 0$ $X_5 = 25$

Note:- Phase (1) min .of w

Phase (2) min . or max. of Z

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Example:- (5)

$$\text{Min. } Z = 2X_1 + 3X_2 + X_3$$

Subject: $2X_1 + X_2 - X_3 \geq 10$

$X_1 + 3X_2 + 2X_3 \geq 15$

$X \geq 0$

Solution :- subtract slack variable X_4 & X_5

Added artificial variable X_6 & X_7

$$2X_1 + X_2 - X_3 - X_4 + X_6 = 10$$

$$X_1 + 3X_2 + 2X_3 - X_5 + X_7 = 15$$

& min w = $X_6 + X_7$

$W = -3X_1 - 4X_2 - X_3 + X_4 + X_5 + 25$

$-W \quad -3X_1 - 4X_2 - X_3 + X_4 + X_5 = -25$

$-Z + 2X_1 + 3X_2 + X_3 = 0$

Phase (1):-

Min w

	X_1	X_2	X_3	X_4	X_5	X_6	X_7	b
	2	1	-1	-1	0	1	0	10
	1	$\boxed{3}$	2	0	-1	0	1	15
-Z	2	3	1	0	0	0	0	0
-W	-3	-4	-1	1	1	0	0	-25

Answer :-

X_1=3 X_2 =4 X_3 =0 X_4 =0 X_5=0

Z= 18

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Example:- (6) find $X_1 \geq 0$; $X_2 \geq 0$; $X_3 \geq 0$

That minimize the objective function

Min. **Z** = $4X_1 + X_2 + X_3$

Subject: $2X_1 + X_2 + 2X_3 = 4$

$3X_1 + 3X_2 + X_3 = 3$

Solution :- Added artificial variable $X_4 \geq 0$ & $X_5 \geq 0$

$2X_1 + X_2 + 2X_3 + X_4 = 4$

$3X_1 + 3X_2 + X_3 + X_5 = 3$

& min .w $= X_4 + X_5$

W = $-5X_1 - 4X_2 - 3X_3 + 7$

$-W - 5X_1 - 4X_2 - 3X_3 = -7$

$-Z + 4X_1 + X_2 + X_3 = 0$

Phase (1):-

Min. – w

	X_1	X_2	X_3	X_4	X_5	b
	-----	-----	-----	-----	-----	-----
	2	1	2	1	0	4
	3	3	1	0	1	3
-Z	4	1	1	0	0	0
-W	-5	-4	-3	0	0	-7

Ans.

$X_1=0$ $X_2 =0.4$ $X_3 =1.8$

$Z= 2.2$

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Example:- (7)

$$\text{Min. } Z = -3X_1 - 4X_2$$

Subject: $X_1 + 2X_2 \geq 10$ -------- (1)

$X_1 - X_2 = 5$ ----------- (2)

$-X_1 +3 X_2 \leq 20$ ---------- (3)

Solution :- subtract slack variable X_3 from eq. (1)

Added slack variable X_4 to eq. (3)

Added artificial variable X_5 to eq. (1)

Added artificial variable X_6 to eq. (2)

$X \geq 0$

Eq. (1) become

$X_1 + 2X_2 - X_3 + X_5 = 10$ -------- (4)

Eq. (2) become

$X_1 - X_2 + X_6 = 5$ ----------- (5)

Eq. (3) become

$- X_1 +3 X_2 + X_4 = 20$ ---------- (6)

$\therefore$ Min $w = X_5 + X_6$

$X_5 = 10 – X_1 – 2X_2 + X_3$

$X_6 = 5 – X_1 + X_2$

$w = –2 X_1 – X_2 + X_3 + 15$

$–w –2 X_1 – X_2 + X_3 = –15$ --------(7)

$– Z –3X_1 –4X_2 = 0$ -----------(8)

Ans.

$X_1=17.5$ $X_2 =12.5$ $X_3 =32.5$

$Z= – 102.5$

++

Example:- (8)

$$\text{Min. } Z = 4X_1 – X_2 – 3 X$$

$$\text{Subject: } X_1 + X_2 \leq 4$$

$$3 X_1 – X_2 +2 X \leq 8$$

$$2 X_1 +4 X_2 – X \leq 11$$

$X_1 \geq 0$ & $X_2 \geq 0$

X unconstrained (any value)

Solution :- assume $X = X_3 – X_4$

Where $X_3 \geq 0$ & $X_4 \geq 0$

$$\text{Min. } Z = 4X_1 – X_2 –3 X_3 + 3 X_4$$

$$\text{St. } X_1 + X_2 \leq 4$$

$$3 X_1 – X_2 +2 X_3 + 2 X_4 \leq 8$$

$$2 X_1 +4 X_2 – X_3 + X_4 \leq 11$$

$X \geq 0$

$\therefore$ Added slack variables X_5 & X_6 & X_7

$$X_1 + X_2 + X_5 = 4$$

$$3 X_1 – X_2 +2 X_3 + 2 X_4 + X_6 =8$$

$$2 X_1 +4 X_2 – X_3 + X_4 + X_7 = 11$$

Ans.

$X_1=0$ $X_2=4$ $X_3=6$

$X_4 = X_5 = X_6 = 0$, $X_7 = 1$

$X = X_3 - X_4 = 6 - 0 = 6$

min. $Z = -22$

++

Example:- (9)

$$\text{Min. } Z = -3X_1 - 4X_2$$

$$X_1 - 2X_2 \geq 10$$

$$-X_1 + X_2 = 5$$

$$2X_1 + 2X_2 \leq 20$$

Solution :-

$$X_1 - 2X_2 - X_3 + X_5 = 10$$

$$-X_1 + X_2 + X_6 = 5$$

$$2X_1 + 2X_2 + X_4 = 20$$

Where X_3 & X_4 are slack variables

X_5 & X_6 are artificial variables

Min . $w = X_5 + X_6$

$= X_2 + X_3 + 15$

$-w + X_2 + X_3 = -15$

$-Z -3X_1 -4X_2 = 0$

	X_1	X_2	X_3	X_4	X_5	X_6	b
	1	-2	-1	0	1	0	10
	-1	1	0	0	0	1	5
	2	2	0	1	0	0	20
-Z	-3	- 4	0	0	0	0	0
-W	0	1	1	0	0	0	-15

The constraints have no feasible solution because $w \neq 0$ and all coefficients in $w \geq 0$

Example:- (10)

$$\text{Min. } \mathbf{Z} = 8X_1 - 2X_2 - 3X_3$$

$$\text{Subject: } \quad 5X_1 - 5X_2 - 3X_3 \leq 15$$

$$2X_1 - X_2 - X_3 \leq 10$$

$$X \geq 0$$

Solution :-

$$5X_1 - 5X_2 - 3X_3 + X_4 = 15$$

$$2X_1 - X_2 - X_3 + X_5 = 10$$

Where X_4 & X_5 are slack variables

	X_1	X_2	X_3	X_4	X_5	b	
	-----	-----	-----	-----	-----	-----	
	5	-5	-3	1	0	15	(-15/3)
	2	-1	-1	0	1	10	(-10/1)
-Z	8	-2	$\boxed{-3}$	0	0	0	

The variable X_3 is un bounded

Note:- all coefficients of X in obj. fnc. Are > 0

Except at X_3 & X_2 are all value of X_3 & X_4 are negative

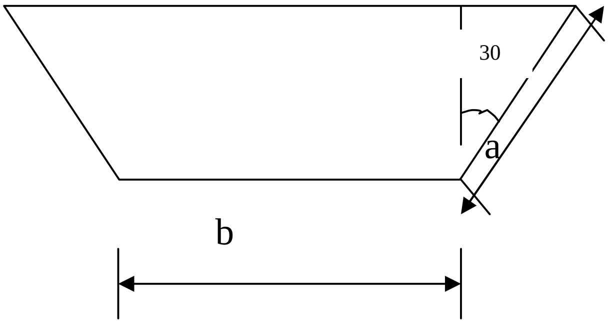
Some application to optimization problems

1.1) suppose you are given the following trapezoidal channel show that the best hydraulic section is one where a =b

when :

1- the wetted perimeter is considered constant .

2- the area considered constant .



1.2) A Baghdad furniture company manufactures four types of table . Each table is first constructed in the carpentry shop & is the next sent to the finishing shop , where it is waxed & polished . the number of labour required in each shop is shown below :

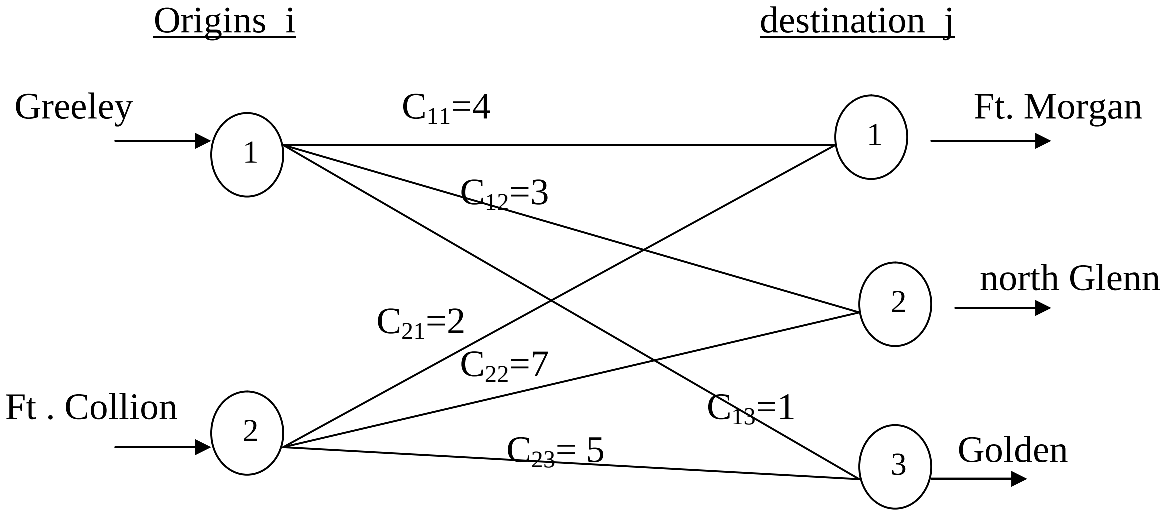
Shop	Table 1	Table 2	Table 3	Table 4
Carpentry	2	2	3	5
Finishing	1	1	2	20
Profit	6 $	10$	9$	20$

Because of limitation in capacity of the plant , no more than 3000 man hours are available in the carpentry shop & 2000 in the finishing shop & polishing shop . the Profit from the sale of each item is shown above:

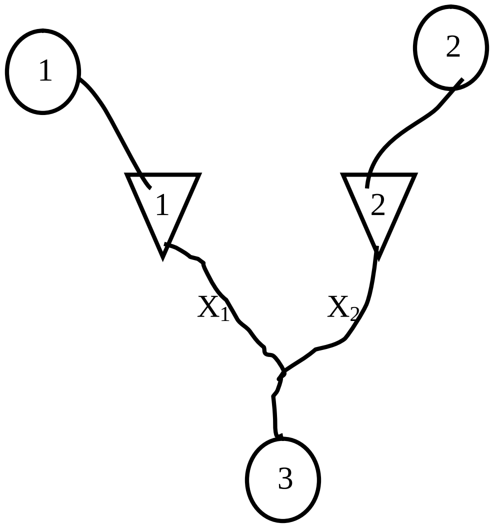
At least 20 tables of type 1 , 40 tables of type 2 , 15 tables of type 3 & no more than 5 tables of type 4 are to be produce .

The Baghdad furniture company would like to know the optimal product maximum (i.e. the quantities to make of each type product which will maximize Profit) . don’t solve , just set up the problem.

1.3) A Contractor has six units of heavy construction equipment each available in Ft . Collion &Greeley & he has construction jobs , in Ft. Morgan , north Glenn ,& Golden that require four, six & two units of such equipment , respectively. The unit shipping cost **C** i,j between cities i &j (i= 1,2 & j = 1,2,3) are shown on the lines linking respective origins & destination in the figure below . formulate the optimization problems which minimize the cost ?



1.4) streams 1 & 2 each possessed of a reservoir , join to form a common stream 3 as shown below:



The total benefits derived from annual release X_1 & X_2 from each reservoir are:

$$B = 5\ X_1 + 3\ X_2$$

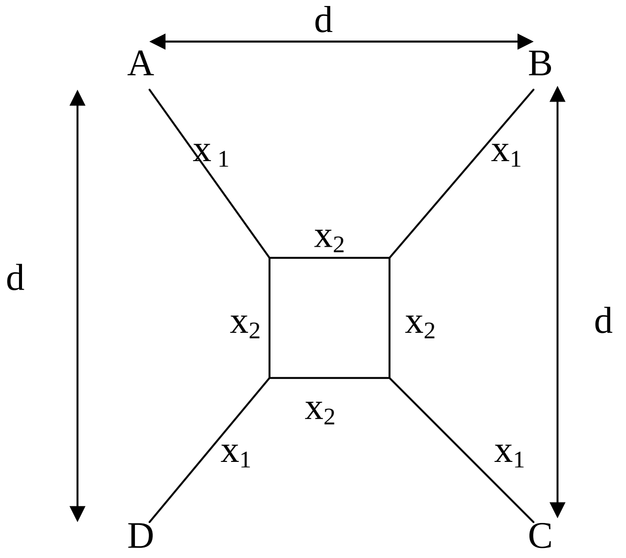
The maximum capacity of the reservoir (1) is (11) MAF & the maximum capacity of reservoir (2) is (10) MAF . initial storage in each of the reservoir at the beginning of the year is (7) MAF ; annual inflow from stream 1 & 2 are 5 MAF & 4 MAF respectively ; and the maximum capacities of channels 1 , 2 & 3 are 6 MAF , 5 MAF & 9 MAF respectively . formulate the optimization problem for maximizing annual benefit ?

1.5) A farm is assumed to consist of one type of soil. Two general classes of crops are grown on the farm (i.e , corn & wheat) with two varieties (i.e , high yield variety & normal different practices of irrigation water supply (i.e ,full irrigation , 3/4 full irrigation , & 1/2 full irrigation) . the available resources are land , water fertilizer & wan power . the farmer would like to know the optimal amount of land that should allocated to each crop to maximize his net return.

1.6) Four towns A , B ,C & D are at a distance of d miles between every two neighboring ones, A net work of roads is proposed for linking the four towns as shown in the figures below . the residents of each town want a network which will minimize the distance of travel between any two towns , but the road contractor wants a network which will minimize the total mileage of roads connecting all Four towns formulate the optimization problems if :

a- the wishes of the residents are taken into consideration

b- the opinion of the road contractor is followed.



1.7) A firm manufactures three products (A, B, C) each of which requires time on all of the four manufactures facilities I , II , III , IV. The manufacturing times & profit margins per unit amount of the produces are shown below :

	Time (hours)				
	I	II	III	IV	profit($)
A	1	3	1	2	3
B	6	1	3	3	6
C	3	3	2	4	4

If the production time available on the facilities I , II , III & IV are 84 , 42 , 21 , & 42 hours respectively , determine which products should be made & in what quantities ? (you may assume there to be an unlimited market for each products Set – up times prior to a change of products being manufactures are negligible & maximization of profit is the only consideration) .

1.8) A manufactures of soft drink has two bottling machines A & B .
Machine A is designed for $^1/_2$ liter bottles & machine B for 1 liter bottles ,but each can be used for both size with some loss in efficiency as shown in the table which gives the rates at which the machines work

Machine	$^1/_2$ liter bottles	1 liter bottles
A	50 per minute	20 per minute
B	40 per minute	30 per minute

Both machines run for 6 hours each day of a 5 day week .
The profit on a $^1/_2$ liter bottle is 4 cents & on 1 liter bottle 10 cents .
The weekly production cannot exceed 50000 liters & the market will only take at most 44000 $^1/_2$ liter bottles & 30000 1 liter bottles .
The manufactures wishes to use his bottling plant so as to maximize his profit. Formulate this Problem as a linear programming Problem& find an optimum solution .

1.9) A manufacture of central heating components make radiators in 4 models, the constraints on his production are the limits on his labor force (in man hours) ,& the steel sheet from which the Radiators are pressed . The sheet delivered each week by a regular supplier . The data in the table give information on the four models. He sets up the Problem as a linear programming Problem which profit maximization as his objective .Obtain the Problem he Formulates and solve it using the simplex method.

Radiators model	A	B	C	D	Available
Man hours needed	0.5	1.5	2	1.5	500 hours
steel sheet needed (m)	4	2	6	8	2500 m
profit/ radiators (&)	5	5	12.5	10	

1.10) A small firm produces two types of bearing A & B which each have to be processed on three machines namely lathes , grinders & drill processes . The time taken for each stage in the production processed is shown in the table below . The firm wishes to produces bearings in quantities in order to maximize its profit formulate this Problem as a linear programming Problem & obtain the solution using the simplex method ,verify the solution graphically .

	Time required (in hours)			
Bearing type	Lathe	Grinder	Drill press	profit/ bearing
A	0.01	0.02	0.04	80Ø
B	0.02	0.01	0.01	125Ø
Total time Available/ week in hours	160	120	150	

1.11) A firm can advertise its products using four media , television , radio newspapers & poster . From various advertising experiments which they have carried out .In the past they have estimated that there are increased profits of $10 , $3, $7 , & $4 per dollar spent on the advertising via has media.

The al location of the advertising budget to the various media is the subject to the following restrictions :

i) The total budget must not exceed $500000.

ii) The policy is to spend at most 40% of the budget on television & at least 20% on posters .

iii) Because of the appeal of the produces to teen – agers the policy is to spent at least half as much on radio as on television . formulate the Problem of allocating the available money to the various media as a linear programming Problem as use in the simplex method to obtain a solution .

1.12) Oxfam tries to construct a diet for refugees containing at least 20 units of protein ,30 units of carbohydrates , 10 units of fat & 40 units of vitamins what is the cheapest way to achieve this given the prices and contents per (kg) (or per liter) of the 5 available foods are as shown below?

	Bread	Soga meat	Dried fish	Fruit	Milk(sub)
Protein	2	12	10	1	2
carbohydrates	12	0	0	4	3
fat	1	8	3	0	4
vitamins	2	2	4	6	2
cost	12	36	32	18	10

1.13) The ESBP oil company is involved in purchasing crude oil from a number of different sources , W, X , Y , & Z & refining it into different grades A , B , & C of lubricating oil ready for sale . The specifications for the grades of lubricating oil impose restriction on the proportions of each crude oils which may be used in the refining process .There are also limitations to the amount of each grade of lubricating oil which can be sold .

	Specifications	Market available (gallons)
Grade A	At least 10% W At most 25% Z	90000
Grade B	At least 15% W	100000
Grade C	At least 20% X At most 30% Y	120000

The costs of the crude oil & the selling price for each grade of lubricating oil are given in the table below.

Cost Per gallon		selling price Per gallon	
W	75	A	90
X	72	B	87
Y	60	C	84
Z	67		

Assuming that the crude oil is available in ultimate quantities formulate the Problem of maximizing profit as a linear programming Problem & find the optimum solution .

1.14) A weaving shop has to be manned 24 hours per day by weavers according to the following table .

Time of day	2 - 6	6 -10	10-14	14-18	18-22	22- 2
Min number of weavers req.	4	8	10	7	12	4

Each weavers is to work eight consecutive hours per day .The objective is to find the smallest number of weavers req. To comply with about requirements .
Show that the manning schedule is to constructions which may be written .

$$\begin{array}{llllllll}
X_1 & & & & & X_6 - X_7 & & =4 \\
X_1+X_2 & & & & & -X_8 & & =8 \\
 & X_2+X_3 & & & & -X_9 & & =10 \\
 & & X_3+X_4 & & & -X_{10} & & =7 \\
 & & & X_4+X_5 & & -X_{11} & & =12 \\
 & & & & X_5+X_6 & -X_{12} & & =4
\end{array}$$

Explain the physical significance of each variable & Show that [X1, X2, X3, X5, X7, X12] form a feasible based . Express this linear programming Problem in canonical form & hence an optimum solution using the simplex method.

"problems in linear programming"

1♣ A company manufactures line of (40 x 40cm.) tiles & (15 x 15cm.) tiles. The major manufacturing departments are the mixing machine department, standardizing dept .& compacting department ; which makes the two types of tiles ,& separate final assembly lines for the two types of tiles.
The monthly capacities of three department are as follows:

Dept . capacity	Tiles(40x40cm.)		Tiles(15x15cm.)
mixing machine	1500	or	4500
standardizing	1000	or	3000
compacting	2000	or	4000
capacity	800	or	3000
profit	$10		$3

Determine the optimal combination of the out put using graphical method of linear programming?

2♣ A construction factory makes three types of blocks , half hollow , solid & hollow , out of three materials sand , gravel & cement (with different useful of mixing). The contributions of the three products are showing in the following table .

Products	sand	gravel	cement	profit, $
Half hollow blocks	1	3	2	3
solid blocks	1	4	2	5
Hollow blocks	1	3	6	4
Amount available Units	8	19	14	

Determine the optimal products mixing ?

3♣ A building constructor products types of houses speculative building , detached & semidetached houses .The customer is offered several choices of architect rural design sold &layout for each types .The proportion of each type of design sold in the past is shown below . The profit on detached house & a semidetached house is $1000 & $800 respectively .

Choices of design	detached	semidetached
type A	0.1	0.33
type B	0.4	0.67
type C	0.5	------

The builder has the capacity to build 400 houses per year . Because of the limited supply of bricks available for type B design a 200 house limit with this design is imposed .
Determine how many detached & a semidetached house should be constructed in order to maximize profit& state the optimum produce?

4♣ A constructor has one mechanical excavator & one bulldozer which are available for work on either of two adjacent sites .On one site clay over burden is being excavated for a ballast pit owner & on the other ballast is being removed under sub-contract to another client . The constructor experience leads him to believe that he can make ($60) profit for every [1000m^3] of ballast he removes . A comprehensive work study assesses the resources required to remove [1000m^3] of clay to be [8 hours] use of the excavator , [4 hours] use of bulldozer& [50 man - hours] of laborers time .In the case of the excavation of [1000m^3] of ballast ,the resources are required for[4 hours, 5 hours, & 13 man - hours] respectively.
The contractor employees work on a[40 hours] week .
The mechanical equipment is also available for to[40 hours] week. In addition to the mechanical equipment[5 laborers] are available for up to[40 hours] each in any one week in order to assist with the work .
When not employed on the excavation use can be made of the laborers elsewhere .
How should the contractor use his resources in order maximize his profit during one working week?

5♣A certain contractor has [9 projects] available to him to bid . The contractor evaluates all of the projects as being potentially profitable & there for would like to under take them all . However since construction of all of the projects is expected to occur simultaneously & since the contractor is limited as to available man power, he believes it unwise to bid all of the projects.
The projects available to the contractor consists of [6 retail stores]&[3 gas stations]. The contractor estimates his profit from the construction of a retail store as [$7000] & that from a gas station as[$5000]. In the vicinity in which the contractor operates, [4 crafts] of labor have limited man power. In addition to lack of time lost Because of seasonal weather characteristics, the constructor estimates that during the time period available to him for actual construction of the various projects, he has available [35000 mason hours] ,[50000 laborer hours] ,[25000carpenter hours] & [25000 iron worker hours] .

The contractor estimates the man - hours required to build are tail store & a gas station to be as follows:

Crafts	one retail store	one gas station
Mason hrs. required	5600	4800
Laborer hrs. required	7500	5600
Carpenter hrs. required	----	4000
Iron worker hrs. required	3000	3000

From the pervious in formation ,how can the contractor decided which project he should , assuming that he will win the contract on the projects be bids to maximize his profits?

6♣ The cement company has the opportunity supply various concrete mixes to a near by construction project. The company Because it is a low – cost operation ,has the option of selection that it supply minimum of [600] cubic meter of [3-4-2] (3 ports cement to 4 parts gravel & 2 parts sand) each day at one site &[200] cubic meter of [3-0-2] each day at another site.

The concrete company will realize a profit contribution of $3.0 per cubic meter from the [3-4-2] mixture & $5.0 per cubic meter from the [3-0-2] mixture .

The plant can obtain for each day's operation a total of 3600 cubic meter of cement ,4000 cubic meter of gravel & 3000 cubic meters of sand.

The plant is situated on a stream so that water is no Problem . The plant manager needs to know what proportion of his production should be the [3-4-2] mix & what proportion of the[3-0-2] mix in order to realize the highest possible profit .set up a linear programming model to help the manager .determine the optimal production strategy

7♣ Factory produces two types of concrete block a ton of the first type requires 2 batches of gravel , 2 bags of cement & 4 batches of sand . a ton of the second type of block require 4 batches of gravel & 2 bag of cement but no sand .There are 20 batches of gravel ,12 bags of cement , & 16 batches of sand available .How many tons of each type of block should be manufactured so that maximum profit may be achieved ? It is assumed that the first type of block yield $2 profit per ton & 2nd type yields $3 profit per ton .Solve this Problem using the graphical method.

linear programming

References:

1 . Bunday , B.D (1984), " Basic Linear programming "
Edward Arnold , London .
2 . Gass , S.B(1984) , " Linear programming"
5 the .ed .Mc Graw - ltill ,New york .
3 . Greeberg ,H. (1971) ," Integery programming"
Academic Press , New york .

problems:

EX.(1) : Min . $Z = -2X_1 - 4X_2$

St. $5X_1 + 7X_2 \leq 5$

$3X_1 + 2X_2 \leq 6$

EX.(2) : Max . $Z = 2X_1 + 4X_2$

St. $3X_1 + 4X_2 \leq 17$

$2X_1 + 5X_2 \leq 16$

EX.(3) : Max . $Z = 5X_1 + 10X_2 + 15X_3$

St. $2X_1 + 3X_2 + 4X_3 \leq 10$

$3X_1 + 2X_2 + 5X_3 \leq 11$

$5X_1 + 10X_2 - 5X_3 \leq 12$

EX.(4) : Min . $Z = 2X_1 + 3X_2 + 8X_3$

St. $X_1 - 5X_2 - 10X_3 \geq 10$

$X_1 + 3X_2 + 4X_3 \leq 15$

EX.(5) : Min . $Z = 2X_1 - 3X_2 + X_3$

St. $2X_1 + X_2 - X_3 \geq 10$

$X_1 + 3X_2 + 2X_3 \geq 15$

EX. (6) : Min . **Z** = $2X_1 + 3X_2 + X_3$

St: $2X_1 + X_2 - X_3 \geq 10$

$X_1 + 3X_2 + 2X_3 \geq 15$

EX. (7) : Min . **Z** = $4X_1 + X_2 + X_3$

St: $2X_1 + X_2 + 2X_3 = 4$

$3X_1 + 3X_2 + X_3 = 3$

EX. (8) : Max . **Z** = $-2X_1 + 3X_2 + X_3 - 2X_4$

St: $-3X_1 + 2X_2 - X_3 + 3X_4 = 2$

$-X_1 + 2X_2 + X_3 + 2X_4 = 3$

EX. (9) : Max . **Z** = $3X_1 + 4X_2$

St: $X_1 + 2X_2 \geq 10$

$X_1 - X_2 = 5$

$-X_1 + 3X_2 \leq 20$

EX. (10) : Min . **Z** = $-3X_1 - 4X_2$

St: $X_1 - 2X_2 \geq 10$

$-X_1 + X_2 = 5$

$2X_1 + 2X_2 \leq 20$

EX. (11) : Min . **Z** = $4X_1 - X_2 - 3X_3 + 3X_4$

St: $X_1 + X_2 \leq 4$

$3X_1 - X_2 + 2X_3 - 2X_4 \leq 8$

$2X_1 + 4X_2 - X_3 + X_4 \leq 11$

EX. (12) : Min . $Z = -X_1 + 2X_2 - X_3$

St: $X_1 - 2X_2 + 3X_3 = 1$

$X_2 - X_3 \leq 2$

$X_1 + X_3 = 3$

EX. (13) : Min . $Z = X_2 + X_3$

St: $2X_1 - 5X_2 + 7X_3 \geq 0$

$X_1 - X_2 + 2X_3 = 2$

$X_1 - 2X_2 + 3X_3 \leq 1$

EX. (14) : Min . $Z = -2X_1 - 3X_2 - 4X_3$

St: $X_1 - 2X_2 + X_3 \geq 1$

$2X_1 - X_2 - 2X_3 = 5$

$-X_1 + X_2 + 3X_3 \leq 6$

EX. (15) : Min . $Z = -2X_1 - X_2$

St: $2X_1 + 4X_2 \geq 4$

$-X_1 + 3X_2 = 10$

$X_1 + X_2 \leq 6$

$X_1 - X_2 \leq 3$

♣*Transportation problems:-*

Transportation problems are a special class of linear programming problems which can be solved by a purpose built algorithm known as the " Transportation method "

The objective is to find the Transportation route which gives minimum cost , or maximum profit .

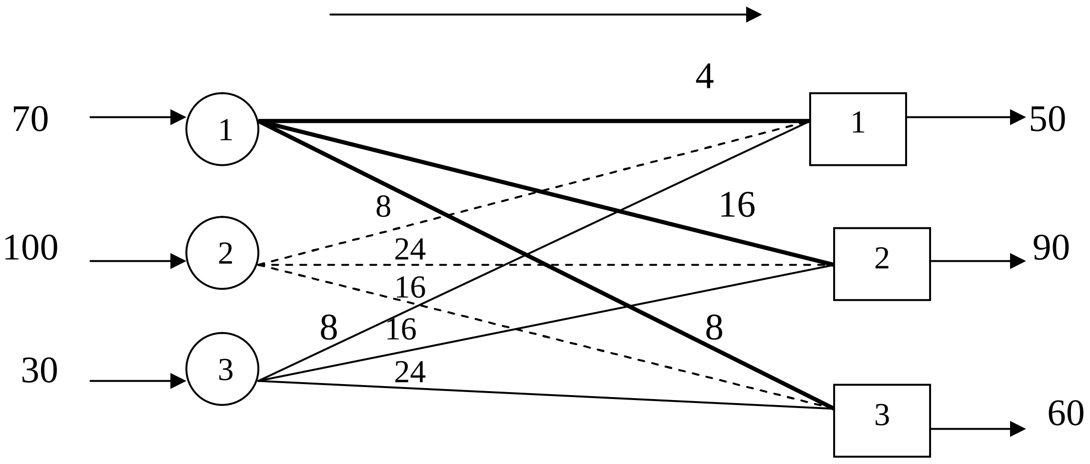
The problem is defined by the a mount and location of the available supplies and the quantities demanded .

further more , a value (usually cost) associated with effort required to transport supplies from their origin to their destination must also be determined.

Example(1) m origins and n destinations showing below :-

$C_{i,j}$ = cost of transporting of one unit

$$Z_{min} = \sum_{i=1}^{m} \sum_{j=1}^{n} C_{i,j} \, . \, X_{i,j}$$



Origins (m)
i= 1,2,…., m

destination (n)
j= 1,2,…..,n

Linear programming formulation :-

Objective function = Total cost of transport

$$Z_{min.} = \sum_{i=1}^{m} \sum_{j=1}^{n} C_{i,j} \; . \; X_{i,j}$$

Subject to the constraints

$$\sum_{j=1}^{n} X_{i,j} = S_i > 0 \quad (\text{ for } i= 1, 2,3 \ldots\ldots,m)$$

$$\sum_{i=1}^{m} X_{i,j} = d_j > 0 \quad (\text{ for } j = 1, 2,3 \ldots\ldots,n)$$

& all ;

$$\sum_{i=1}^{m} S_i = \sum_{j=1}^{n} d_j$$

$$X_{i,j} \geq 0 \quad (\text{ for all } (i , j))$$

Where

$X_{i,j}$ = amount transported from origin (i) to destination (j)

$C_{i,j}$ = cost of transporting 1 unit of resource from origin (i) to destination (j)

S_i = number of units available at origin (source) (i)

d_j = number of units required at destination (j)

Z = Total transportation cost

m = number of origion (j)

n = number of destination (j)

since

$$\sum_{i=1}^{m} S_i = \sum_{i=1}^{m} \sum_{j=1}^{n} X_{i,j} = \sum_{j=1}^{n} \sum_{i=1}^{m} . X_{i,j} = \sum_{j=1}^{n} d_j$$

there are only (m+n – 1) independent constraints and hence (m+n – 1) basic variables in a basic feasible solution .

The transportation matrix:-

The example (1) may be displayed in a transportation matrix such as the one shown in table (1):

In the transportation matrix , each row represents an origin and each column destination .

the cell at the intersection of each column & each row includes the transportation cost to link that origin destination.

Table (1)

Origin (i)	Destination (j)			S_i From
	1	2	3	
1	4	16	8	70
2	8	24	16	100
3	8	16	24	30
d_j To	50	90	60	200 200

The initial feasible solution (minimum cost method) :

In this method as much al location as possible is made to the cell with the minimum cost . then as much al location as possible is made to the cell with the next lowest cost – see table 2 –

Table (2)
INITIAL BASIC FEASIBLE SOLUTION

		1	2	3	From (available)
	Vj U i	– 4	8	0	
1	8	50–w 4	16	20+w 8	70
2	16	+w 8	60 24	40–w 16	100
3	8	8	30 16	24	30
To (required)		50	90	60	200

Total cost = 2920

The cost of the solution can be obtained by multiplying the allocation in the basic cells by the associated unit transportation cost s as shown below:-

Basic cell		Quantity	unit cost	Total cost
j	i	Xi, j	Ci,j	Ci,j + Xi, j
1	1	50	4	200
2	2	60	24	1440
2	3	30	16	480
3	1	20	8	160
3	2	40	16	640

Total transportation cost 2920

Note:- NO. of Basic variable = m+n – 1 = 3+3 – 1 = 5

Testing a solution for possible improvement:

Having obtained an initial solution ,the next step is to ascertain whether this is the optimal solution . this can be achieved by examining the non- basic cells in the transportation matrix to determine if it is possible to make a replacement to one of them & also reduce the total transportation cost .one procedure will be examined

Modified Distribution Method [U – V method]

For any feasible solution ,find number ui for available (from) i & vj for required (To) j such that :

$ui + vj = C_{i,j}$ for every basic $X_{i,j}$ ------- (1)
these numbers can be positive , negative or zero then :

$\overline{C}_{i,j} = C_{i,j} - (ui + vj)$ for all non –basic $X_{i,j}$ ------- (2)

If all the $\overline{C}_{i,j}$ are non- negative ,then the current basic feasible solution is optimal . If not , there exists anon basic variable $X_{p,q}$, such that:

$\overline{C}_{p,q} = \min. C_{i,j} < 0$ ------- (3)

and $X_{p,q}$ is made a basic variable to improve the vale of the objective function.
To apply the U–V method to example (1) , we have to compute six numbers
$u_1 , u_2 , u_3 , v_1 , v_2 , v_3$
In the current basic feasible solution (Table 2) , the basic variables are $x_{1,1}$, $x_{2,2}$, $x_{3,2}$, $x_{1,3}$, $x_{2,3}$
Using eq (1), we get five equation s :
$u_1 + v_1=4$; $u_2 + v_2 = 24$
$u_3 + v_2=16$; $u_1 + v_3 = 8$
$u_2+ v_3= 16$

Since the system has five equation s in six unknowns, there exists an infinite number of possible solutions to get a particular solution we can set any of the variable zero and solve for the rest .
Setting $v_3 =0$, we get $v_1= - 4$ $v_2= 8$ $u_1 =8$
$u_2 = 16$ & $u_3 = 8$
This computation can easily be accomplished using table (2) directly rather than writing the five eqs. Separately.

Optimality test :

For every non basic variable $X_{i,j}$, the unit cost $C_{i,j}$ is compared with the sum of ui & vj . If all the $u_i + v_j \leq C_{i,j}$ then we have an Optimal solution ; other wise , the non - basic variable with the least relative cost value is chosen.

In our example (1)

$$C_{p,q} = \text{min.} \ (C_{i,j} – u_i – v_j) = C_{2,1} – u_2 – v_1$$
$$= 8 – 16 \ – (– 4) = – 4$$

Hence the non basic variable $X_{2,1}$ is introduced in to the basis as the new basic variable , determine the maximum increase in $X_{2,1}$, we assign $X_{2,1}$ on unknown non-negative value w .

In order to satisfy the constrains , w has to be added or subtracted from the basic variables.

So that the row sums & column sums are equal to the corresponding available (supplies) & requires (demands) .

Referring to table (2) , we see that both the basic variables $X_{1,1}$& $X_{2,3}$ are decreased by w while $X_{1,3}$ is increased by w .

Now w is increased as the solution remains non – negative .

The maximum of w is limited by those basic variables which start decrease with w .

The basic variable which becomes zero first is removed from the basis *

In our example (1) , the maximum w is equal to 40 (i.e 40 – w =0) and the basic variable $X_{2,1}$ is replaced by $X_{2,3}$

Table (3) gives the new basic feasible solution , with anew set of value of ui & vj

* In some problems, it is possible for more than one basic variable to become zero , simultaneously.

Under such cases , the rule is to select any one of them (but only one) to leave the basis

No.1 basic variables = n+ m – 1
(see example (2))

Table (3)
Basic Feasible Solution

		1	2 j	3	From available
	Vj u i	–16	0	–12	
1	20	10–w 4	+w 16	60 8	70
2 i	24	40+w 8	60–w 24	16	100
3	16	8	30 16	24	30
To (required)		50	90	60	200

The total cost = 2760 is not optimal , since

Min. ($C_{i,j} - u_i - v_j$) = $C_{1,2} - u_1 - v_2$

= 16 – 20 – (0) = – 4

Now $X_{1,2}$ is introduced as basic variable at a non- negative value w , this produces the change in the values of the basic variables .
the maximum values of (w) is 10 & $X_{1,2}$ replaces $X_{1,1}$ in the basis . the new basic feasible solution , is given by table (4)

Table (4)

		1	2 j	3	From available
	Vj u i	–16	0	–8	
1	16	4	10 16	60 8	70
2 i	24	50 8	50 24	16	100
3	16	8	30 16	24	30
To (required)		50	90	60	200

Total cost = 2760

Table (4) represents an optimal solution , since $\bar{C}_{i,j} \geq 0$ for all non basic variables.

The optimal transportation schedule is to transport

				Cost
10	units	from	1 To 2	10 * 16
60	units	from	1 To 3	60 * 8
50	units	from	2 To 1	50 * 8
50	units	from	2 To 2	50 * 24
30	units	from	3 To 2	30 * 16
200	units		Total	2720

Alternative optimal solution :

It will be need that the improvement index $\bar{C}_{i,j}$ for $\bar{C}_{2,3}$= 16 – 24 – (– 8) =0 In the last solution Table (4) .

A zero improvement index implies that if this rout were bought in to the solution . the allocation would change but the total transportation cost would remain constant . Thus, an alternative optimal solution exists , to complete the solution to the transportation this other optimal solution should be found , as this solution may be more convenient than the optimal solution already found ($X_{1,2}$ = 60 , $X_{1,3}$ = 10 , $X_{2,1}$ = 50 , $X_{2,3}$ = 50 , $X_{3,2}$ =30 & total transportation cost =2720)

Example (2- a) :

		1	2	3	4	From
	Vj u i	–5	–9	–10	0	
1	6	10 1	15	12	0 6	10
2	11	16	10 2	5	0 11	10
3	13	10	8	10 3	0 13	10
4	4	7	9	14	10 4	10
To		10	10	10	10	

$\bar{C} \geq 0$ cost = 100 (final sol.)

Example (2- b) : For more than one basic variable become zero [$X_{2,3}$, $X_{3,2}$] removed one basic variable only.

		1	2	3	4	From
	Vj / u i	–20	–15	0	–124	
1	126	110	115	10 126	0	10
2	130	107	13+w 115	2–w 130	0	15
3	124	10 104	2–w 109	+w 116	13 0	25
To		10	15	12	13	

$\bar{C} = -8$ w = 2 cost = 4273

Dummy

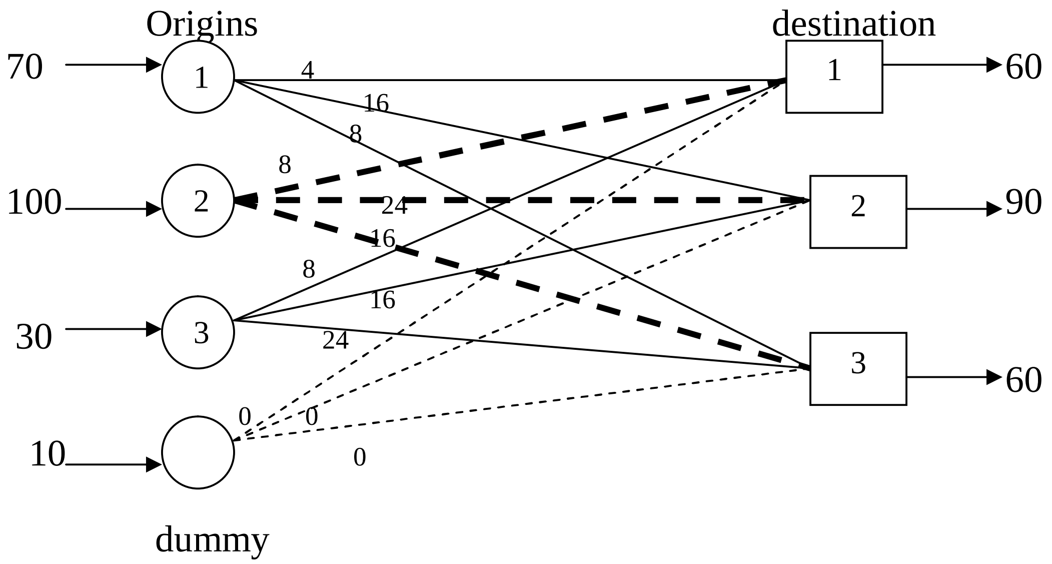
		1	2	3	4	From
	Vj / u i	–12	–7	0	–116	
1	126	110	115	10–w 126	+w 0	10
2	122	107	15 115	130	0	15
3	116	10 104	0 109	2+w 116	15–w 0	25
To		10	15	12	13	

$\bar{C} = -10$ w = 10 cost = 4257

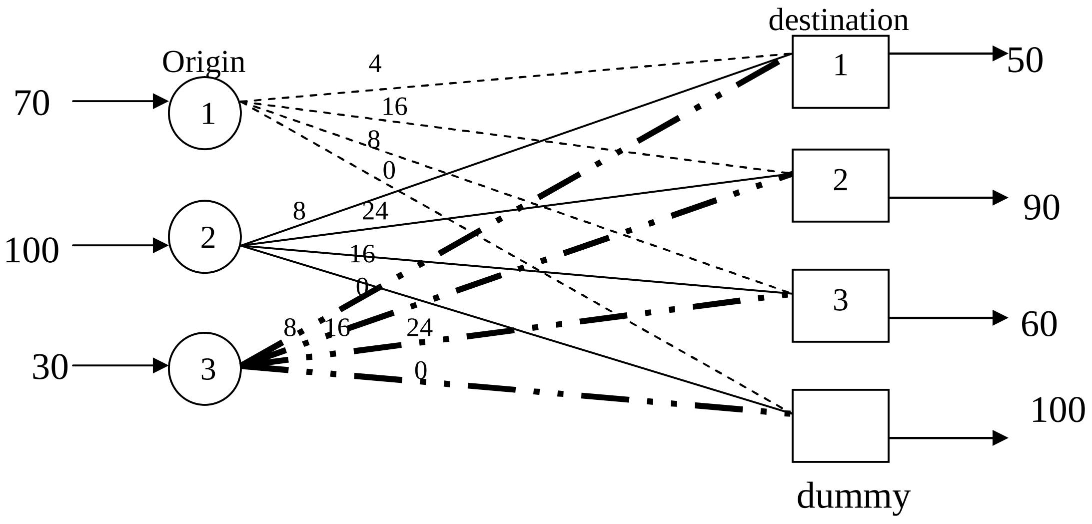
Unbalanced transportation problems :

Unbalanced transportation problems in which supply and demand are unequal $\sum_{i=1}^{m} S_i \neq \sum_{j=1}^{n} d_j$ can be solved by introducing dummy origins or destination . The optimal solution indentifies the requirement which cannot be satisfied or the location of available supplies which remain used . the cost of transport from the origin to each destination is zero .

Examples of dummy origins & destination are shown below .



(a) dummy origin



(b) dummy destination

Example (3) : Unbalanced transportation

(a) dummy origin

		1	2	3	From
	Vj u i	– 4	8	0	
1	8	60 4	16	10 8	70
2	16	8	60 24	40 16	100
3	8	8	30 16	24	30
4	0	0	0	10 0	10
To		60	90	60	

$\bar{C}$ = – 8 , Col. = 2 , Row.= 4 Total cost = 2880

(b) dummy destination

		1	2		3	From
	Vj U i	– 4	8	0	– 8	
1	8	50 4	16	20 8	0	70
2	16	8	60 24	40 16	0	100
3	8	8	30 16	24	100 0	130
To		50	90	60	100	

$\bar{C}$ = – 8 , Col. = 4 , Row.= 2 Total cost = 2920

Maximization transportation problems :

Transportation problems are usually posed as cost minimization problems . some times however , transportation profits rather than costs are associated with each route and the objective is to maximize the total profit. To solve such problems ,the profit in each cell are replaced by the amount by which the profit for that cell falls short of the largest profit in the transportation matrix [$C_{i,j} = - C_{i,j}$]

maximize profit = – minimum cost

* see example (4)

Remark: some of the cost elements are not possible (set at M or an infinity large value) as shown in the following example.

Example (5):

Consider the problem of scheduling the weekly production of a certain item for the next 4 weeks . the production cost of the item is $10 for first 2 weeks , & $ 15 for the last 2 weeks . the weekly demands are 300 , 700, 900 & 800 which must be met . The plant can produce a maximum of 700 units each week .

In addition the company can employ overtime during the second and third week .the increases the weekly production by an additional 200 units , but the production cost increases by $ 5 per item .

Excess production can be stored at a unit cost of $ 3 per week.

How should be production be schedule minimize the total cost ?

Example (4) : Maximization problem

Let $C_{i,j} = - C_{i,j}$

		1	2	3	4	From
	Vj u i	– 6	0	– 4	– 3	
1	–10	1 –16	0 –10	–14	–11	1
2	–11	–14	0 –11	1 –15	–15	1
3	–15	–15	1 –15	–13	–12	1
4	–12	–13	0 –12	–14	1 –15	1
To		1	1	1	1	

$\bar{C} = -1$, w = 0 , Cost = – 61 No. of basic variable = 7

Example (5) : some of the $C_{i,j}$ are not possible (set at M or an infinity large value)

		1	2	3	4	Dummy	From
	Vj u i	– 9	– 6	– 3	0	–15	
1 N	19	300 10	13	200 16	200 19	0	700
2 N	16	M	700 10	0 13	16	0	700
2 0	21	M	15	18	200 21	0	200
3 N	18	M	M	700 15	18	0	700
3 0	23	M	M	20	200 –w 23	w 0	200
4 N	15	M	M	M	200 +w 15	500 –w 0	700
To		300	700	900	800	500	

$\bar{C}$ = – 8 W = 200 Cost = 39300

No. of basic variable = 10

Assignment problem :

A special form of the transportation problem occurs when there is just item at each of several depth (origin or from) points & one item is required at each of several destination (To) .the problem here is to assign and the item to the destinations in an optimal manner and we speak of this at an assignment .

Example (6) :

A car hire company has one car at each of five depots a , b , c , d & e . A customer in each of the five towns A , B , C , D & E required a car . the distance are given in the following table.

		A	B	C	D	E	Depots
	Vj u i	–30	–65	– 30	0	15	
a	185	160	130	175	190	1 200	1
b	160	135	120	1 130	0 160	175	1
c	170	140	110	155	1 170	0 185	1
d	80	1 50	50	80	0 80	110	1
e	100	55	1 35	0 70	80	105	1
Towns		1	1	1	1	1	

C = – 20 Col. = 4 Row = 5

Total cost = 585

w=0 , col. = 4 Row = 2

Ans: the total cost = 570

EXERCISES

1♣ A Concrete transit –mix company owns three plants with capacities & production costs as follows:

Plant No.	Daily Capacity (m^3)	production cost ($\$/m^3$)
I	160	10
II	160	9
III	80	13

The company is under contract to supply concrete for abridge contraction and is scheduled to deliver to the various job sites the following quantities of concrete

	Job Site	Amount (m^3)
1)	South bank pier	100
2)	South bank abutment	40
3)	North bank abutment	80
4)	North bank pier	140

Based on distance , traffic , and site delays , the following transportation costs are estimated :

Job Site No.	Transportation Cost ($\$/m^3$) Plant No. I	II	III
1	1	1	3
2	1	2	3
3	2	1	2
4	3	2	1

Scheduled tomorrows production to minimize total cost to the company . [This example is taken from Meredith et al]

2♣ A contractor has five locations A , B , C , D , & E on a road contract to which crushed stone is to be delivered .

The stone , which is all of the same quality, is to be supplied from three quarries 1 , 2 & 3 . The table shows the relative costs per cubic meter of transporting stone from each source to each location , the quantity of stone which is required at each location , and the quantity which will be available of each quarry .

Treat this as a transportation problem & allocate quantities of stone from each quarry to each location for the least value of total transportation cost .

	Location					Quarry output (m^3)
Quarry	A	B	C	D	E	
	Relative transportation costs					
1	9	10	12	11	10	120
2	12	8	6	3	7	255
3	10	9	4	13	6	150
	45	105	150	135	90	
	Quantity required at each location (m3)					

3♣ The Brunel Gravel Company has received a contract to supply gravel for three new construction projects , located in the towns of I , II , & III.

Construction engineers have estimated the required amount which would be needed at three construction projects.

Project	Location	Weekly requirement
A	I	72
B	II	102
C	III	41
		Total 215

The Brunel Gravel Company has three gravel pits located in towns 1, 2, & 3 .The gravel required for the construction projects can be supplied by these three plants . Brunel's chief dispatcher has calculated the amounts of gravel which can be supplied by each plant & the unit transportation costs.

Plant	Location	Amounts available, week (truck loads)
W	1	56
X	2	82
Y	3	77
	Total available	215

cost per truck load

From	To project (A)	To project (B)	To project (C)
Plant W	$ 4	$ 8	$ 8
Plant X	$ 16	$ 24	$ 16
plant Y	$ 8	$ 16	$ 24

Give the amounts required at each project and the amounts available at each Plant , the company's problem is to scheduled shipments from each Plant to each project in such manner as to minimize the total transportation cost . Solve using the transportation method .

4♣ In the above problem , suppose that Plant W has a capacity of 76 truck loads per week rather than 56 . The company would be able to supply 235 truck loads per week. However , the project requirements remain the same . Solve this un balanced transportation problem.

5 ♣ Fig 4-15 represents a railways net work .The point labeled a , b, c , & d represent collieries and those labeled A , B & C represents the coke ovens of a steel works .The rail mileages from these points to the junction of the net work , & from one junction to another , are shown by the number on the lines . The collieries can supply the following amounts .

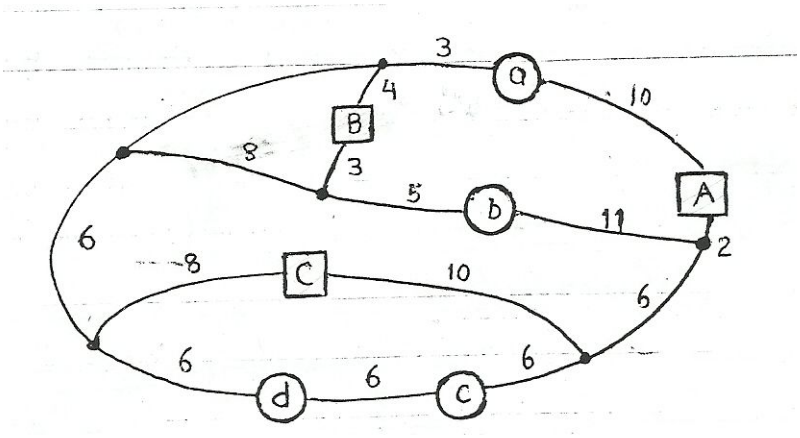


Fig .4-15 Railways Network

a	3000 tons/ month
b	2000 tons/ month
c	8000 tons/ month
d	3000 tons/ month

The requirements for coke oven are:

A	2000 tons/ month
B	4000 tons/ month
C	6000 tons/ month

If the transportation cost is £3 per ton mile ,find that distribution of coal from the collieries to coke ovens which minimize the total transportation costs.

6♣ A Company produces a special type of precast concrete component for use in prefabricated contraction . The company has three factories A , B & C which supply five contractors (working on different sites) with the precast Components. The production capacities of the factories & the demands of contractors – assumed constant – & distribution costs are given below .How should the Components be distributed in order to supply- the contractors with their demands in the cheapest way?

Factory	(production capacity) (000s per month)
A	5
B	10
C	10

			Contractor		
Factory	a	b	c	d	e
A	5	7	10	5	3
B	8	6	9	12	4
C	10	9	8	10	15
Contractor requirements (000s)	3	3	10	5	4

7 ♣ Industries I_1 , I_2 & I_3 require 80 , 30 , & 90 (x10^5) cu.ft of water per day respectively to be supplied from three reservoirs R_1 , R_2 , & R_3

Reservoirs R_1 can supply 100 (x10^5) cu.ft per day

Reservoirs R_2 can supply 25 (x10^5) cu.ft per day

Reservoirs R_3 can supply 75 (x10^5) cu.ft per day

Given the following transportation cost matrix , show how to obtain an allocation of water to minimize the total transportation cost.

Transportation cost matrix

	I_1	I_2	I_3
R_1	5	10	2
R_2	3	7	5
R_3	6	8	4

8 ♣ An engineering firm produces a steel structural component for use in offshore oil production plat forms .Three work centre's are required to manufacture , assemble & package the product .Four location are available within the plant .The materials handling cost at each location for the work centre is given by the following cost matrix .Determine the location of work centre's that minimizes the total materials handling costs .

	Location			
	1	2	3	4
Manufacturing	18	18	16	13
Assembly	16	11	X	15
Packaging	9	10	12	8

The symbol X implies that assembly cannot be performed in location 3.

9 ♣ A civil engineering consulting firm has a back log of four contracts .work on these project must be started immediately . Three project leaders are available for assignment to these contracts .Because of varying work experience of the project leaders the profit to the consulting firm will vary Used on the assignment .

	Contract			
Project leaders	1	2	3	4
A	13	10	9	11
B	15	17	13	20
C	6	8	11	7

The unassigned contract can be assigned by sub – contracting the work to an outside consultant .Determine the assignment which optimizes the over all profits.

10 ♣ An aggregate producer has four identical mobile crushing –screening plant & four sources of row material which he can use during the coming construction –season . Given the following profit matrix , how many plants should be assigned to each site .

Profit matrix

No. of plants assigned	Raw – materials site 1	2	3	4
1	47	39	24	35
2	81	62	47	51
3	105	62	47	61
4	132	91	87	68

11 ♣ The following matrix gives the cost of using each of five different earth-movers to perform five separate jobs on a civil engineering site .Assign one job to each earth- movers in order that the total cost is optimized .

Earth-movers	Job A	B	C	D	E
1	10	5	9	18	11
2	13	19	6	12	14
3	3	2	4	4	5
4	18	9	12	17	15
5	11	6	14	19	10

12 ♣ A construction company has to move four large cranes from old construction sites to new construction sites . The distance (in miles) between the old location and the new are given in the following matrix.

old construction Sites	New construction sites N_1	N_2	N_3	N_4
O_1	25	30	17	43
O_2	20	23	45	30
O_3	42	32	18	26
O_4	17	21	40	50

Any of the cranes work equally well on any of the new sites . Determine a plan for moving the cranes that will minimizes the total distance involved in the move.

13 ♣ An engineering firm wishes to assign five of its personnel to the task of designing five projects . Given the following time estimates required by each designer to design a given project , find the assignment which minimizes total time.

	Projects				
Engineering	1	2	3	4	5
1	3	10	3	1	8
2	7	9	8	1	7
3	5	7	6	1	4
4	5	3	8	1	4
5	6	4	10	1	6

14 ♣ A contractor wants to hire some special excavation equipment for a large earth moving job. Two types of suitable excavators are available and each requires the following man power :

	Operators
Excavator 1	5
Excavator 2	2

The contractor can use no more than 17 operators. After taking rental and lab our costs into consideration the contractor finds he can make profit of $10 per hour on excavator 1 & $3 per hour on excavator 2 . Use Gomorys cutting plan method to find how many units of each types he should hire to maximize his hourly profit (N.B. Having worked out the first cut constraint , rework the simplex method from the beginning) .check your answer graphically .

15 ♣ Large orders for four steel components to be used on North sea oil rigs are be assigned to four man -machine centers. some machines are better suited to produce certain components and their operators are more proficient at producing some components then others . The costs to produce each components at each center are:

Component	MMC1	MMC2	MMC3	MMC4
C_1	12	9	11	13
C_2	8	8	9	6
C_3	14	16	21	13
C_4	14	15	17	12

i) Which components (C) should be assigned to each man- machine center (MMC) ?

ii) Following a union agreement man –machine center (MMC2) cannot be used to produce components C_1 . How dose his affect the assignment of components to machines ?

iii) Assume anew machine (MMC5) has been added to the facilities above . one old machine is to be phased out .The operator one the old machine will operator the new one if it can lead to on assignment Which is less expensive than the assignment made in part (i) .
The estimated cost using the new machine is

Components	C_1	C_2	C_3	C_4
Production cost at MMC5	11	7	15	10

Ignoring the union agreement in (ii) , should the new machine be used ? If so , Which components should it Produce?

Transportation problems

EX.1 ♦

The transportation matrix

	1	2	3	From
1	4	16	8	70
2	8	24	16	100
3	8	16	24	30
To	50	90	60	

Ans. : 2720

EX.2 ♦

The transportation matrix

	1	2	3	4	From
1	1	15	12	6	10
2	16	2	5	11	10
3	10	8	3	13	10
4	7	9	14	4	10
To	10	10	10	10	

Ans. : 100

EX.3 ♦

The transportation matrix

	1	2	3	4	5	From
1	2	3	4	5	0	15
2	3	2	5	2	0	20
3	4	1	2	3	0	25
To	8	10	12	15	15	

Ans. :80

EX.4 ♦

The transportation matrix

	1	2	3	4	From
1	1	15	12	6	10
2	16	2	5	11	10
3	10	8	3	13	10
4	7	9	14	4	10
5	0	0	0	0	40
To	20	20	20	20	

Ans. :100

EX.5 ♦

The transportation matrix

	1	2	3	4	5	From
1	1	15	12	6	0	20
2	16	2	5	11	0	20
3	10	8	3	13	0	20
4	7	9	14	4	0	20
To	10	10	10	10	10	

Ans. :100

EX.6 ♦

The transportation matrix

	1	2	3	From
1	8	6	3	15
2	9	11	8	16
3	6	5	7	11
4	3	10	9	13
To	17	20	18	

Ans. :298

EX.7 ♦

The transportation matrix

	1	2	3		From
1	2	2	2	1	3
2	10	8	4	1	7
3	7	6	6	8	5
To	4	3	4	4	

Ans. :53

EX.8 ♦

The transportation matrix

	1	2	3	4	5	From
1	1	0	3	4	2	15
2	5	1	2	3	3	25
3	4	8	1	4	3	20
To	20	12	5	8	15	

Ans. :121

EX.9♦

The transportation matrix

	1	2	3	4	From
1	110	115	126	0	10
2	107	115	130	0	15
3	104	109	116	0	25
To	10	15	12	13	

Ans. :4190

EX.10 ♦

The transportation matrix

	1	2	3	4	From
1	–1	–15	–12	–6	10
2	–16	–2	–5	–11	10
3	–10	–8	–3	–13	10
4	–7	–9	–14	– 4	10
To	10	10	10	10	

Ans. :580

2- Non –linear Optimization

Basic concepts : The general non-linear Optimization Problem can be formulate as follows :
Find the values of n variables X_1 , X_2 ,….. X_n which satisfy m equation &/or in equalities (constraint function) .

Z minimize = f (X_1 , X_2 ,….. X_n)
or maximize

Subject to
$g_j(x) \geq 0$; j =1,2,…,m_1
$g_j(x) = 0$; j= m_{1+1} , m_{1+2},…., m_2
$g_j(x) \leq 0$; j= m_{2+1} , m_{2+2},…., m

if m= 0 the Problem is said to be un constrained .
The variables X_i may also include lower and upper bounds on the variable that is:

$$L_i \leq X_i \leq U_i \quad ; i = 1 , 2 ,\dots , n$$

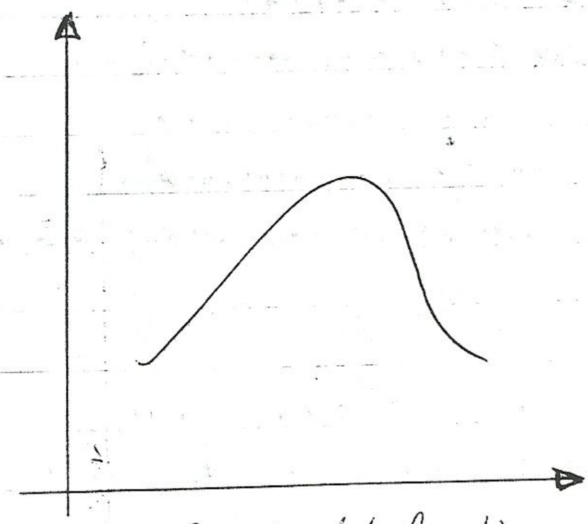
Where L_i & U_i are lower & upper limits for Xi variable.

Uni modal function : It has only one peak (valley)as shown in fig(1) .

Multi modal function : It has two or more (peaks) as shown in fig(2).

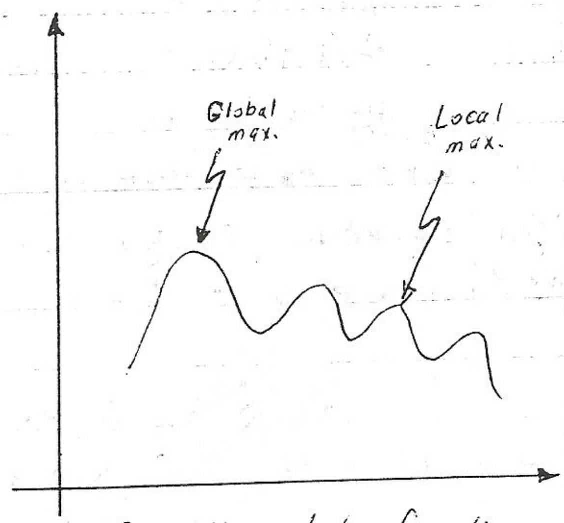
Saddle point: Is one which appear to be optimal solution from a local view point but is in fact in forior to some other point in the solution , see fig(3) .
This point is called stationary point or a point of inflection .

Global & local optimum : The best optimizing valuation is called a global optimum Most non –linear programming algorithms yield solutions which are local optimum .This is caused primarily by algorithmic produre which depend on the local properties of the non –linear programming Problem .See fig(3) .



A unimodal function

Fig.(1)



A multimodal function

(Fig.2.)

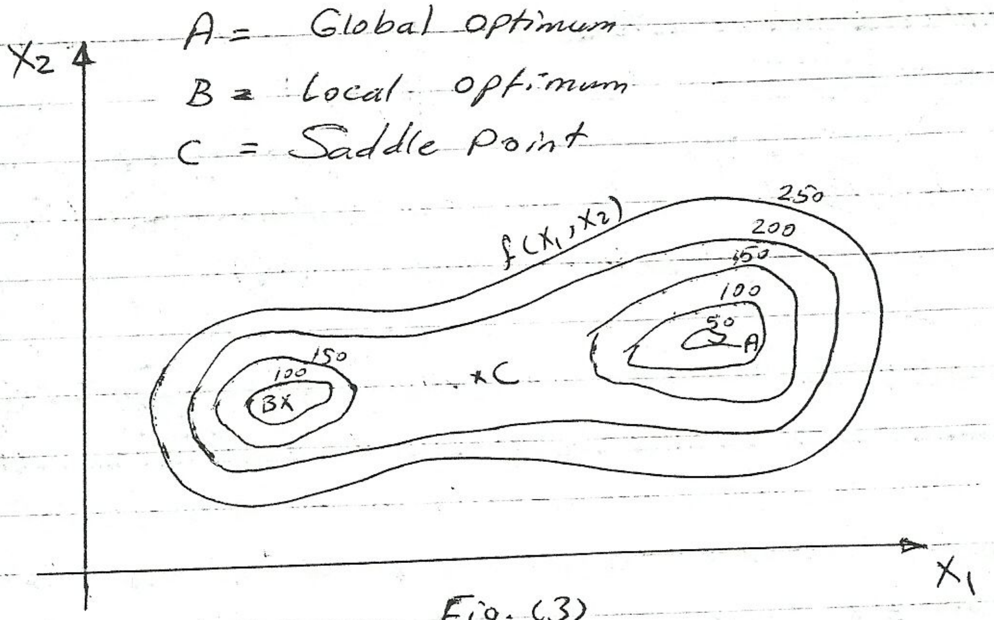
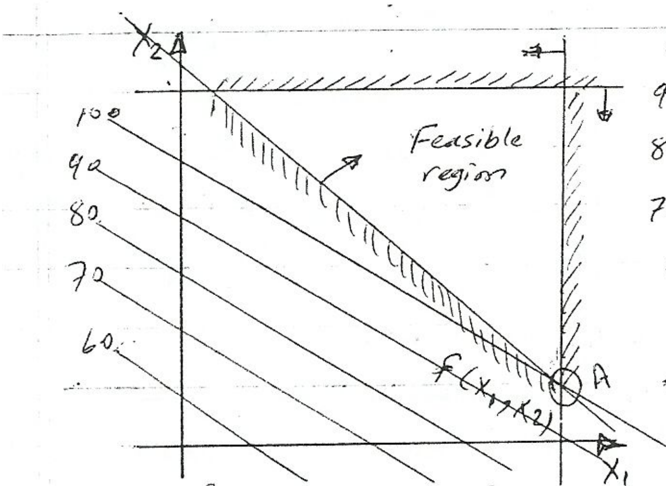


Fig. (3)

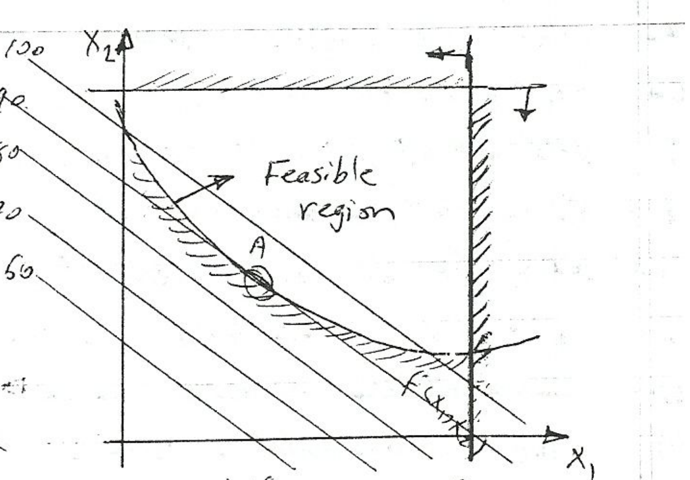
A multimodal function with a saddle point, Global & Local optimum solution.

Effect of non –linear on the solution :

Linear programming is a special Problem of the non –linear programming . the various effect of non –linear of the objective function & constraints are shown in the fig below :



Objective function Linear , Constraints are linear . Solve by linear or Non- linear programming (Min. at A= 100)



Objective fnc. is linear. Constraints are Non-linear . Solve by Non-linear programming only (Min. at A = 90)

Non –linear programming :

1- In direct methods (classical methods).
2- Standard Non –linear programming :

A ♦Un constrained Optimization .
a . Direct search methods :
  1- Hook & Jeeves method.
  2- Rosen bork method.
  3- Nelder & Meads method.
b . Gradient methods .
  1 . Steepest Descent method.
  2 . David on– Fletcher –Powell method.
  3 . Fletcher –Reeves method.

B ♦ constrained Optimization .
a . complex method   b . Rosen bork method.

C ♦ Sequential un constrained Optimization
a .Penalty fnc .
b . The SUMT method of fiacco & Mc- cormick.

♦*Lagrange Multiplier Methods*:

The general non –linear programming Problem can be converted to a Problem that contains only equality constraints as follows :

Min .Z (X_i) ; i =1,2,… ,n
St.
$h_j(X_i) = 0$; j= 1,2,….,m
$g_j(X_i) \geq 0$; j = (m+1) , (m+2) ,….,P

Solution: let $g_j(X_i) - \sigma_j^2 = 0$; j = m+1 , m+2 ,…..,p

where σ_j^2 is slack variable to convert the inequality constraints to equality constraints$(\sigma_j \geq 0)$

$$L(x, w, \sigma^2) = Z(X_i) + \sum_{j=1}^{w} w_j \cdot h_j(X) + \sum_{j=m+1}^{P} w_j \left(g_j(X_i) - \sigma_j^2\right)$$

$$\frac{\partial L}{\partial x} = 0 \quad ; \quad \text{for} \quad i = 1,2, \ldots ,n$$

$$\frac{\partial L}{\partial w_j} = 0 \quad ; \quad \text{for} \quad j = 1,2, \ldots ,p$$

$$\frac{\partial L}{\partial \sigma_j} = 0 \quad ; \quad \text{for} \quad j = m+1, m+2, \ldots, p$$

$$w_j \leq 0 \quad ; \quad \text{for} \quad j = 1,2, \ldots ,p$$

Note :

1- For max .Z (X_i) replace $w_j \leq 0$ with $w_j \geq 0$.

2- If the above set of equations is non–linear set it is preferable to select another method of Optimization .

3- The number of function decision variable (n) should be greater than the number of equality constraints (m).

[i.e $m < n$]

4- (n+m) variable must be devided & solved .

5- We must check to see X * is a minimum & not stationary points (or maximum)because Lagrange function it self exhibits a saddle point with respect to X & W at the Optimum .

Example (1):

$$\text{Min .Z} = 4X_1^2 + 5X_2^2$$

$$\text{St.} \quad 2X_1 + 3X_2 = 6$$

Solution:

$n = 2,\ m = 1$; $\therefore$ $m < n$ o.k.

$L(x, w) = 4X_1^2 + 5X_2^2 + w(2X_1 + 3X_2 - 6)$

$\frac{\partial L}{\partial X_1} = 8X_1 + 2W_1 = 0$ ------------- (1)

$\frac{\partial L}{\partial X_2} = 10X_2 + 3W_1 = 0$ ------------- (2)

$\frac{\partial L}{\partial w} = 2X_1 + 3X_2 - 6 = 0$ ------------- (3)

From eq .(1) $\Rightarrow$ $X_1 = \frac{-1}{4} W_1$

From eq .(2) $\Rightarrow$ $X_2 = \frac{-3}{10} W_1$

eq .(3) becomes $\Rightarrow$ $2\left[\frac{-1}{4} W_1\right] + 3\left[\frac{-3}{10} W_1\right] - 6 = 0$

$\therefore$ $W_1 = -4.286$; $X_1^* = 1.071$; $X_2^* = 1.286$

$Z^* = 4[1.071]^2 + 5[1.286]^2 = 12.853$

Example (2): Solve the following Problem by the Lagrange multiplier

Min .Z(X) $= X_1 . X_2$

St. g (X) ; $25 - X_1^2 - X_2^2 \geq 0$

Solution:

n =2 ; m=1 ; $\therefore$ m < n 0.k

$$L(X; w; \sigma_1) = X_1 . X_2 + W_1[25 - X_1^2 - X_2^2 - \sigma_1^2 \;]$$

$$\frac{\partial L}{\partial X_1} = X_2 - 2\, W_1 .\, X_1 = 0 \qquad \text{--------------(1)}$$

$$\frac{\partial L}{\partial X_2} = X_1 - 2\, W_1 .\, X_2 = 0 \qquad \text{--------------(2)}$$

$$\frac{\partial L}{\partial w_1} = 25 - X_1^2 - X_2^2 - \sigma_1^2 = 0 \qquad \text{--------------(3)}$$

$$\frac{\partial L}{\partial \sigma_1} = 2\, W_1 .\, \sigma_1 = 0 \qquad \text{---------------(4)}$$

The simultaneous Solution for $W_1=0$ & for $W_1 \neq 0$ are listed :

W_1	X_1	X_2	σ_1	Z(X)	Remarks
0	0	0	5	0	saddle
– 0.5	3.54	– 3.54	0	– 12.5	minimum
– 0.5	–3.54	3.54	0	– 12.5	minimum
0.5	3.54	3.54	0	12.5	maximum
0.5	– 3.54	–3.54	0	12.5	maximum

(Economics and management of water resources')

Reference :

1 – Edgar T.F., and Himmel blan D.M ., (1989) "Optimization Of Chemical processes"
McGraw –Hill Book company ,New York ,pp(652) .

2 – simth A.,A., Hinton F., & Lewis R.,W(1984) ,"civil Engineering systems analysis & Design " , John Wiley & sons new York ,pp(473).

Problems :

1 ♦ The work ability of on experimental paving material is found to be a function of time . Experimental measures of density may be approximated by the expression :

$$Z = -3t^4 + 4t^3 + 2$$

Determine the optimal time ?

2 ♦ The percentage removal of impurities in a water treatment process can be improved by the addition of two agents .
The percent removed can be expressed as :

$$R= 60 + 8 X_1 + 2 X_2 - X_1^2 - 0.5X_2^2$$

Where X_1 & X_2 are the percentage dose of two agents .
Determine the optimal level of dose for each agent ?

3 ♦ A sedimentation tank is circular in plan with vertical sides above ground and conical hopper bottom below ground . The slope of the conical part being 3 vertically to 4 horizontally . Determine the proportions to hold a volume of 4070 cu.. meter for minimum area of the bottom & sides?

4 ♦ A temporary precast concrete operation is to be set up to produce 70 , 80 & 120 units respectively for use in the months of June, July & August. Due to the need to employ more formwork & over time lab our when production rates are high ,the production cost is a quadratic function of the number of units produced a month, i,e.

$$\text{production cost / month} = 1.75\ X^2$$

where X = number of units produced in that month .Some economy can be achieved by spreading production more evenly & storing units . . Due to double handling ,the cost of storage for one month (or any part) is a linear function of the number of units stored .i.e .

$$\text{storage cost} = 10\ X$$

Determine an optimum production schedule to minimize the total cost ?

5♦ An open channel cross - section is to probation as asymmetrical triangle such that the section area is 46 sq .meter also , as the velocity is to be maintained below a value of (2 m /sec). The wetted perimeter must be not less than 30m .

If excavation costs (C_1 = \$ 35/m^3) & the lining of the inclined bank costs (C_2= \$ 21.5 / m^2) determine the proportions for minimum cost?

6 ♦ water supply for a community is available from any or all of three potential sources . The cost of each supply described as a quadratic function of the flow rate Thus .

$C_1 = 6000 + 100\ Q_1 + 10\ Q_1^2$

$C_2 = 5000 + 80\ Q_2 + 20\ Q_2^2$

$C_3 = 7000 + 120\ Q_3 + 5\ Q_3^2$

Where C is the cost & Q is the flow rate . Determine the optimal development policy to meet a total demand of 100 units . Discuss briefly how the method of solution & the resulting optimal policy might be modified if the following constraints apply :

$$Q_1 \le 50\ ,\ Q_2 \le 40\ ,\ Q_3 \le 65$$

7 ♦ A proposed water supply to a factory comprises a pipeline terminating in a balancing tank adjacent to the factory .

The purpose of the balancing tank is to be supplied with 0.28 m^3/sec .over an 8 hour period in each 48 hours . The flow is zero for the remaining 40 hours .Details of the costs & other specification . are given below obtain an optimal design by selecting 2 or 3 values of diameter & solving for the other variables .state whether or not the objective function is convex or concave with respect to pipe diameter & state the significance of this on the selection of an optimal solution .

Project specification :

Pipe length = 1830 m

Flow capacity Q(m^3/sec) = 2.75 $D^{5/2}$

The tank is to be square in plan with verticals sides & an open top . Calculate costs in terms of the net internal dimensions with no freeboard allowance .

Pipe cost/m length= \$45$\sqrt{D}$ (m)

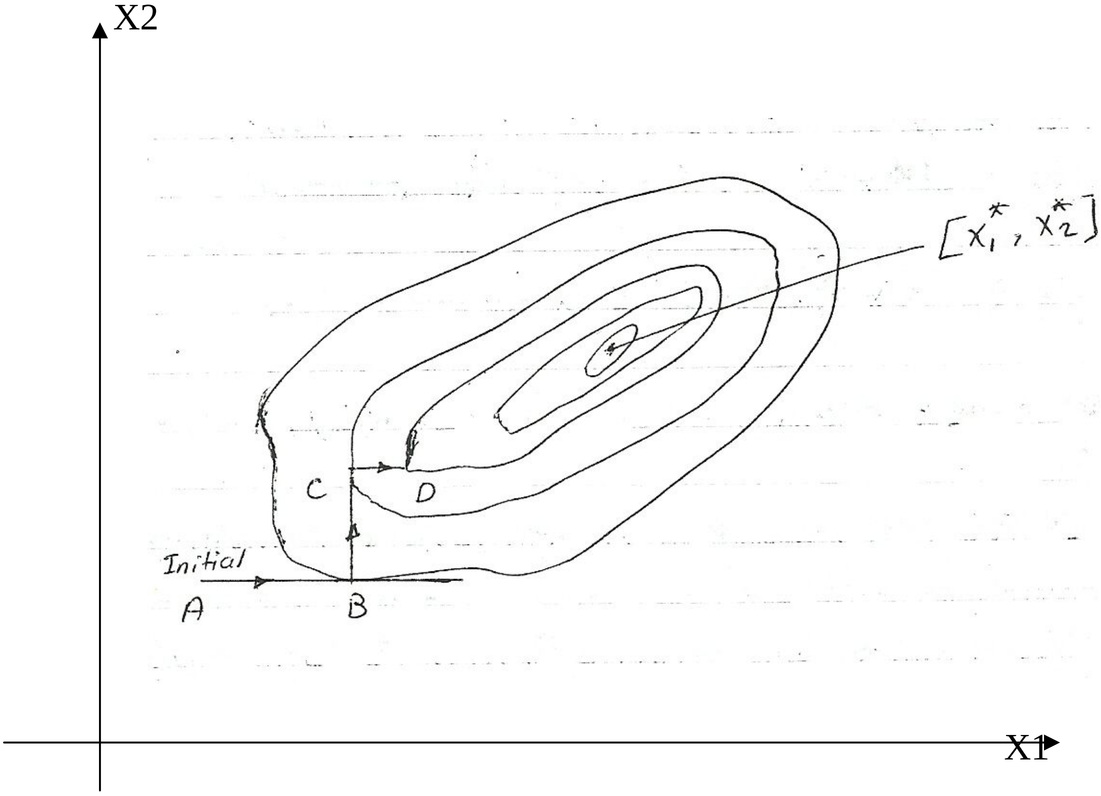
Tank base cost/m^2 = \$10.50

Tank wall cost/m^2 = \$32.25

♦ *Direct search methods :*

consider a function of two variables , its contour lines are shown in fig. below . its minimum at (x_1* , x_2*) the crudest search method is the alternating variable search method we start at some point A & search direction of the x_1– axis for the min. in this direction, & thus find B at which the tangent to the contour is parallel to the x_1– axis , from B we then search in the direction of the x_2– axis & so produced to C & then to D by searching parallel to x_1– axis , etc.

In this way we produced to the optimum point(x_1* , x_2*) . it is clearly possible to extend the idea to function of n variables.



♣ *Test the method !*

A number of functions have been constructed to test the method is used such as :

1) Rosen brocks fnc. $f (x_1 , x_2) = 100 (x_2 – X_1^2)^2+ (1– x_1)^2$;

$x* =(1,1)$

2) powells fnc.

$f (x) = (x_1 +10 x_2)^2+5 (x_3– x_4)^2+ (x_2– 2x_3)^4 +10(x_1– x_4)^2$; $x* =(0 ,0 ,0 ,0)$

3) the two dimensional exponential fnc .

$$f (x_1 , x_2) = \sum_a \left[(e^{–ax1} – e^{–ax2}) – (e^{–a} – e^{–10a})\right]^2$$

where a =0.1 (0.1)1 ; $x* =(1,10)$

c) if $X^{(1)} = X^{(0)}$; i,e no function reduction has been achieved , the exploration is repeated about the same base point $X^{(0)}$ but with reduce step length . i.e

$p = (\frac{DX1}{2}, \frac{DX2}{2}, \ldots\ldots., \frac{DXn}{2})$

d) if $X^{(1)} \neq X^{(0)}$ we make a pattern move

3- pattern move produced is as follows:

a) it seem sensible to move from base point $X^{(1)}$ in the direction $X^{(1)} - X^{(0)}$ since that move has already led to a reduction in the function value . so ,we evaluate the function at the next pattern point.

$X^{(2)} = X^{(0)} + 2(X^{(1)} - X^{(0)})$

$X^{(2)} = 2 X^{(1)} - X^{(0)}$

call this point base (2)

b) then continue with exploratory moves about base(2)

4- terminate the process when the step length has been reduced to a predetermined small value

Example :-

Minimize $f (X) = 3 X_1^2 + X_2^2 - 12 X_1 - 8 X_2$

Solution : start initial value $X^{(0)} = (1, 1)$

Initial change vector P= (0.5 , 0.5)

Exploration search :

---------------------- $X^{(0)} = (1, 1)$; $f (X^{(0)}) = f (1,1) = - 16.0$

$X_1^{(1)} = X_1^{(0)} + D X_1 = 1+0.5 = 1.5$; f (1.5,1) = – 18.25 (success)

$X_2^{(1)} = X_2^{(0)} + D X_2 = 1+0.5 = 1.5$; f (1.5,1.5) = – 21.0 (success)

The exploration search is successful ; hence ;

$X^{(1)} = (1.5 ,1.5)$; $f (X^{(1)}) = - 21.0$

Base (1) = $X^{(1)} = (1.5 ,1.5)$ & set 1 = $X^{(0)} = (1, 1)$

A pattern move is now employed :

$X^{(2)} = 2\ X^{(1)} - X^{(0)} =$ Base (1) – set 1

$X_1^{(1)} = 2 \times 1.5 - 1 = 2$

$X_2^{(2)} = 2 \times 1.5 - 1 = 2$

$X^{(2)} = (2, 2)$; $f(X^{(2)}) = -24.0$

Base (2) = (2, 2)

Exploration search :

------------------------- $X^{(2)} = (2, 2)$; $f(X^{(2)}) = -24.0$

$X_1^{(3)} = 2.0 + 0.5 = 2.5$; $f(2.5,2) = -23.25$ (failure)

$X_1^{(3)} = 2.0 - 0.5 = 1.5$; $f(1.5,2) = -23.25$ (failure)

$X_2^{(3)} = 2.0 + 0.5 = 2.5$; $f(2, 2.5) = -25.75$(success)

Hence :

$X^{(3)}$ (2, 2.5) & $f(X^{(3)}) = -25.75$

Base (1) = $X^{(3)}$ =(2 ,2.5) &

set (1) $X^{(1)} = (1.5 , 1.5)$

A pattern move is now employed :

$X^{(4)} = 2\ X^{(3)} - X^{(1)} =$ Base (1) – set 1

$X_1^{(4)} = 2 \times 2 - 1.5 = 2.5$

$X_2^{(4)} = 2 \times 2.5 - 1.5 = 3.5$

$X^{(4)} = (2.5, 3.5)$; $f(X^{(4)}) = -27.0$

Exploration search :

------------------------- $X^{(4)} = (2.5, 3.5)$; $f(X^{(4)}) = -27.0$

$X_1^{(5)} = 2.5 + 0.5 = 3$; $f(3,3.5) = -24.75$ (failure)

$X_1^{(5)} = 2.5 - 0.5 = 2$; $f(2 , 3.5) = -27.75$ (success)

$X_2^{(5)} = 3.5 + 0.5 = 4$; $f(2, 4) = -28.0$(success)

Base (1) = $X^{(5)}$ =(2 ,4) & set (1) = (2.0 , 2.5)

A pattern move is now employed :

$$x^{(k)} = 2\,x^{(k-1)} - x^{(k-3)}$$

$$x_1^{(6)} = 2 \times x_1^{5} - x_1^{3} = 2 \times 2 - 2 = 2.0$$

$$x_2^{(6)} = 2 \times 4 - 2.5 = 5.5$$

Hence;

$x^{(6)} = (2.0, 5.5)$; $f (x^{(6)}) = - 25.75$

An exploration search is compared to $f (x^{(5)}) = - 28.0$

$x_1^{(7)} = 2.0 + 0.5 = 2.5$; $f (2.5 , 5.5) = - 25$ (failure)

$x_1^{(7)} = 2.0 - 0.5 = 1.5$; $f (1.5 , 5.5) = - 25$ (failure)

$x_2^{(7)} = 5.5 + 0.5 = 6.0$; $f (2.0 , 6.0) = - 24$ (failure)

$x_2^{(7)} = 5.5 - 0.5 = 5.0$; $f (2 , 5.0) = - 27$ (failure)

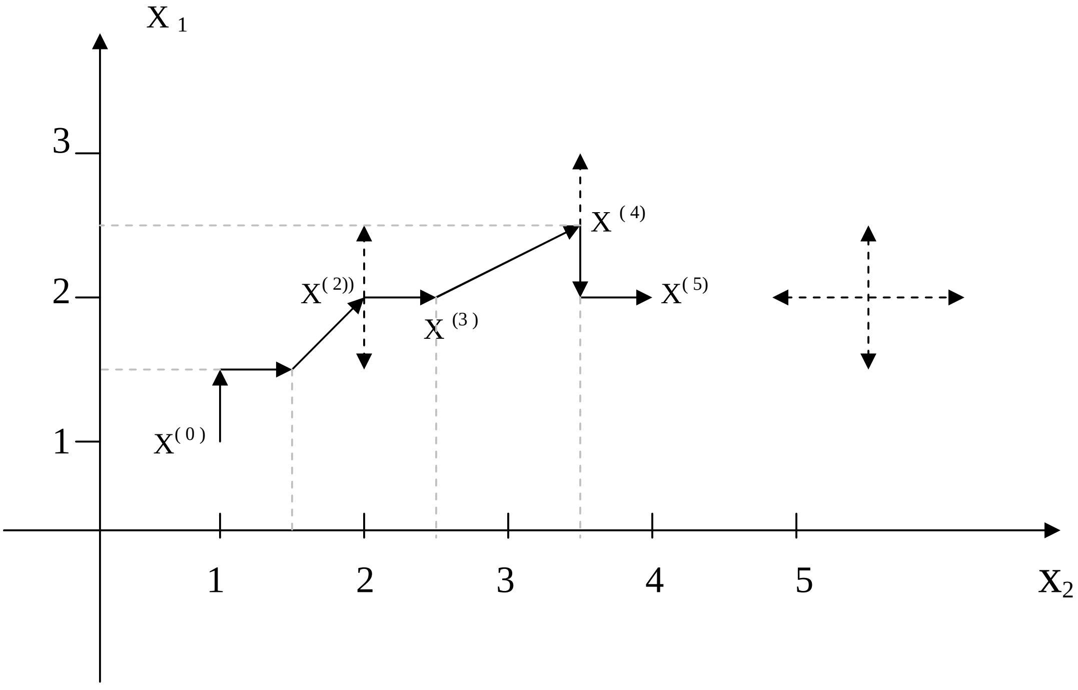
At this , the pattern search is decmed a failure .

The algorithm now returns to point $x^{(5)}$ & is began as if $x^{(5)}$ were $x^{(0)}$ initially .if a pattern search about $x^{(5)}$ is successful ,then a pattern move will be made in the direction of improvement .

If the pattern search about $x^{(5)}$ fails , then the step size vector

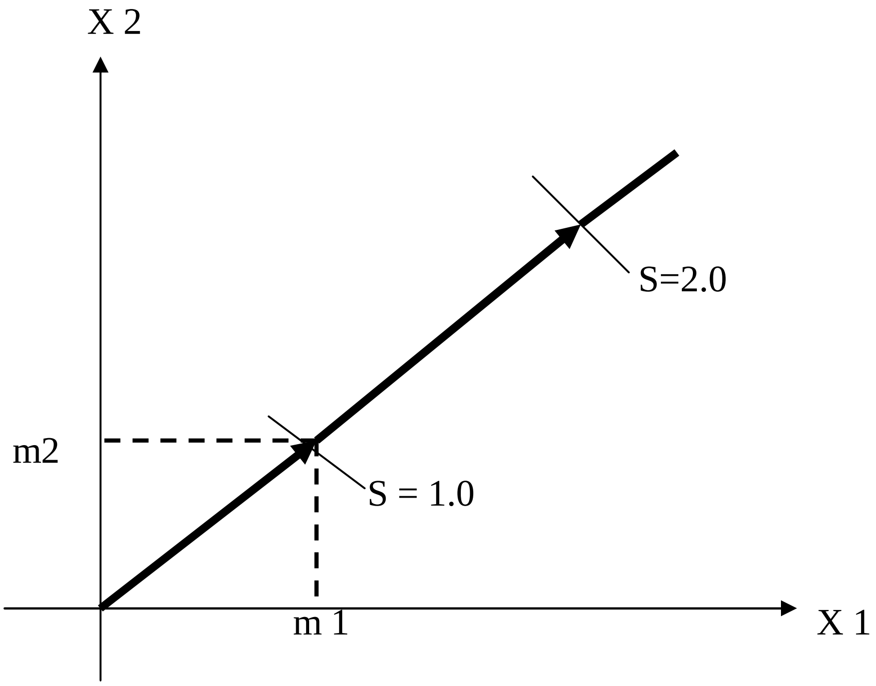
$P = (DX_1 , DX_2 , \ldots\ldots, DX_n)$ is change to $p = (\frac{DX1}{2}, \frac{DX2}{2}, \ldots\ldots, \frac{DXn}{2})$ & the process begins again

<u>Note:-</u> in this example , the procedure will terminate at point $x^{(5)}$ since the global minimizing solution is $x^* =(2.0 , 4.0)$



♦*Gradient method (the method of steepest descent) :*

Suppose from appoint $x_i^{(j)}$, we move to a neighbor point
$x_{i+1}^{(j+1)} = x_i^{(j)} + S_x m_i^{(j+1)}$
where m_i is the direction of move for the components (see fig.)
S: the step length



Suppose we wish to take a small step (ds) in such away that the objective function y = f(x) increases or decreases as much as possible . the distance of move is given by

$$ds = \sqrt{dx_1^2 + dx_2^2 + \ldots\ldots\ldots\ldots + dx_n^2} \qquad \text{------- (1)}$$

Assume y to be differentiable , the change in y associated with a set of displacement dx_i is given by :

$$dy = \sum_{i=1}^{n} (\frac{\partial y}{\partial xi}) dxi \quad \text{or} \quad \frac{dy}{ds} = \sum_{i=1}^{n} (\frac{\partial y}{\partial xi}) \frac{dxi}{ds} \qquad \text{------- (2)}$$

Now a particular set of displacements will make $\frac{dy}{ds}$ as large or small as possible. This is the direction of steepest ascent or descent.

Maximize / Minimize $\frac{dy}{ds} = \sum_{i=1}^{n} (\frac{\partial y}{\partial xi}) \frac{dxi}{ds}$

Subject to $ds = \sqrt{\sum_{i=1}^{n} dx_i^2} = 1$

Using the Lagrange function

Maximize / Minimize: $\sum_{i=1}^{n} (\frac{\partial y}{\partial xi}) \frac{dx_i}{ds} - \lambda \left[1 - \sum_{i=1}^{n} dx_i^2 \right]$

Differentiating with respect to $\frac{dx_i}{ds}$

$\frac{\partial y}{\partial x_i} - 2\lambda \left(\frac{dx_i}{ds} \right) = 0$; i= 1, 2 ,3 ,4 , …….., n ------(3)

& with respect to Lagrange function multiplier :

$$\sum_{i=1}^{n} \left(\frac{dx_i}{ds} \right)^2 = 1$$

$\therefore \frac{1}{4\lambda^2} \sum_{i=1}^{n} \left(\frac{dy}{dx_i} \right)^2 = 1$ or $2\lambda = \sqrt{\sum_{i=1}^{n} \left(\frac{dy}{dx_i} \right)^2}$ ------ (4)

$\frac{\partial x_i}{\partial s} = \frac{\partial y}{\partial x_i} . \frac{1}{2\lambda}$ i= 1, 2, 3, ………., n

$$x_i^{(i+1)} = x_i^{(i)} + \left[\frac{\partial y}{\partial x_i} . \frac{1}{2\lambda} \right] \quad s = x_j^{(j)} + m_i^{[j+1]}.s$$

Eq. (3) a move gives the greater increase in y is given by:

$m_i^{(j+1)} = \frac{\frac{\partial y}{\partial x_i}}{\sqrt{\sum_{i=1}^{n} \left(\frac{\partial y}{\partial x_i} \right)^2}}$ i= 1, 2,3 , …….., n

& eq. (3) giving the maximum decrease in y is:

$m_i^{(j+1)} = \frac{-\frac{\partial y}{\partial x_i}}{\sqrt{\sum_{i=1}^{n} \left(\frac{\partial y}{\partial x_i} \right)^2}}$ i= 1, 2,3 , …….., n

♦♦ Example:-

Minimize $F = 2(x_1+x_2)^2 + 2(x_1^2+x_2^2)$

Start point $\overrightarrow{x_0^{(0)}} = [5,2]$; F =156.9

Solution :-

$$\frac{\partial F}{\partial x_1} = 4(x_1+x_2) +4\, x_1 = 8\, x_1+4\, x_2$$

$$\frac{\partial F}{\partial x_2} = 4(x_1+x_2) +4\, x_2 = 4\, x_1+8\, x_2$$

Iteration (1):-

$$\frac{\partial F}{\partial x_1}\Big|_{x_0^{(0)}} = 8\times 5 +4 \times 2=48$$

$$\frac{\partial F}{\partial x2}\Big|_{x_0^{(0)}} = 4\times 5 +8 \times 2=36$$

$$m_i^{(j+1)} = \frac{-\frac{\partial F}{\partial x_i}}{\sqrt{\sum_{i=1}^{n}\left(\frac{\partial F}{\partial x_i}\right)^2}}$$

$$m_1^{(1)} = \frac{-48}{\sqrt{(48)^2+(36)^2}} = -0.8$$

$$m_2^{(1)} = \frac{-36}{\sqrt{(48)^2+(36)^2}} = -0.6$$

$x_1^{(1)} = x_1^{(1)} + S\,.\,m_1^{(1)} = 5- 0.8\, S$

$x_2^{(1)} = x_2^{(1)} + S\,.\,m_2^{(1)} = 2- 0.6\, S$

$F(x) = 2\,[\,5- 0.8\, S + 2 - 0.6\, S]^2 + 2\,[\,(5- 0.8\, S)^2 + (2 - 0.6\, S)^2\,]$

$= 2[\,7- 1.4\, S]^2 + 2\,[\,(5- 0.8\, S)^2 + (2 - 0.6\, S)^2\,]$

$F(x) = \frac{dF}{dS} = 4(7- 1.4\, S)(- 1.4) + 4[\,(5- 0.8\, S)(- 0.8) + (2 - 0.6\, S)(- 0.6)\,]=0$

$2.96\, S =15 \Rightarrow S = \frac{15}{2.96} =5.07$

$x_1^{(1)} =5- 0.8\, S = 5- 0.8\,(5.07) = 0.944$

$x_2^{(1)} = 2 - 0.6\ S = 2 - 0.6\ (5.07) = -1.042$

F (0.944 , – 1.042) = 3.977

Iteration (2):-

$$\frac{\partial F}{\partial x_1}\Big|_{x^{(2)}} = 8\ x_1 + 4\ x_2 = 8\ (0.944) + 4\ (-1.042)$$

$$= 3.384$$

$$\frac{\partial F}{\partial x2}\Big|_{x^{(2)}} = 4\ x_1 + 8\ x_2 = 4\ (0.944) + 8\ (-1.042)$$

$$= -4.56$$

$$m_1^{(2)} = \frac{-3.384}{\sqrt{(3.384)^2 + (-4.56)^2}} = -0.5959$$

$$m_2^{(2)} = \frac{-(-4.56)}{\sqrt{(3.384)^2 + (-4.56)^2}} = 0.803$$

$x_1^{(2)} = x_1^{(1)} + S\ .\ m_1^{(2)} = 0.944 - 0.5959\ S$

$x_2^{(2)} = x_2^{(1)} + S\ .\ m_2^{(2)} = -1.042 + 0.803\ S$

$F(x) = 4\ [-0.098 + 0.2071\ S]\ (0.2071) + 4\ [\ (0.944 - 0.5959\ S)(\ -0.5959\ ) + (-1.042 + 0.803\ S)\ (0.803)\ ] = 0$

$-1.4195 + 1.0428\ S = 0 \Rightarrow\ S = 1.361$

$x_1^{(2)} = 0.944 - 0.5959\ (1.361) = 0.133$

$x_2^{(2)} = -1.042 + 0.803\ (1.361) = 0.0501$

F (0.133 , 0.0501) = 0.11

Iteration (3):-

$$\frac{\partial F}{\partial x_1}\Big|_{x^{(3)}} = 8\ x_1 + 4\ x_2 = 1.2962$$

$$\frac{\partial F}{\partial x_2}\Big|_{x^{(3)}} = 4\ x_1 + 8\ x_2 = 0.9352$$

$$m_1^{(3)} = \frac{-1.2692}{\sqrt{(1.2692)^2 + (0.9352)^2}} = -0.8051$$

$$m_2^{(3)} = \frac{-0.9352}{\sqrt{(1.2692)^2 + (0.9352)^2}} = -0.5932$$

$x_1^{(3)} = x_1^{(2)} + S \,.\, m_1^{(3)} = 0.133 - 0.8051\ S$

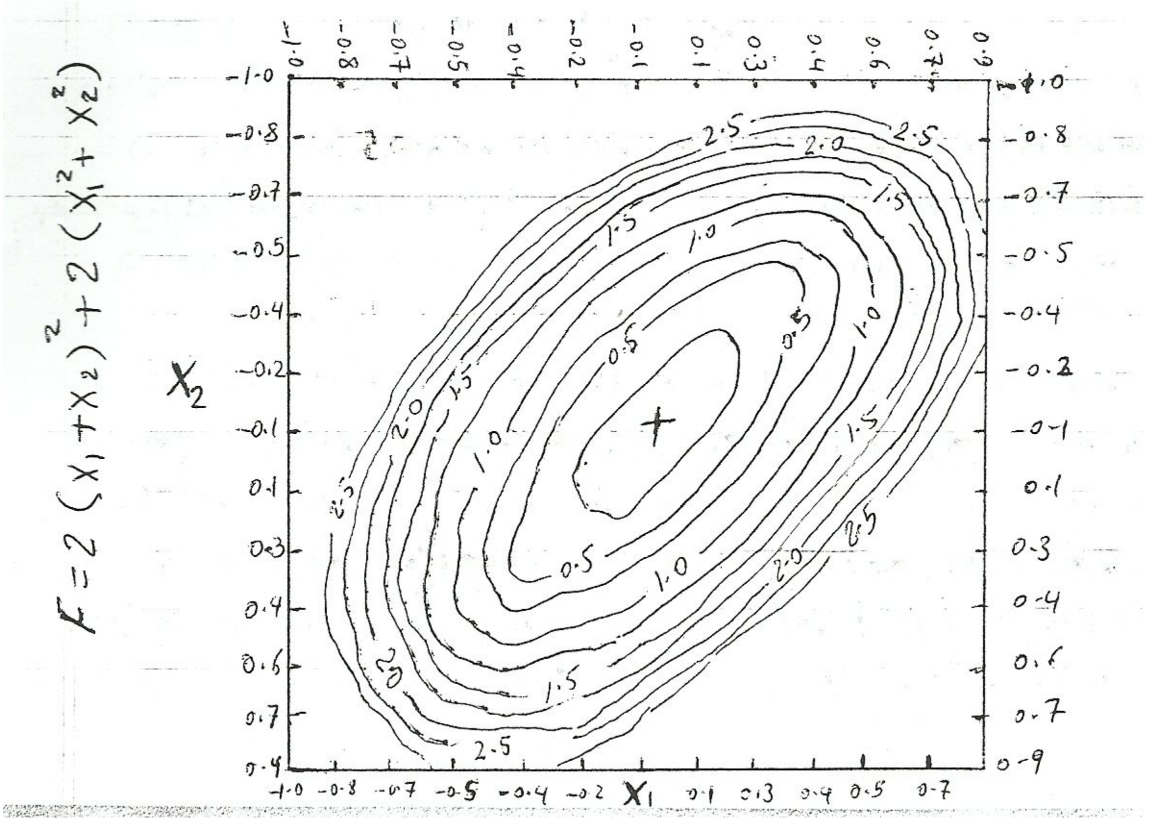
$x_2^{(3)} = x_2^{(2)} + S \,.\, m_2^{(3)} = 0.0501 - 0.5932\ S$

$S = 0.13 \Rightarrow\ x_1^{(3)} = 0.089\ \&\ x_2^{(3)} = -0.027$

F (0.0289 , – 0.027) = 0.13

Optimal X* (0, 0) & F (0, 0) = 0

Iteration	m_1	m_2	optimal S	X_2	X_1	F
0	--	--	--	2	5	156.9
1	– 0.8	– 0.6	5.07	– 1.042	0.944	3.977
2	– 0.5959	0.803	1.361	0.0501	0.133	0.11
3	– 0.8051	– 0.5932	0.13	– 0.027	0.0289	0.13
4	– 0.7752	0.6317	0.039	– 0.002		0.00



((EXERCISES))

1♣ A two bar cantilever is to be designed to support the vertical load W(=500KN) at a distance L(=3m) from the wall as shown in fig. (1) the members AB & BC are to be tubular in section of diameter D and wall thickness T. the working stresses in the two members are given by :

f (tension) = 125 N/mm^2

f (compression) = 125 (1– 0.01 L/D) N/mm^2

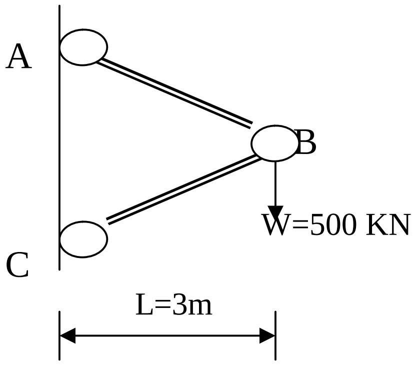


Fig.(1) tow bar cantilever

the vertical deflection of point B should be not greater than 15 mm assuming a modulus of elasticity of 200 KN/mm^2 .

Determine a design which will minimize the total weight of material assuming D & T to be constant .

2♣ A roof member is to formed as a folded plate with the cross – section shown in fig . (2) the member is to be support a super imposed load of 3500 N /m^2 over a simply supported span of (4 m) . the maximum allowable working stress is 100N/mm^2 & the mind – span deflection due to super imposed load only is to be to not greater than (1/ 180) of the span .

The properties of one complete wave of the member may be approximated by the following expressions :

$A = 2b .t + 2 (h – t) (t / \sin\varphi)$

$I = 2b .t (h/2)^2 + (t / \sin\varphi) (h–t)^3 / 6$

Assuming an elastic modules of 200 KN/mm2 design the dimensions for minimum weight

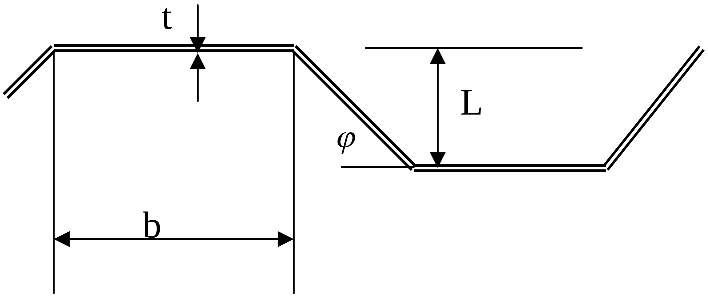


Fig.(2) folded plate roof member

3 ♣ Fig. (3) shows the cross- section of amass concrete wall to retain a height of H=3m of soil . for simplicity the pressure distribution on both front & rear faces may be assumed to be linear & given by:

$P_{active} = 0.3\ \gamma\ h$

$P_{passive} = 1.8\ \gamma\ h$

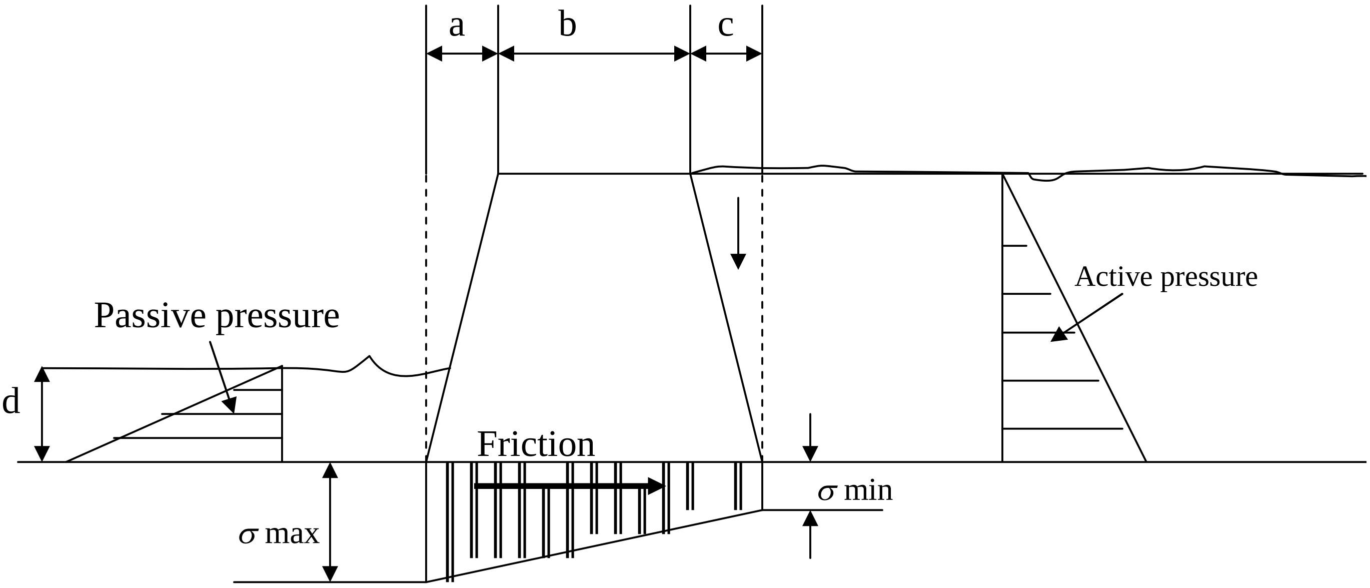


Fig. (3) : Mass concrete retaining wall

Where γ is the specific weight of the soil (14 KN/m^3) friction on the rear face may be ignored but on the base a coefficient of (0.35) should be used . the wall is to be designed for the following criteria:

σ max $\leq$ 100 kpa

σ min ≥ 0

Factor of safety against sliding = 2

The cost of construction may be estimated on the basis of concrete & form work using the following rates:

Concrete	\$ 120 /m^3
Front face forms	\$ 30 /m^3
Rear face forms	\$ 15 /m^3

Proportion the dimensions a , b , c & d for minimum cost (weight of concrete = 23 KN/m3 approximately) .

4 ♣ Fig. (4) shows the arrangement of a sheet piling waling beam , & struts to brace an excavation . the pressure distribution (greatly simplified for this analysis) is assumed to be uniform from the surface down to a point A.

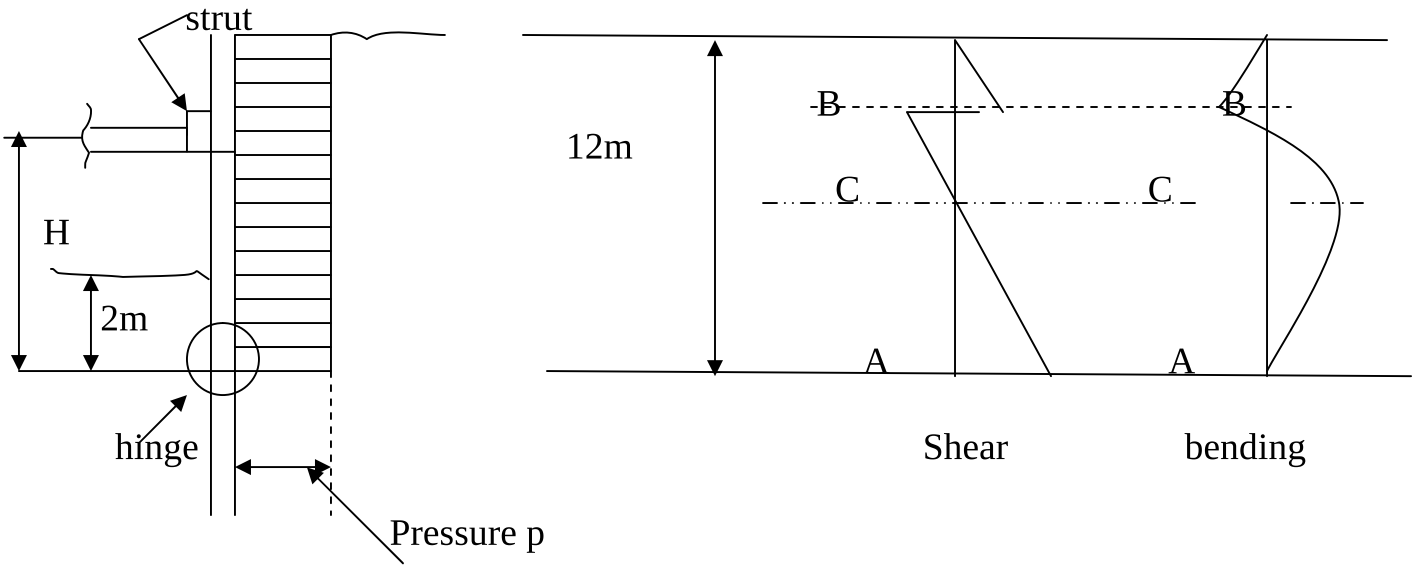


Fig. (4) sheet piling system

Which represent a virtual hinge . The pressure is 30 KN/m^2 .costs for the three components are given (per meter length of trench) in terms of the force action resisted the members thus:

Piling : C_p (\$/m) = 60 + M_p /40 (M_p in KN .m /m of trench)

Wale : C_w (\$/m) = 4 + M_w /20 (M_w in KN .m)

Struts : C_s (\$/m) = 124 / S + F /20 (F in KN /m of trench)

Where M_p & M_w are the moments in the piling & waling beam respectively , F is the thrust per meter of trench , & S is the strut spacing in meters.

Determined the optimal values of the height H (m) & strut spacing S(m) & hence the minimum cost per meter of trench (Adapted from stark & Nicholls)

5 ♣ A catchments has been found to have an instantaneous unit respons function as defined below , the values being at equal time intervals of 1 hour.

u( ) =	0	0.0	0.0464	0.0976	0.1585
	0.1464	0.1196	0.0988	0.0773	0.0572
	0.0428	0.0379	0.0304	0.0237	0.0173
	0.0132	0.0106	0.0079	0.0059	0.0036
	0.0025	0.0018	0.0009		

It is intended to model this as the response of a cascade of 4 linear reservoirs & a linear channel in series if the lag times of the reservoirs are respectively K_1 , K_2 ,K_3 , & K_4 & the channel lag is K_0 the response can be expressed as follows:
$u(t_1) = 0$ for $t_1 = t - K_0 < 0$

$$u(t_1) = \sum_{j=1}^{n}\left[k_j^{n-1}.e^{-t.k_j}/\prod_{i=1}^{A}\left(K_j - K_i\right)\right] \quad \text{for} \quad t_1 = t - K_0 > 0$$

in which n=4 for this case
Determined the optimal lag values which best represents the observed response (Smith & Kimmetl).

6 ♣ solve the prob. Of section 1.4 by direct search .

Example (1) :

Minimize $Z(x)= [X_1 – 10]^2 + [X_2 – 5]^2$

Subject to : $X_1 +2 X_2 =14$

Rosen brock minimization procedure

Min. $Z=[(X_1 – 10)^2 + (X_2 – 5)^2 + 1 \times 10^2 \times (X_1 + 2 X_2 – 14)^2]$

Parameters :

MAXK = 1000 MKAT=506 MCYC=50 NSTEP=1

ALPHA=3.00 BETA=0.5 EPSY=1×10^{-9}

Total No. of stages = 17

Total No. of function evaluation= 352

Final value of objective function = 7.18567

parameters	Initial values	Optimized values	Lower cons.	Upper cons.
X_1 X_2	6.00 3.00	8.808056 2.602023	0.00 0.00	20.00 10.00

++

Example (2) :

Minimize $Z(x)= [X_1 – 10]^2 + [X_2 – 5]^2$

Subject to : $X_1 +2 X_2 \geq 14$

Rosen brock minimization procedure

PARAMETERS :

MAXK = 1000 MKAT=506 MCYC=50 NSTEP=1

ALPHA=3.00 BETA=0.5 EPSY=1×10^{-9}

Total No. of stages =4

Total No. of function evaluation= 90

Penalty function =0.0

Final value of objective function = $2.4761 * 10^{-10}$

parameters	Initial values	Optimized values	Low cons.	Up. cons.
X_1 X_2	6.00 3.00	10.00 4.999984	0.00 0.00	20.00 10.00

Example (3) :

$$\text{Minimize } Z(x)= [X_1 - 10]^2 + [X_2 - 5]^2$$

$$\text{Subject to : } X_1 +2 X_2 \leq 14$$

Rosen brock minimization procedure

Parameters :

MAXK = 1000 MKAT=506 MCYC=50 NSTEP=1

ALPHA=0.00 BETA=0.5 ESPY=$1*10^{-9}$

Total No. of stages =14

Total No. of function evaluation= 313

Penalty function = $1.4302 * 10^{-2}$

Final value of objective function = 7.18563

parameters	Initial values	Optimized values	Low. cons.	Up. cons.
X_1	6.00	8.802032	0.00	20.00
X_2	3.00	2.604964	0.00	10.00

++

Example (4) :

$$\text{Minimize } Z(x)= [X_1 - 10]^2 + [X_2 - 5]^2$$

$$\text{Subject to : } X_1 +2 X_2 =14$$

Rosen brock minimization procedure

$$\text{Min. } Z=[(X_1 - 10)^2 + (X_2 - 5)^2 + 1 \times (\exp \left| X_1 + 2 X_2 - 14 \right|) - 1]$$

Parameters :

MAXK = 1000 MKAT=3000 MCYC=50 NSTEP=1

ALPHA=3.00 BETA=0.5 ESPY=$1*10^{-9}$

Total No. of stages =11 ; Total No. of function evaluations =222

Penalty function = 1.10272 ; final value of objective function =6.62944

parameters	Initial values	Optimized values	Low. cons.	Up. cons.
X_1	6.00	8.948646	0.00	20.00
X_2	3.00	2.897293	0.00	10.00

Rosenbrock unconstrained procedure :

To minimize the objective function

$f(X_1 , X_2 , X_3 ,......., X_n)$, the algorithm of the procedure on the following steps :

1)A starting point $x_1^{(0)}, \quad x_2^{(0)}, ..., x_n^{(0)}$ & initial step size (λ_i , i= 1,2,.......,n) are selected & the objective function evaluated.

2) The 1st variable X_1 is stepped a distance λ_1 parallel to the axis & the function evaluated. If the value of obj. fnc. decreased, the move is termed a success & λ_1 increased by a factor α $(\alpha \geq 1)$. If the value of obj. fnc. increased , the move is termed a failure & λ_1 decreased by a factor β $(0 < \beta < 1)$ & the direction of movement reversed . The value of α & β recommend ended by Rosen brock are $\alpha = 3.0$ & $\beta = 0.5$.

3) The next variable X_i (i = 1,2,.......,n) is in turn stepped a distance λ_i parallel to the axis. The same acceleration or deceleration in step (2) are applied for all variable. In consecutive repetitive sequences until one process & one failure have been encountered at least in all N- direction.
i.e. continue the search sequentially along the directions
$S_1^{(j)} , S_2^{(j)} , ..., S_i^{(j)} , ..., S_n^{(j)}, ...$
Until at least one step has been successful & one step has failed in each of the N-direction $S_1^{(j)} , S_2^{(j)} , ..., S_i^{(j)} , ..., S_N^{(j)}$
4) compute the new set of direction $S_1^{(j+1)} , S_2^{(j+1)} , ..., S_i^{(j+1)} , ..., S_n^{(j+1)}$
For use in next or (j+1)the stage of minimization procedure :
a-<u>compute a set of independent direction :</u>
p1 ,p2 ,.........., pi ,......pn as :

$$p_{N.N} = [p_1 ,p_2 ,.........., p_i ,......p_n]$$
$$= [S_1^{(j)} , S_2^{(j)} , ..., S_i^{(j)} , ..., S_N^{(j)}]$$

$$\begin{bmatrix} A_1 & 0 & 0 & ..... & 0 \\ A_2 & A_2 & 0 & ..... & 0 \\ A_3 & A_3 & A_3 & ...... & 0 \\ ..... & ..... & ...... & ...... & ...... \\ A_N & A_N & ..... & ..... & ...... \end{bmatrix}$$

In which :

Ai = the algebraic sum 0f all the successful step length in the corresponding direction $S_i^{(j)}$, i= 1 ,2 ,….., N

P_1 = The vector joining the starting point & the final point obtained after the sequence of the search in the jth stage .

P_2 ,….,P_n = the algebraic sum. of the successful step length in all directions except the first one & so on .

These linearly independent vectors P_1 , P_2 ,….,P_N can be used to generate a new set of orthogonal directions

b)Set $D_1^{(j)} = P_1^{(j)}$, therefore :

$$S_i^{(j)} = \frac{D_1^{(j)}}{\sqrt{D_1^{(j)T} . D_1^{(j)}}}$$

c) compute: $D_i^{(j)} = P_i^{(j)} - \sum_{m=1}^{i-1} \left[\left(B_{i+1}^{(j)T} . B^{(i)} \right) S_m^{(i)} \right]$

with:

$$S_i^{(j)} = \frac{D_1^{(j)}}{\sqrt{D_1^{(j)T} . D_1^{(j)}}} \qquad i= 1, 2, 3,\ldots\ldots,N$$

5)Take the best point obtained in the present stage & repeat the same procedure of searching from step (2) on words i.e.

New $[\overrightarrow{X}_{L,i}{}^{(j)}]$ = Old $[\overrightarrow{X}_{L,i}{}^{(j)}]$ + λ_i [$S_{L,i}{}^{(j)}$]

Where :

L = variable index ; L = 1 , 2,… ,N

I = direction index ; i = 1 , 2 , …,N

J =stage index

6) The procedure terminates when the convergence criterion is satisfied:

T : is the transformation

i.e. $D.T.D = [P_1 . P_2] . \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = [P_1^2 + P_2^2]$

Example :

$$\text{Min } f(x) = X_1 - X_2 + 2X_1^2 + 2X_1 . X_2 + X_2^2 \ .$$

$$\text{Initial Base } X = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \ \& \quad F(x) = 0.$$

$$S_1^{(0)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad ; \quad S_2^{(0)} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad ; \quad \lambda_i = \begin{bmatrix} 0.8 \\ 0.8 \end{bmatrix}$$

$\beta = 0.5 \quad ; \quad \alpha = 3.0$

Solution: (1)

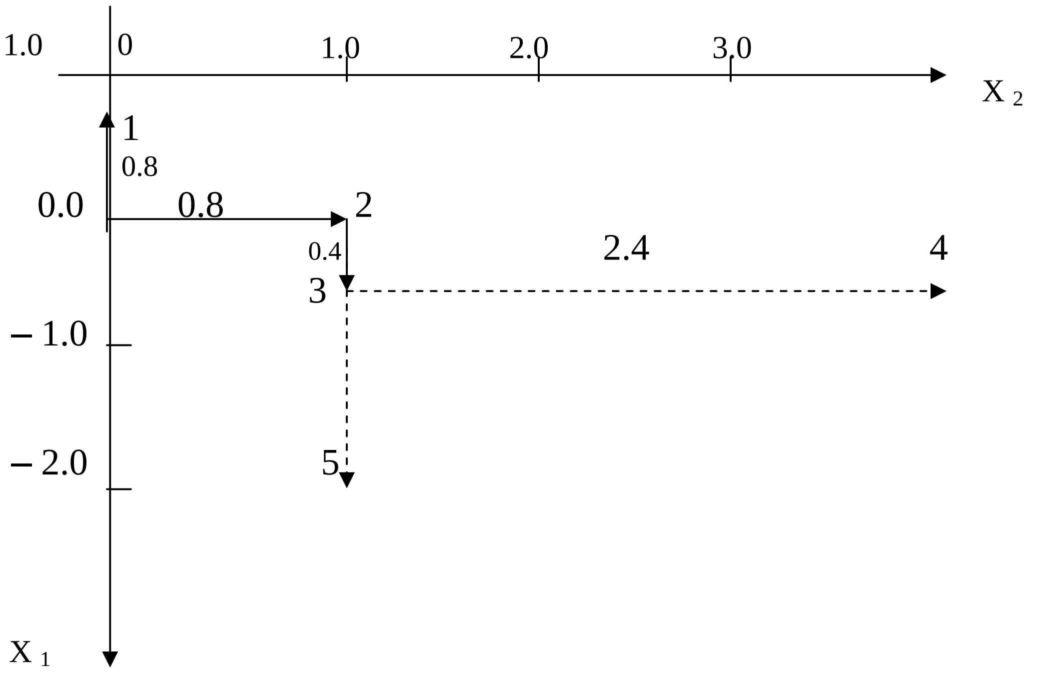
Stage (1) :

$$[\ \vec{X}\ ]_1 = [\ \vec{X}\ ]_{base} + \lambda_1 [\ \vec{S}_1^{(0)}\ ]$$

$$= \begin{bmatrix} 0 \\ 0 \end{bmatrix} + 0.8 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.8 \\ 0 \end{bmatrix}$$

$$[\ \vec{X}\ ]_2 = [\ \vec{X}\ ]_{base} + \lambda_2 [\ \vec{S}_2^{(0)}\ ]$$

$$= \begin{bmatrix} 0 \\ 0 \end{bmatrix} + 0.8 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0.8 \end{bmatrix}$$



point	X_1	X_2	λ_1	λ_2	F(x)	Remark
0	0.0	0.0	-	-	0.0	Initial
1	0.8	0.0	0.8	-	2.08	Failure
2	0.0	0.8	-	0.8	– 0.16	Success
3	– 0.4	0.8	– 0.4	-	– 0.88	Success
4	– 0.4	3.2	-	2.4	4.4	Failure
5	–1.6	0.8	–1.2 A_1= –0.4	- A_2=0.8	0.8	Failure

<u>Note :</u> New base point X = $\begin{bmatrix} -0.4 \\ 0.8 \end{bmatrix}$ & F(x) = – 0.88 we have at least one failure & at least one success in each direction $S_1^{(0)}$, $S_2^{(0)}$

(2) New set of search direction

$$P_{2x2} = [\ p_1 , p_2\ ] = [\ S_1^{(0)} . S_2^{(0)}\ ] \begin{bmatrix} A_1 & 0 \\ A_2 & A_2 \end{bmatrix}$$

$$P_{2x2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} . \begin{bmatrix} -0.4 & 0 \\ 0.8 & 0.8 \end{bmatrix} = \begin{bmatrix} -0.4 & 0 \\ 0.8 & 0.8 \end{bmatrix}$$

$$D_1^{(1)} = P_1^{(1)} = \begin{bmatrix} -0.4 \\ 0.8 \end{bmatrix} \ \& \ \ P_2^{(1)} = \begin{bmatrix} 0 \\ 0.8 \end{bmatrix}$$

$$S_1^{(1)} = \frac{D_1^{(1)}}{\sqrt{D_1^{(1)^T} . D_1^{(1)}}} = \frac{1}{\sqrt{(-0.4)^2 + (0.8)^2}} . \begin{bmatrix} -0.4 \\ 0.8 \end{bmatrix} = \begin{bmatrix} -0.447 \\ 0.894 \end{bmatrix}$$

$$D_2^{(1)} = P_2^{(1)} - \sum_{m=1}^{1} \left[\left(p_2^{(1)^T} . S_m^{(1)} . S_m^{(1)} \right) \right] = \begin{bmatrix} 0 \\ 0.8 \end{bmatrix} - [0 \quad 0.8] \begin{bmatrix} -0.447 \\ 0.894 \end{bmatrix} \begin{bmatrix} -0.447 \\ 0.894 \end{bmatrix}$$

$$= \begin{bmatrix} 0.32 \\ 0.16 \end{bmatrix} *$$

$$S_2^{(1)} = \frac{D_1^{(j)}}{\sqrt{D_1^{(j)^T} . D_1^{(j)}}} = \frac{1}{\sqrt{(0.32)^2 + (0.16)^2}} . \begin{bmatrix} 0.32 \\ 0.16 \end{bmatrix}$$

$$= \begin{bmatrix} 0.894 \\ 0.447 \end{bmatrix}$$

$$* \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \cdot \begin{bmatrix} X_1 & y_1 \\ X_2 & y_2 \\ X_3 & y_3 \end{bmatrix} = \begin{bmatrix} Z_1 & W_1 \\ Z_2 & W_2 \\ Z_3 & W_3 \end{bmatrix}$$

$Z_1 = a_1X_1 + a_2X_2 + a_3X_3$.
$Z_2 = b_1X_1 + b_2X_2 + b_3X_3$.
$Z_3 = c_1X_1 + c_2X_2 + c_3X_3$.
$W_1 = a_1y_1 + a_2y_2 + a_3y_3$.
$W_2 = b_1y_1 + b_2y_2 + b_3y_3$.
$W_3 = c_1y_1 + c_2y_2 + c_3y_3$.

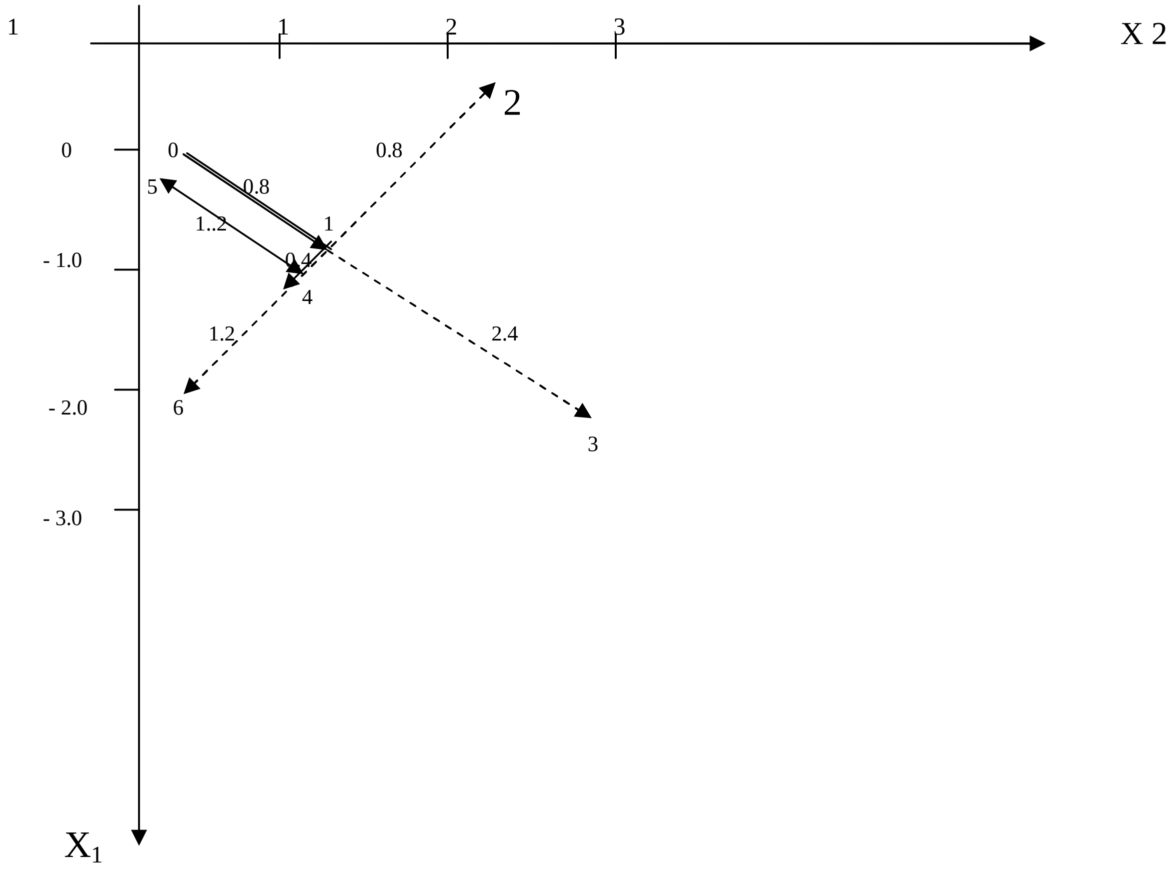
$$\begin{bmatrix} 0 \\ 0.8 \end{bmatrix} - [\ 0 \times (-0.447 + 0.8\ (\ 0.894)\ ] \cdot \begin{bmatrix} -0.447 \\ 0.894 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0.8 \end{bmatrix} - [0.7152] \cdot \begin{bmatrix} -0.447 \\ 0.894 \end{bmatrix} = \begin{bmatrix} 0 - 0.7152(-0.447) \\ 0.8 - 0.7152(0.894) \end{bmatrix} = \begin{bmatrix} 0.32 \\ 0.16 \end{bmatrix}$$

Stage (2): Base value $X = \begin{bmatrix} -0.4 \\ 0.8 \end{bmatrix}$ & $F(x) = -0.88$

$S_1^{(1)} = \begin{bmatrix} -0.447 \\ 0.894 \end{bmatrix}$, $S_2^{(1)} = \begin{bmatrix} 0.894 \\ 0.447 \end{bmatrix}$

$X_1 = [\ X\ ]_B + \lambda_1\ [S_1^{(1)}\ ]$ & $X_2 = [\ X\ ]_B + \lambda_2\ [S_2^{(1)}\ ]$



point	X_1	X_2	λ_1	λ_2	F(x)	Remark
0	-0.4	0.8	--	--	- 0.88	B- value
1	-0.758	1.516	0.8	--	-1.125	Success
2	-0.042	1.873	-	0.8	1.439	Failure
3	-1.831	3.662	2.4	--	1.212	Failure
4	-1.116	1.337	-	-0.4	-1.159	Success
5	-0.579	0.263	- 1.2	--	-0.408	Failure
6	-2.189	0.88	--	- 1.2	3.731	Failure
			0.8	-0.4		

New base point $X = \begin{bmatrix} -1.116 \\ 1.337 \end{bmatrix}$ & $F(x) = -1.159$

we have at least one failure & at least one success in each direction $S_1^{(1)}$, $S_2^{(1)}$

4) New set of search direction S^2

$$P_{2x2} = [\ p_1 , p_2\ ] = [\ S_1^{(1)} .\ S_2^{(1)}\ ] \begin{bmatrix} A_1 & 0 \\ A_2 & A_2 \end{bmatrix}$$

$$P_{2x2} = \begin{bmatrix} -0.447 & 0.894 \\ 0.894 & 0.447 \end{bmatrix} . \begin{bmatrix} 0.8 & 0 \\ -0.4 & -0.4 \end{bmatrix} = \begin{bmatrix} -0.7152 & -0.3576 \\ 0.5364 & -0.1788 \end{bmatrix}$$

$$D_1^{(2)} = P_1^{(2)} = \begin{bmatrix} -0.7152 \\ 0.5364 \end{bmatrix} \ \& \ P_2^{(2)} = \begin{bmatrix} -0.3576 \\ -0.1788 \end{bmatrix}$$

$$S_1^{(2)} = \frac{D_1^{(2)}}{\sqrt{D_1^{(2)T} . D_1^{(2)}}} = \frac{1}{\sqrt{(-0.7152)^2 + (0.5364)^2}} . \begin{bmatrix} -0.7152 \\ 0.5364 \end{bmatrix}$$

$$= \begin{bmatrix} -0.8 \\ 0.6 \end{bmatrix}$$

$$D_2^{(2)} = P_2^{(2)} - \sum_{m=1}^{1} \left[\left(p_2^{(2)T} . S_m^{(2)} . S_m^{(2)} \right) \right]$$

$$= \begin{bmatrix} -0.3576 \\ -0.1788 \end{bmatrix} - [-0.3576 - 0.1788] \begin{bmatrix} -0.8 \\ 0.6 \end{bmatrix} . \begin{bmatrix} -0.8 \\ 0.6 \end{bmatrix} = \begin{bmatrix} -0.2147 \\ -0.2861 \end{bmatrix} *$$

$$S_2^{(2)} = \frac{D_2^{(2)}}{\sqrt{D_2^{(2)T} . D_2^{(2)}}} = \frac{1}{\sqrt{(-0.2147)^2 + (-0.2861)^2}} . \begin{bmatrix} -0.2147 \\ -0.2861 \end{bmatrix}$$

$$= \begin{bmatrix} -0.6 \\ -0.8 \end{bmatrix}$$

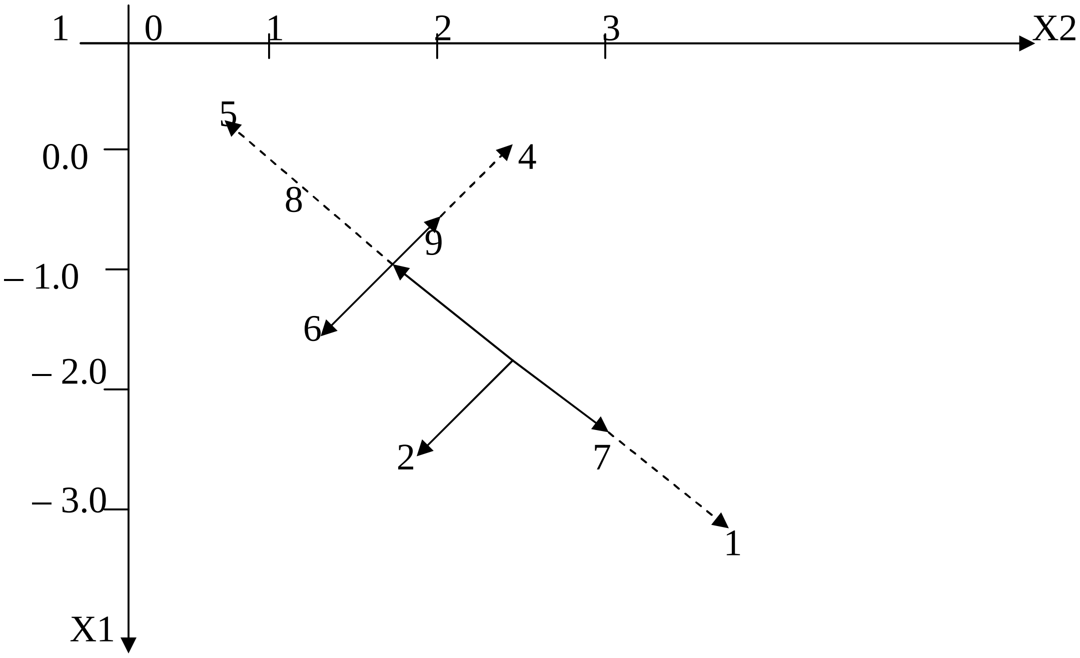
$$* -0.3576 - \{[L(-0.3576)(-0.8) + (-0.1788)(0.6]].[-0.8]\} = -0.2147$$

$$[\quad] . [\ 0.6\ ] = -0.2861$$

5) : Base value $X = \begin{bmatrix} -1.116 \\ 1.337 \end{bmatrix}$ & F(x) = – 1.159

$S_1^{(2)} = \begin{bmatrix} -0.8 \\ 0.6 \end{bmatrix}$; $S_2^{(2)} = \begin{bmatrix} -0.6 \\ -0.8 \end{bmatrix}$

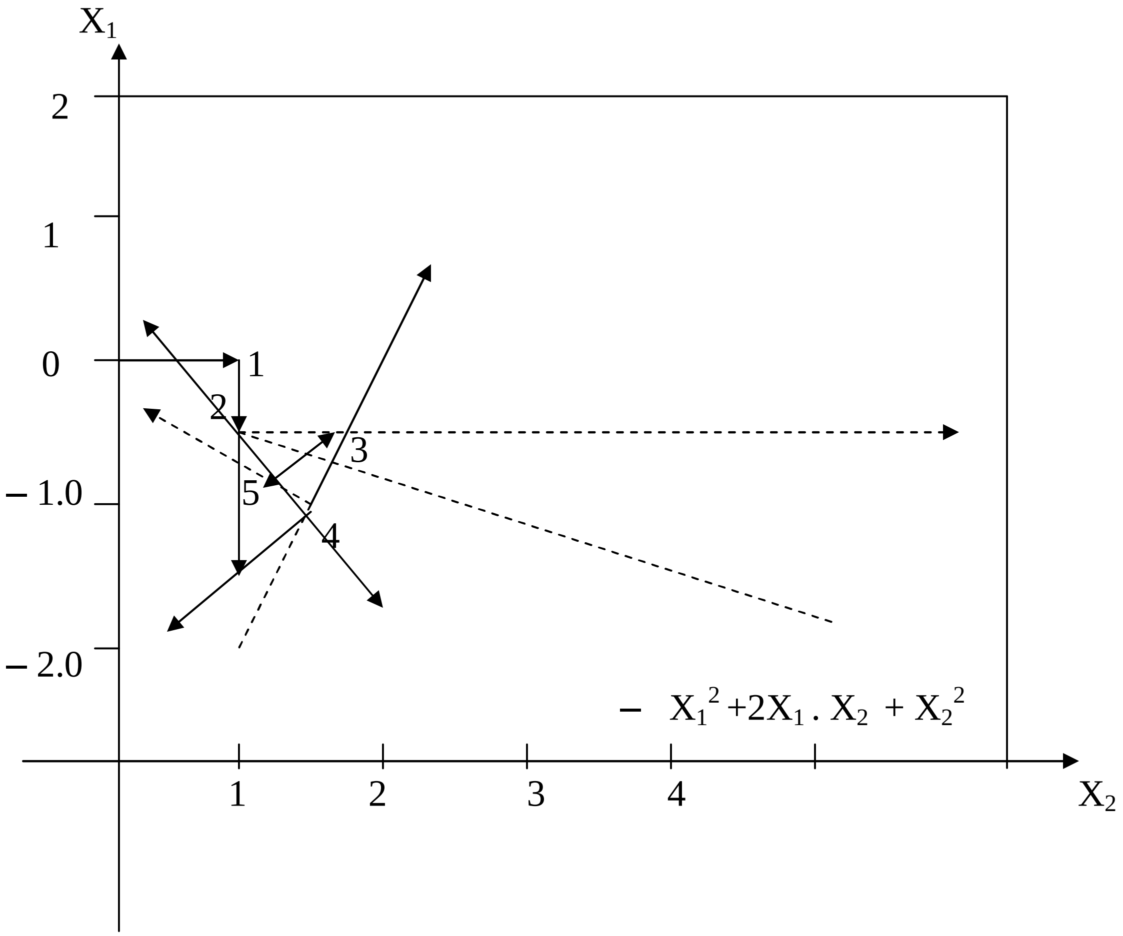
$X_1 = [\ X\ ]_B + \lambda_1\ [S_1^{(2)}\ ]$ & $X_2 = [\ X\ ]_B + \lambda_2\ [S_2^{(2)}\ ]$

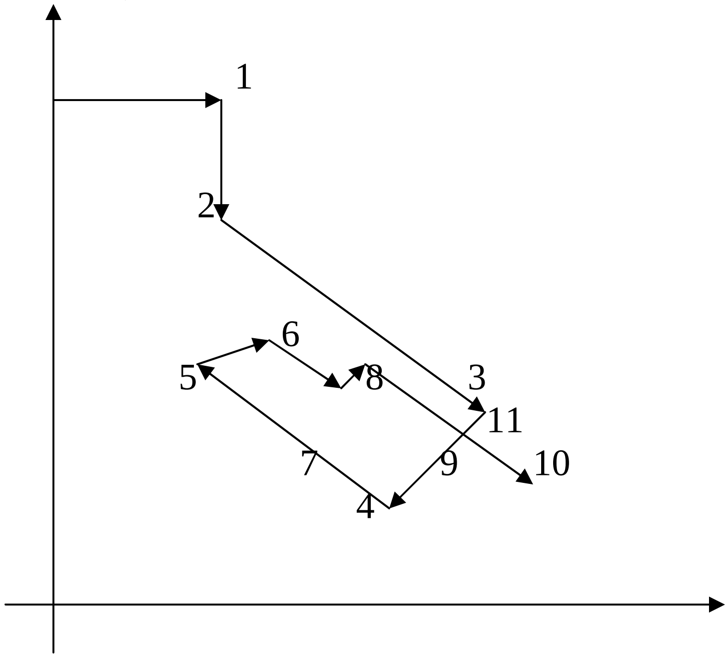


point	X_1	X_2	λ_1	λ_2	F(x)	Remark
0	-1.116	1.337	-	-	-1.159	B-value
1	-1.756	1.317	0.8	-	-0.487	Failure
2	-1.596	0.697	-	0.8	1.062	Failure
3	-0.796	1.097	- 0.4	-	-1.169	Success
4	-0.556	1.417	-	- 0.4	- 0.922	Failure
5	0.164	0.377	- 1.2	-	0.108	Failure
6	-0.416	0.937	-	0.2	-1.014	Failure
7	-1.276	1.457	0.6	-	-1.072	Failure
8	-0.736	1.177	-	-0.1	-1.177	Success
9	-0.496	0.997	-0.3	-	-0.996	Failure
10	-0.556	1.417	-	-0.2	-0.992	Failure

New base point X = $\begin{bmatrix} -0.736 \\ 1.177 \end{bmatrix}$ & F(x) = – 1.177

A_1 = – 0.4 , A_2 = – 0.1





Example :

minimize $f(x) = X_1 + X_2 + 2X_1^2 + 2\, X_1 .\ X_2\ + X_2^2$

Initial Base $X = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ & [X = 0]

$S_1^{(0)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$; $S_2^{(0)} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$; $\lambda_i = \begin{bmatrix} 0.8 \\ 0.8 \end{bmatrix}$

$\beta = 0.5$; $\alpha = 3.0$

Solution : Rosen. Minimization procedure

Stage (1)

point	X_1	X_2	DX_1	DX_2	F(x)	Remark
0	0.0	0.0	-	-	0.0	Initial
1	0.8	0.0	0.8	-	2.08	Failure
2	0.0	0.8	-	0.8	0.16	Success
3	-0.4	0.8	0.4	-	-0.88	Success
4	0.4	3.2	-	2.4	4.4	Failure
5	-1.6	0.8	1.2	-	0.8	Failure
			$A_1 = -0.4$	$A_2 = 0.8$		

V(1 ,1) = – 0.447 V(2,1) = 0.894

V(1,2) = 0.894 V(2,2) = 0.447

Stage (2)

point	X_1	X_2	DX_1	DX_2	F(x)	Remark
0	-0.4	0.8	-	-	-0.88	Initial
1	-0.758	1.516	0.8	-	-0.125	Success
2	-0.042	1.873	-	0.8	0.439	Failure
3	-1.831	8.662	2.4	-	0.212	Failure
4	-1.116	1.337	-	-0.4	-1.159	Success
5	0.579	0.263	- 1.2	-	-0.408	Failure
6	2.189	0.88	-	0.2	0.731	Failure
			$A_1 = 0.8$	$A_2 = -0.4$		

V(1 ,1) = – 0.8 V(2,1) = 0.6

V (1,2) = 0.6 V(2,2) = – 0.8

Stage (3)

point	X_1	X_2	DX_1	DX_2	F(x)	Remark
0	-1.116	1.337	-	-	-1.159	B-value
1	-1.756	1.817	0.8	-	-0.187	Failure
2	-1.596	0.697	-	0.8	1.162	Failure
3	-0.796	1.097	- 0.4	-	-1.169	Success
4	-0.556	1.417	-	-0.4	-0.922	Failure
5	0.164	0.377	- 1.2	-	0.108	Failure
6	-0.916	0.937	-	0.2	-1.014	Failure
7	-1.276	1.457	0.6	-	-1.072	Failure
8	-0.736	1.177	-	-0.1	-1.177	Success
9	-0.496	0.997	-0.3	-	-0.996	Failure
10	-0.556	1.417	-	-0.3	-0.992	Failure
			$A_1 = -0.4$	$A_2 = -0.1$		

V(1 ,1) = 0.922 V(2,1)= 0.388
V(1,2)= 0.388 V(2,2) = 0.922

Stage (4)

point	X_1	X_2	DX_1	DX_2	F(x)	Remark
0	-0.736	1.177	-	-	-1.177	B-value
1	0.002	0.866	0.8	-	-0.111	Failure
2	-0.425	1.914	-	0.8	0.058	Failure
3	-1.104	1.332	- 0.4	-	-1.165	Failure
4	-0.891	0.808	-	-0.4	-0.898	Failure
5	-0.551	1.099	0.2	-	-1.046	Failure
6	-0.658	1.361	-	0.2	-1.092	Failure
7	-0.828	1.215	-0.1	-	-1.208	Success
8	-0.867	1.123	-	-0.1	-1.173	Failure
9	-1.104	1.332	0.2	-	-1.165	Failure
10	-0.808	1.262	-	0.05	-1.211	Success
11	-0.670	1.203	0.15	-	-1.14	Failure
12	-0.750	1.40	-	0.15	-1.165	Failure
			$A_1 = -0.1$	$A_2=0.05$		

V(1 ,1) = – 0.65 V(2,1)= 0.759
V(1,2) = 0.759 V(2,2) = 0.651
$A_1 = -0.1$ $A_2=0.05$

Example : minimize $Z(X) = [X_1 - 10]^2 + [X_2 - 10]^2$

Initial Base $X = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

$S_1^{(0)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$; $S_2^{(0)} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$; $\lambda = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\beta = 0.5$; $\alpha = 3.0$

Solution : Rosen Brock Minimization procedure

Stage (1)

point	X_1	X_2	DX_1	DX_2	F(x)	Remark
0	3.0	1.0	-	-	65000	Initial
1	4.0	1.0	1.0	-	52000	Success
2	4.0	2.0	-	1.0	45000	Success
3	7.0	2.0	3.0	-	18000	Success
4	7.0	5.0	-	3.0	9000	Success
5	16.0	5.0	9.0	-	36000	Failure
6	7.0	14.0	-	9.0	90000	Failure
			$A_1 = 4.0$	$A_2 = 4.0$		

V(1 ,1) = 0.707 V(2,1) =– 0.707

V(1,2) = 0.707 V(2,2) = 0.707

Stage (2)

point	X_1	X_2	DX_1	DX_2	F(x)	Remark
0	7.0	5.0	-	-	9.0	B-value
1	7.707	5.717	1.0	-	5.757	Success
2	7.0	6.404	-	1.0	11.00	Failure
3	9.828	7.828	3.0	-	8.029	Failure
4	8.061	5.354	-	-0.5	3.886	Success
5	7.0	4.23	-1.5	-	9.500	Failure
6	9.121	4.23	-	-1.5	1.272	Success
7	9.652	4.823	0.75	-	0.153	Success
8	12.834	1.641	-	-4.5	19.311	Failure
			$A_1 = 1.75$	$A_2 = -2.00$		

V(1 ,1) = 0.998 V(2,1) = – 0.067

V(1,2) = – 0.067 V(2,2) = – 0.998

Stage (3)

point	X_1	X_2	DX_1	DX_2	F(x)	Remark
0	9.652	4.923	--	--	0.153	B-value
1	10.649	4.757	1.0	--	0.481	Failure
2	9.585	3.825	--	1.0	1.552	Failure
3	9.153	4.856	0.5	--	0.738	Failure
4	9.685	5.322	--	- 0.5	0.203	Failure
5	9.901	4.807	0.25	--	0.047	Success
6	9.884	4.557	--	0.25	0.209	Failure
7	10.649	4.757	0.75	--	0.481	Failure
8	9.909	4.931	--	- 0.25	0.013	Success
9	9.535	4.956	0.375	--	0.218	Failure
10	9.934	5.305	--	-0.375	0.098	Failure
			A_1 =0.25	A_2= – 0.125		

V(1 ,1) = 0.922 V(2,1) = – 0.387

V(1,2) = 0.87 V(2,2) = 0.22

Stage (4)

point	X_1	X_2	DX_1	DX_2	F(x)	Remark
0	9.909	4.931	--	--	0.013	B-value
1	10.832	5.318	1.0	--	0.793	Failure
2	9.523	5.854	--	1.0	0.956	Failure
3	9.448	4,738	- 0.5	--	0.373	Failure
4	10.103	4.470	--	- 0.5	0.291	Failure
5	10.140	5.028	0.25	--	0.020	Failure
6	9.813	5.162	--	0.25	0.061	Failure
7	9.794	4.883	- 0.125	--	0.056	Failure
8	9.958	4.816	--	- 0.125	0.036	Failure
9	9.967	4.955	0.063	--	0.003	Success
10	9.943	5.013	--	0.063	0.003	Failure
11	10.140	5.028	0.188	--	0.02	Failure
12	9.979	4.927	--	- 0.031	0.006	Failure
13	9.881	4.919	- 0.094	--	0.021	Failure
14	9.961	4.970	--	0.016	0.002	Success
15	10.004	4.988	0.047	--	0.000	Success
16	9.986	5.031	--	0.047	0.001	Failure
			A_1 =0.11	A_2= 0.016		

V(1 ,1) = 0.858 V(2,1) = – 0.513

V(1,2) = 0.513 V(2,2) = 0.858

Stage (5)

point	X_1	X_2	DX_1	DX_2	F(x)	Remark
0	10.004	4.988	--	--	0.000	B-value
1	10.862	5.501	1.0	--	0.995	Failure
2	9.491	5.846	--	1.0	0.975	Failure
3	9.575	4.731	- 0.5	--	0.253	Failure
4	10.261	4.559	--	- 0.5	0.263	Failure
5	10.219	5.116	0.25	--	0.061	Failure
6	9.876	5.203	--	0.25	0.056	Failure
7	9.897	4.924	- 0.125	--	0.061	Failure
8	10.068	4.881	--	- 0.125	0.019	Failure
9	10.058	5.020	0.063	--	0.004	Failure
10	9.972	5.042	--	0.063	0.003	Failure
11	9.977	4.972	- 0.031	--	0.001	Failure
12	10.020	4.961	--	- 0.031	0.002	Failure
13	10.018	4.996	0.016	--	0.000	Failure
14	9.996	5.001	--	0.016	0.000	Success
15	9.990	4.997	- 0.008	--	0.000	Failure
16	9.972	5.042	--	0.047	0.003	Failure
	10.000	5.003	0.004	--	0.000	Success
	10.012	4.983	--	-0.023	0.000	Failure
	10.010	5.009	0.012	--	0.000	Failure
			A_1=0.004	A_2=0.016		

V(1 ,1) = – 0.290   V(1,2) = 0.957
V(2 , 1) = – 0.957   V(2,2) = – 0.290

Constrained function of several variable :-

When a function of several variables is to be optimized subject to constrains , the model takes the following form :

Minimize $Z = Z(X_i)$; $i = 1,2,\dots,n$

Subject to :

$h_j(X_i) = 0$; $j = 1,2,\dots,p$

$g_j(X_i) \le 0$; $j = (p+1), (p+2), \dots, m$

The type of constraint may be either equality or inequality .

It may be linear or non- linear , explicit or implicit.

It should be noted that inequality constraints may be written either as " ≥ " or " ≤ " constraints simply by reversing the sign of the function $g_j(X_i)$.

a)) Equality constraints:

Example :- Minimize $Z = (X_1 - 10)^2 + (X_2 - 5)^2$

Subject to : $X_1 + 2X_2 = 14$

To apply the device of penalty terms the objective function is modified to the following form :

Minimize $Z_{art} = (X_1 - 10)^2 + (X_2 - 5)^2 + 10^2 \, [X_1 + 2X_2 - 14]$

Where is absolute value

The penalty term $[\; 10^2 \;\; |X_1 + 2X_2 - 14| \;]$ will have anon- zero value , point not satisfying the constraint .

The constraint multiplier of 10^2 is same what at bitrary but certain problem may be sensitive to the scaling of the penalty term in relation to the objective function .

It will be appreciated that the adoption of too large a multiplying constant on the penalty term may mask to a great extent the influence of the real objective function making a search procedure in accurate or even impossible ,as a general guide, the penalty term should increase the objective function by only one or two orders of magnitude in the vicinity of the solution .

Even with this precaution it is good practice to evaluate the final optimum value of the objective function by means of the real objective , un modified by penalty terms.

b)) <u>In equality constraints:</u> the treatment of in equality constraint is similar to the equality constraints , see the following example :-

Example :- Minimize $Z = (x_1 - 10)^2 + (x_2 - 5)^2$

Subject to : $x_1 + 2 x_2 \geq 14$

The constraint may be re- written as :

$$14 - x_1 - 2 x_2 \leq 0$$

And the modified objective function becomes:

Min $.Z_{art} = (x_1 - 10)^2 + (x_2 - 5)^2 + \delta_1 .10^2 (14 - x_1 - 2 x_2)^2$

Where:-

$\delta_1 = 1$ for $14 - x_1 - 2 x_2 > 0$

$\delta_1 = 0$ for $14 - x_1 - 2 x_2 \leq 0$

Example :- Minimize $Z = (x_1 - 10)^2 + (x_2 - 5)^2$

Subject to : $x_1 + 2 x_2 \leq 14$

Or : $x_1 + 2 x_2 - 14 \leq 0$

Thus the equivalent un constraint function becomes:

Min $.Z_{art} = (x_1 - 10)^2 + (x_2 - 5)^2 + \delta_1 .10^2 (x_1 + 2 x_2 - 14)$

Where:-

$\delta_1 = 1$ for $x_1 + 2 x_2 - 14 > 0$

$\delta_1 = 0$ for $x_1 + 2 x_2 - 14 \leq 0$

c)) A transitional penalty term:
consider the problem represented by the following model:

Minimize $Z = Z(X_i)$; $i = 1,2,\ldots,n$
Subject to : $g_j(X_i) \leq 0$; $j = 1,2,\ldots,m$

This may be converted to the unconstraint objective function:

$$\text{Minimize } .Z_{art} = Z(X_i) + \sum_{j=1}^{m} \delta_i . C_i . \left[e^{g_j(x_i)} - 1 \right]$$

Where:-

$\delta_j = 1$ for $g_j(X_i) > 0$

$\delta_j = 0$ other wise $j = 1,2,\ldots,m$

& C_j = a weight factor for the jth constraint.

By this device , the penalty term has zero value at $g_j(X_i) = 0$, & disappears also for $g_j(X_i) < 0$, but the magnitude of the penalty increase in a smooth exponential transition as the extent of constraint violation increases .