Chapter 1

Vector Algebra

1.1 Definitions

A **vector** consists of two components: *magnitude* and *direction* .
(e.g. force, velocity, pressure)

A **scalar** consists of *magnitude* only.
(e.g. mass, charge, density)

1.2 Vector Algebra

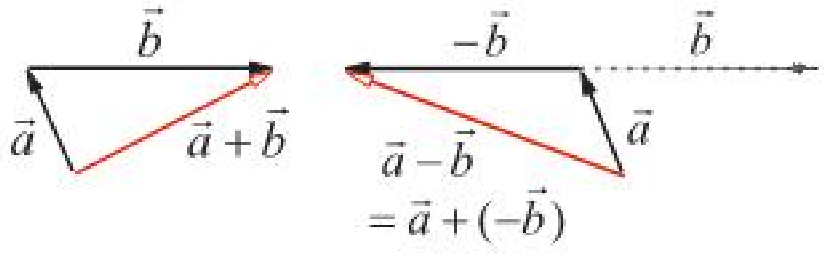


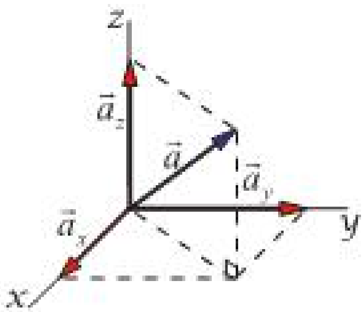
Figure 1.1: Vector algebra

$$
\begin{aligned}
\vec{a}+\vec{b} &= \vec{b}+\vec{a} \\
\vec{a}+(\vec{c}+\vec{d}) &= (\vec{a}+\vec{c})+\vec{d}
\end{aligned}
$$

1.3 Components of Vectors

Usually vectors are expressed according to **coordinate system**. Each vector can be expressed in terms of *components*.

The most common coordinate system: **Cartesian**



$$\vec{a} = \vec{a}_x + \vec{a}_y + \vec{a}_z$$

Magnitude of $\vec{a} = |\vec{a}| = a$,

$$a = \sqrt{a_x^2 + a_y^2 + a_z^2}$$

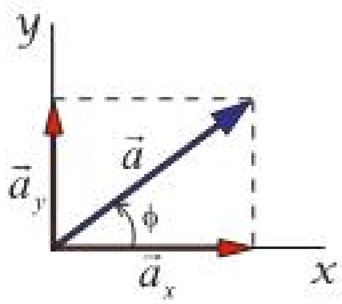


Figure 1.2: ϕ measured anti-clockwise from position x-axis

$$\begin{aligned} \vec{a} &= \vec{a}_x + \vec{a}_y \\ a &= \sqrt{a_x^2 + a_y^2} \\ a_x &= a\,cos\phi;\ a_y = a\,sin\phi \\ tan\phi &= \frac{a_y}{a_x} \end{aligned}$$

Unit vectors have magnitude of 1

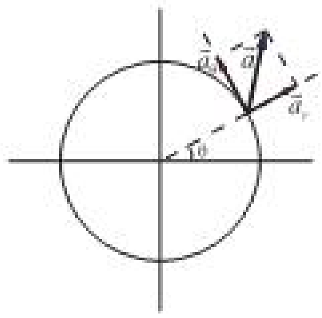
$$\hat{a} = \frac{\vec{a}}{|\vec{a}|} = \text{unit vector along } \vec{a} \text{ direction}$$

$$\begin{array}{cccl} \hat{i} & \hat{j} & \hat{k} & \text{are unit vectors along} \\ \updownarrow & \updownarrow & \updownarrow & \\ x & y & z & \text{directions} \end{array}$$

$$\vec{a} = a_x\,\hat{i} + a_y\,\hat{j} + a_z\,\hat{k}$$

Other coordinate systems:

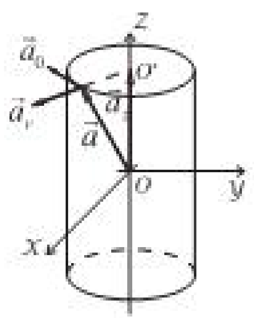
1. Polar Coordinate:



$$\vec{a} = a_r\,\hat{r} + a_\theta\,\hat{\theta}$$

Figure 1.3: Polar Coordinates

2. Cylindrical Coordinates:

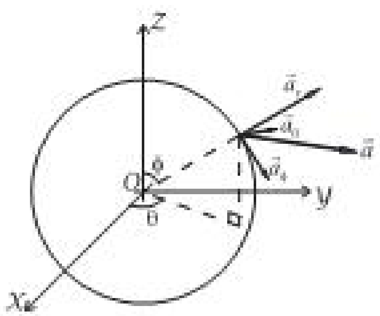


$$\vec{a} = a_r\,\hat{r} + a_\theta\,\hat{\theta} + a_z\,\hat{z}$$

$\hat{r}$ originated from nearest point on z-axis (Point O')

Figure 1.4: Cylindrical Coordinates

3. Spherical Coordinates:



$$\vec{a} = a_r\,\hat{r} + a_\theta\,\hat{\theta} + a_\phi\,\hat{\phi}$$

$\hat{r}$ originated from Origin O

Figure 1.5: Spherical Coordinates

1.4 Multiplication of Vectors

1. Scalar multiplication:

If	$\vec{b}$=m $\vec{a}$	$\vec{b}, \vec{a}$ are vectors; m is a scalar
then	b=m a	(Relation between magnitude)
	b_x=m a_x b_y=m a_y	$\}$ Components also follow relation

i.e.

$$\begin{array}{rcrcrcr} \vec{a} & = & a_x \ \hat{i} & + & a_y \ \hat{j} & + & a_z \ \hat{k} \\ m\vec{a} & = & ma_x \ \hat{i} & + & ma_y \ \hat{j} & + & ma_z \ \hat{k} \end{array}$$

2. Dot Product (Scalar Product):

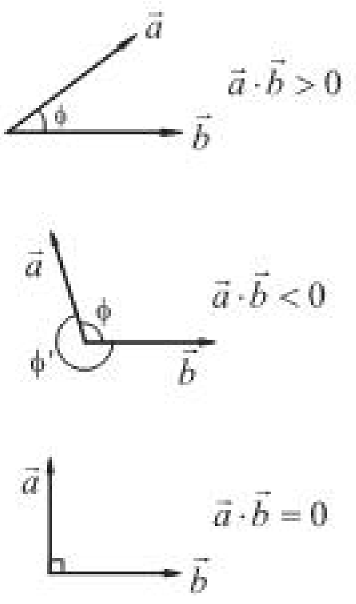


Figure 1.6: Dot Product

$$\boxed{\vec{a} \cdot \vec{b} = |\vec{a}| \cdot |\vec{b}| \; cos\phi}$$

Result is **always** a scalar. It can be positive or negative depending on ϕ.

$$\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$$

Notice: $\vec{a} \cdot \vec{b} = ab\,cos\phi = ab\,cos\phi'$
i.e. Doesn't matter how you measure angle ϕ between vectors.

$$\begin{aligned} \hat{i} \cdot \hat{i} &= |\hat{i}|\,|\hat{i}| \; cos0^\circ \; = 1 \cdot 1 \cdot 1 = 1 \\ \hat{i} \cdot \hat{j} &= |\hat{i}|\,|\hat{j}|\, cos90^\circ = 1 \cdot 1 \cdot 0 = 0 \end{aligned}$$

$$\boxed{\begin{array}{c} \hat{i} \cdot \hat{i} = \hat{j} \cdot \hat{j} = \hat{k} \cdot \hat{k} = 1 \\ \hat{i} \cdot \hat{j} = \hat{j} \cdot \hat{k} = \hat{k} \cdot \hat{i} = 0 \end{array}}$$

If $\quad \vec{a} = a_x \,\hat{i} + a_y \,\hat{j} + a_z \,\hat{k}$
$\quad\quad \vec{b} = b_x \,\hat{i} + b_y \,\hat{j} + b_z \,\hat{k}$

then $\quad \vec{a} \cdot \vec{b} = a_x b_x + a_y b_y + a_z b_z$
$\quad\quad \vec{a} \cdot \vec{a} = |\vec{a}| \cdot |\vec{a}| \; cos0^\circ = a \cdot a = a^2$

3. Cross Product (Vector Product):

If $\quad \vec{c} = \vec{a} \times \vec{b}$,
then $\quad c = |\vec{c}| = a\, b\, sin\phi$

$$\vec{a} \times \vec{b} \neq \vec{b} \times \vec{a} \ !!!$$

$$\boxed{\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}}$$

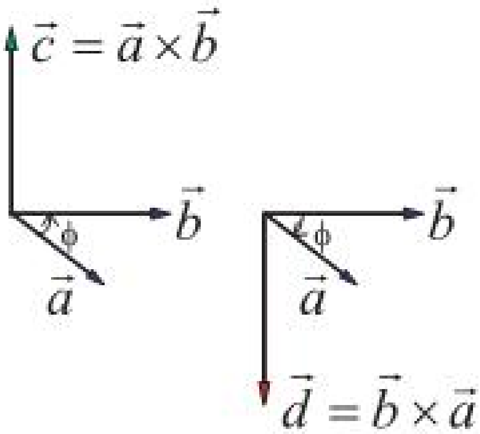


Figure 1.7: Note: How angle ϕ is measured

- Direction of cross product determined from *right hand rule*.
- Also, $\vec{a} \times \vec{b}$ is $\perp$ to $\vec{a}$ and $\vec{b}$, `i.e.`

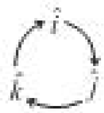
$$\vec{a} \cdot (\vec{a} \times \vec{b}) = 0$$
$$\vec{b} \cdot (\vec{a} \times \vec{b}) = 0$$

- IMPORTANT:

$$\boxed{\vec{a} \times \vec{a} = a \cdot a\, sin0^\circ = 0}$$

$$\begin{aligned} |\hat{i} \times \hat{i}| &= |\hat{i}|\,|\hat{i}|\ sin0^\circ\ = 1 \cdot 1 \cdot 0 = 0 \\ |\hat{i} \times \hat{j}| &= |\hat{i}|\,|\hat{j}|\, sin90^\circ = 1 \cdot 1 \cdot 1 = 1 \end{aligned}$$

$$\boxed{\begin{array}{c} \hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = 0 \\ \hat{i} \times \hat{j} = \hat{k};\ \hat{j} \times \hat{k} = \hat{i};\ \hat{k} \times \hat{i} = \hat{j} \end{array}}$$



$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} = \begin{aligned} &(a_y\, b_z - a_z\, b_y)\,\hat{i} \\ &+(a_z\, b_x - a_x\, b_z)\,\hat{j} \\ &+(a_x\, b_y - a_y\, b_x)\,\hat{k} \end{aligned}$$

4. Vector identities:

$$\begin{aligned}
\vec{a} \times (\vec{b} + \vec{c}) &= \vec{a} \times \vec{b} + \vec{a} \times \vec{c} \\
\vec{a} \cdot (\vec{b} \times \vec{c}) &= \vec{b} \cdot (\vec{c} \times \vec{a}) = \vec{c} \cdot (\vec{a} \times \vec{b}) \\
\vec{a} \times (\vec{b} \times \vec{c}) &= (\vec{a} \cdot \vec{c})\,\vec{b} - (\vec{a} \cdot \vec{b})\,\vec{c}
\end{aligned}$$

1.5 Vector Field (Physics Point of View)

A **vector field** $\vec{\mathcal{F}}(x, y, z)$ is a mathematical function which has a *vector* output for a *position* input.

(**Scalar field** $\vec{\mathcal{U}}(x, y, z)$)

1.6 Other Topics

Tangential Vector

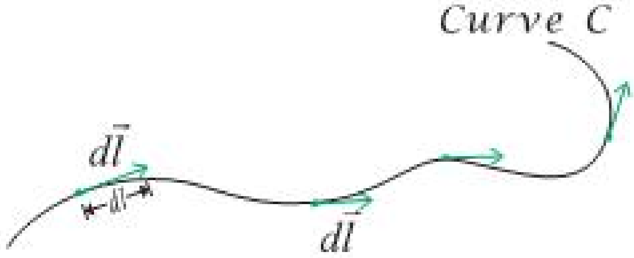


Figure 1.8: $d\vec{l}$ is a vector that is <u>always</u> tangential to the curve C with infinitesimal length dl

Surface Vector

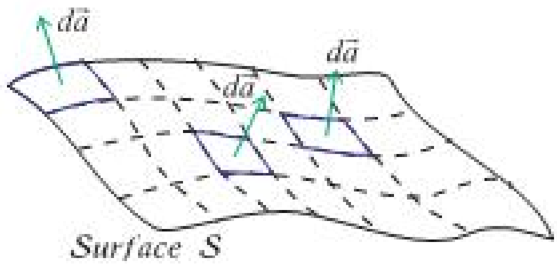


Figure 1.9: $d\vec{a}$ is a vector that is <u>always</u> perpendicular to the surface S with infinitesimal area da

Some uncertainty! $(d\vec{a} \quad versus \quad -d\vec{a})$

Two conventions:

- Area formed from a closed curve

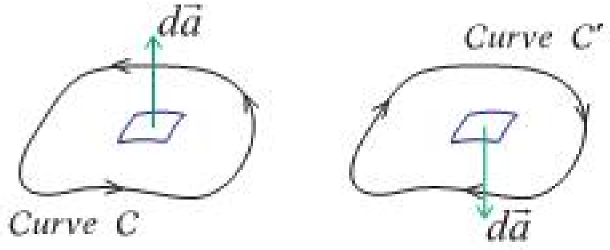


Figure 1.10: Direction of $d\vec{a}$ determined from right-hand rule

- Closed surface enclosing a volume

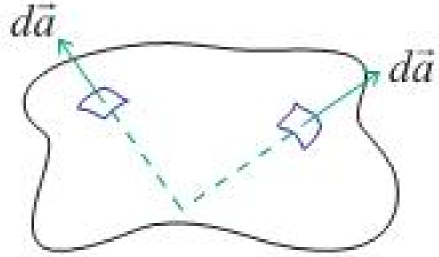


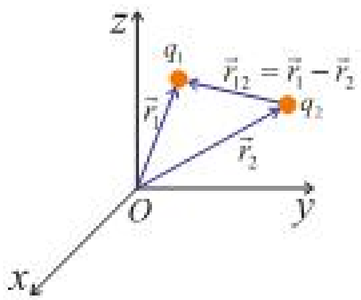
Figure 1.11: Direction of $d\vec{a}$ going from inside to outside

Chapter 2

Electric Force & Electric Field

2.1 Electric Force

The electric force between two **charges** $\boldsymbol{q_1}$ and $\boldsymbol{q_2}$ can be described by **Coulomb's Law**.



$\vec{F}_{12} = Force$ on $\boldsymbol{q_1}$ exerted by $\boldsymbol{q_2}$

$$\boxed{\vec{F}_{12} = \frac{1}{4\pi\epsilon_0} \cdot \frac{q_1 q_2}{r_{12}^2} \cdot \hat{r}_{12}}$$

where $\hat{r}_{12} = \dfrac{\vec{r}_{12}}{|\vec{r}_{12}|}$ is the *unit vector* which locates particle 1 relative to particle 2.

$$\texttt{i.e.} \quad \boxed{\vec{r}_{12} = \vec{r}_1 - \vec{r}_2}$$

- q_1, q_2 are electrical charges in units of *Coulomb*(C)
- Charge is *quantized*
  Recall 1 electron carries $1.602 \times 10^{-19} C$
- ϵ_0 = Permittivity of free space = $8.85 \times 10^{-12} C^2/Nm^2$

COULOMB'S LAW:

(1) q_1, q_2 can be either positive or negative.

(2) If q_1, q_2 *are of same sign*, then the force experienced by q_1 is in direction *away from* q_2, that is, *repulsive*.

(3) Force on q_2 exerted by q_1:

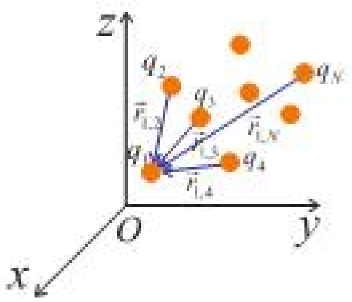
$$\vec{F}_{21} = \frac{1}{4\pi\epsilon_0} \cdot \frac{q_2 q_1}{r_{21}^2} \cdot \hat{r}_{21}$$

BUT:

$$r_{12} = r_{21} = \text{distance between } q_1, q_2$$
$$\hat{r}_{21} = \frac{\vec{r}_{21}}{r_{21}} = \frac{\vec{r}_2 - \vec{r}_1}{r_{21}} = \frac{-\vec{r}_{12}}{r_{12}} = -\hat{r}_{12}$$

$$\therefore \boxed{\vec{F}_{21} = -\vec{F}_{12} \ \textit{Newton's 3rd Law}}$$

<u>**SYSTEM WITH MANY CHARGES**</u>:



The total force experienced by charge q_1 is the *vector sum* of the forces on q_1 exerted by other charges.

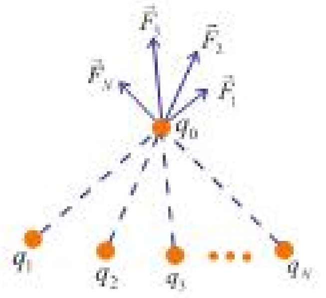
$$\begin{aligned} \vec{F}_1 &= \text{Force experienced by } q_1 \\ &= \vec{F}_{1,2} + \vec{F}_{1,3} + \vec{F}_{1,4} + \cdots + \vec{F}_{1,N} \end{aligned}$$

PRINCIPLE OF SUPERPOSITION:

$$\vec{F}_1 = \sum_{j=2}^{N} \vec{F}_{1,j}$$

2.2 The Electric Field

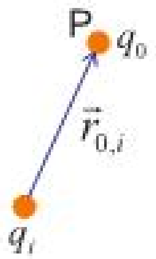
While we need two charges to quantify the **electric force**, we define the **electric field** for any single charge distribution to describe its effect on other charges.



Total force $\vec{F} = \vec{F}_1 + \vec{F}_2 + \cdots + \vec{F}_N$
The **electric field** is defined as

$$\boxed{\lim_{q_0 \to 0} \frac{\vec{F}}{q_0} = \vec{E}}$$

(a) E-field due to a single charge q_i:



From the definitions of **Coulomb's Law**, the force experienced at location of q_0 (point P)

$$\vec{F}_{0,i} = \frac{1}{4\pi\epsilon_0} \cdot \frac{q_0 q_i}{r_{0,i}^2} \cdot \hat{r}_{0,i}$$

where $\hat{r}_{0,i}$ is the unit vector along the direction *from charge* q_i *to* q_0.

$$\begin{aligned} \hat{r}_{0,i} &= \text{Unit vector from charge } q_i \text{ to point P} \\ &= \hat{r}_i \text{ (radical unit vector from } q_i) \end{aligned}$$

Recall $\vec{E} = \lim_{q_0 \to 0} \frac{\vec{F}}{q_0}$
$\therefore$ E-field due to q_i at point P:

$$\vec{E}_i = \frac{1}{4\pi\epsilon_0} \cdot \frac{q_i}{r_i^2} \cdot \hat{r}_i$$

where $\vec{r}_i$ = Vector pointing from q_i to point P,
thus $\hat{r}_i$ = Unit vector pointing from q_i to point P
Note:

(1) E-field is a **vector**.

(2) Direction of E-field depends on **both** position of P and sign of q_i.

(b) E-field due to system of charges:

Principle of Superposition:
In a system with N charges, the **total** E-field due to all charges is the **vector sum** of E-field due to individual charges.

i.e. $$\boxed{\vec{E} = \sum_i \vec{E}_i = \frac{1}{4\pi\epsilon_0}\sum_i \frac{q_i}{r_i^2}\hat{r}_i}$$

(c) Electric Dipole

System of *equal and opposite* charges separated by a distance d.

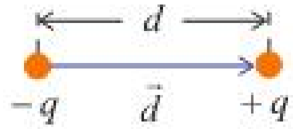
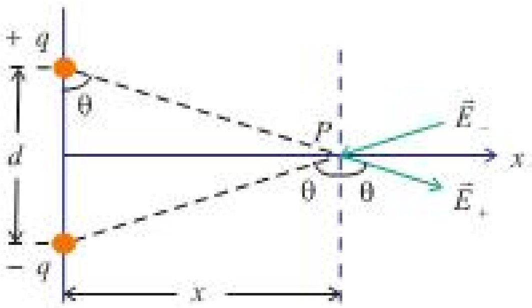


Figure 2.1: An electric dipole. (Direction of $\vec{d}$ from negative to positive charge)

$$\boxed{\begin{array}{c}\text{Electric Dipole Moment}\\ \vec{p} = q\vec{d} = qd\hat{d}\end{array}}$$

$$\boxed{p = qd}$$

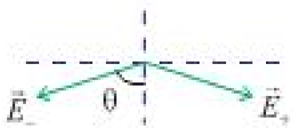
<u>Example:</u> $\vec{E}$ due to dipole along x-axis



Consider point P at distance x along the perpendicular axis of the dipole $\vec{p}$:

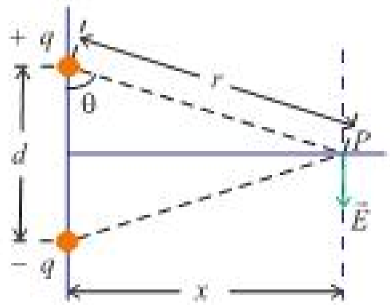
$$\vec{E} \;=\; \underset{\substack{\uparrow\\ \text{(E-field}\\ \text{due to } +q)}}{\vec{E}_+} \;+\; \underset{\substack{\uparrow\\ \text{(E-field}\\ \text{due to } -q)}}{\vec{E}_-}$$

Notice: Horizontal E-field components of $\vec{E}_+$ and $\vec{E}_-$ cancel out.



$\therefore$ Net E-field points along the axis opposite to the dipole moment vector.

Magnitude of E-field $= 2E_+ \cos\theta$



$$\therefore E = 2\Big(\overbrace{\frac{1}{4\pi\epsilon_0}\cdot\frac{q}{r^2}}^{E_+ \text{ or } E_- \text{ magnitude!}}\Big)\cos\theta$$

$$\text{But} \quad r = \sqrt{\left(\frac{d}{2}\right)^2 + x^2}$$

$$\cos\theta = \frac{d/2}{r}$$

$$\therefore E = \frac{1}{4\pi\epsilon_0}\cdot\frac{p}{[x^2+(\frac{d}{2})^2]^{\frac{3}{2}}}$$

$$(p = qd)$$

<u>Special case:</u> When $x \gg d$

$$[x^2+(\frac{d}{2})^2]^{\frac{3}{2}} = x^3[1+(\frac{d}{2x})^2]^{\frac{3}{2}}$$

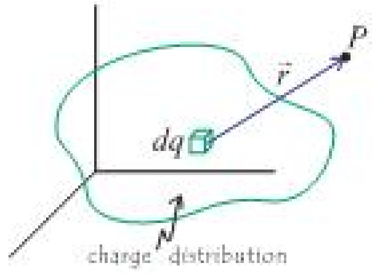
- Binomial Approximation:

$$(1+y)^n \approx 1+ny \qquad \text{if } y \ll 1$$

$$\boxed{\text{E-field of dipole} \doteq \frac{1}{4\pi\epsilon_0}\cdot\frac{p}{x^3} \sim \frac{1}{x^3}}$$

- Compare with $\dfrac{1}{r^2}$ E-field for single charge
- Result also valid for point P along any axis with respect to dipole

2.3 Continuous Charge Distribution



E-field at point P due to dq:

$$d\vec{E} = \frac{1}{4\pi\epsilon_0}\cdot\frac{dq}{r^2}\cdot\hat{r}$$

$\therefore$ E-field due to charge distribution:

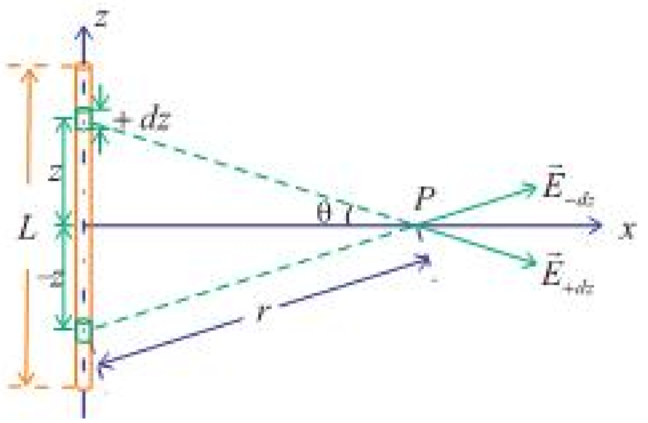
$$\vec{\boldsymbol{E}} = \int\limits_{Volume} d\vec{\boldsymbol{E}} = \int\limits_{Volume} \frac{1}{4\pi\epsilon_0} \cdot \frac{dq}{r^2} \cdot \hat{\boldsymbol{r}}$$

(1) In many cases, we can take advantage of the *symmetry* of the system to simplify the integral.

(2) To write down the small charge element dq:

1-D	$dq = \lambda\, ds$	λ = linear charge density	ds = small length element
2-D	$dq = \sigma\, dA$	σ = surface charge density	dA = small area element
3-D	$dq = \rho\, dV$	ρ = volume charge density	dV = small volume element

Example 1: Uniform line of charge

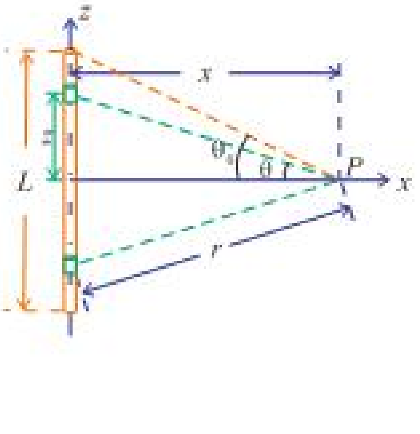


charge per unit length $= \lambda$

(1) Symmetry considered: The E-field from $+z$ and $-z$ directions *cancel along z-direction*, $\therefore$ Only horizontal E-field components need to be considered.

(2) For each element of length dz, charge $dq = \lambda dz$
$\therefore$ Horizontal E-field at point P due to element dz =

$$|d\vec{E}| \cos\theta = \underbrace{\frac{1}{4\pi\epsilon_0} \cdot \frac{\lambda dz}{r^2}}_{dE_{dz}} \cos\theta$$



$\therefore$ E-field due to entire line charge at point P

$$\begin{aligned} E &= \int_{-L/2}^{L/2} \frac{1}{4\pi\epsilon_0} \cdot \frac{\lambda dz}{r^2} \cos\theta \\ &= 2\int_0^{L/2} \frac{\lambda}{4\pi\epsilon_0} \cdot \frac{dz}{r^2} \cos\theta \end{aligned}$$

To calculate this integral:

- First, notice that x is fixed, but z, r, θ all varies.
- Change of variable (from z to θ)

(1) $\begin{array}{ll} z = x\tan\theta & \therefore dz = x\sec^2\theta\, d\theta \\ x = r\cos\theta & \therefore r^2 = x^2\sec^2\theta \end{array}$

(2) When $\begin{array}{lll} z = 0\ , & \theta = 0^\circ & \\ z = L/2 & \theta = \theta_0 & \text{where } \tan\theta_0 = \dfrac{L/2}{x} \end{array}$

$$\begin{aligned} E &= 2\cdot\frac{\lambda}{4\pi\epsilon_0}\int_0^{\theta_0}\frac{x\sec^2\theta\, d\theta}{x^2\sec^2\theta}\cdot\cos\theta \\ &= 2\cdot\frac{\lambda}{4\pi\epsilon_0}\int_0^{\theta_0}\frac{1}{x}\cdot\cos\theta\, d\theta \\ &= 2\cdot\frac{\lambda}{4\pi\epsilon_0}\cdot\frac{1}{x}\cdot(\sin\theta)\Big|_0^{\theta_0} \\ &= 2\cdot\frac{\lambda}{4\pi\epsilon_0}\cdot\frac{1}{x}\cdot\sin\theta_0 \\ &= 2\cdot\frac{\lambda}{4\pi\epsilon_0}\cdot\frac{1}{x}\cdot\frac{L/2}{\sqrt{x^2+(\frac{L}{2})^2}} \end{aligned}$$

$$\boxed{E = \frac{1}{4\pi\epsilon_0}\cdot\frac{\lambda L}{x\sqrt{x^2+(\frac{L}{2})^2}}} \qquad \text{along } x\text{-direction}$$

<u>Important limiting cases:</u>

1. $x \gg L: \quad E \doteqdot \dfrac{1}{4\pi\epsilon_0}\cdot\dfrac{\lambda L}{x^2}$
   But λL = Total charge on rod
   $\therefore$ System behave like a point charge

2. $L \gg x: \quad E \doteqdot \dfrac{1}{4\pi\epsilon_0}\cdot\dfrac{\lambda L}{x\cdot\frac{L}{2}}$

$$\boxed{E_x = \frac{\lambda}{2\pi\epsilon_0 x}}$$

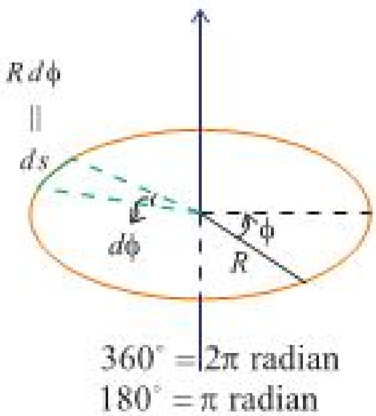
ELECTRIC FIELD DUE TO INFINITELY LONG LINE OF CHARGE

Example 2: Ring of Charge

E-field at a height z above a ring of charge of radius R

(1) Symmetry considered: For every charge element dq considered, there exists $dq\prime$ where the horizontal $\vec{E}$ field components cancel.
$\Rightarrow$ Overall E-field lies along z-direction.

(2) For each element of length dz, charge



$$dq = \underset{\substack{\uparrow\\ \text{Linear}\\ \text{charge density}}}{\lambda} \cdot \underset{\substack{\uparrow\\ \text{Circular}\\ \text{length element}}}{ds}$$

$dq = \lambda \cdot R\, d\phi$, where ϕ is the angle measured on the ring plane

$\therefore$ Net E-field along z-axis due to dq:

$$dE = \frac{1}{4\pi\epsilon_0} \cdot \frac{dq}{r^2} \cdot \cos\theta$$

$$\begin{aligned}\text{Total E-field} &= \int dE \\ &= \int_0^{2\pi} \frac{1}{4\pi\epsilon_0}\cdot\frac{\lambda R\, d\phi}{r^2}\cdot\cos\theta \qquad (\cos\theta = \frac{z}{r})\end{aligned}$$

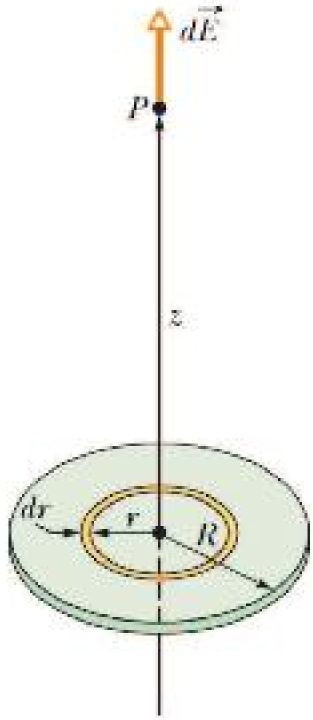
Note: Here in this case, θ, R and r are *fixed* as ϕ varies! BUT we want to convert r, θ to R, z.

$$E = \frac{1}{4\pi\epsilon_0}\cdot\frac{\lambda R z}{r^3}\int_0^{2\pi} d\phi$$

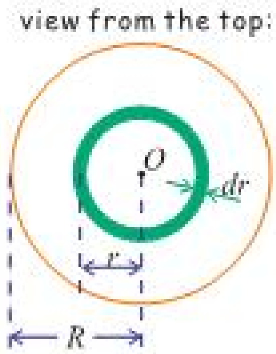
$$\boxed{E = \frac{1}{4\pi\epsilon_0}\cdot\frac{\lambda(2\pi R)z}{(z^2+R^2)^{3/2}}} \qquad \text{along } z\text{-axis}$$

BUT: $\lambda(2\pi R)$ = total charge on the ring

Example 3: E-field from a disk of surface charge density σ



We find the E-field of a disk by integrating concentric rings of charges.



Total charge of ring

$$dq = \sigma \cdot (\underbrace{2\pi r\ dr}_{\text{Area of the ring}})$$

Recall from Example 2:

$$\text{E-field from ring: } dE = \frac{1}{4\pi\epsilon_0} \cdot \frac{dq\ z}{(z^2 + r^2)^{3/2}}$$

$$\begin{aligned} \therefore \quad E &= \frac{1}{4\pi\epsilon_0} \int_0^R \frac{2\pi\sigma r\ dr \cdot z}{(z^2+r^2)^{3/2}} \\ &= \frac{1}{4\pi\epsilon_0} \int_0^R 2\pi\sigma z \frac{r\ dr}{(z^2+r^2)^{3/2}} \end{aligned}$$

- Change of variable:

$$\begin{aligned} & u = z^2 + r^2 && \Rightarrow && (z^2+r^2)^{3/2} = u^{3/2} \\ \Rightarrow \quad & du = 2r\ dr && \Rightarrow && r\ dr = \tfrac{1}{2} du \end{aligned}$$

- Change of integration limit:

$$\begin{cases} r = 0 & , \quad u = z^2 \\ r = R & , \quad u = z^2 + R^2 \end{cases}$$

$$\therefore\ E = \frac{1}{4\pi\epsilon_0} \cdot 2\pi\sigma z \int_{z^2}^{z^2+R^2} \frac{1}{2} u^{-3/2} du$$

BUT: $\displaystyle \int u^{-3/2} du = \frac{u^{-1/2}}{-1/2} = -2u^{-1/2}$

$$\begin{aligned} \therefore\ E &= \frac{1}{2\epsilon_0} \sigma z \left(-u^{-1/2}\right)\Big|_{z^2}^{z^2+R^2} \\ &= \frac{1}{2\epsilon_0} \sigma z \left(\frac{-1}{\sqrt{z^2+R^2}} + \frac{1}{z} \right) \end{aligned}$$

$$\boxed{E = \frac{\sigma}{2\epsilon_0} \left[1 - \frac{z}{\sqrt{z^2+R^2}} \right]}$$

VERY IMPORTANT LIMITING CASE:

If $R \gg z$, that is if we have an infinite sheet of charge with charge density σ:

$$\begin{aligned} E &= \frac{\sigma}{2\epsilon_0}\left[1 - \frac{z}{\sqrt{z^2+R^2}}\right] \\ &\simeq \frac{\sigma}{2\epsilon_0}\left[1 - \frac{z}{R}\right] \end{aligned}$$

$$\boxed{E \approx \frac{\sigma}{2\epsilon_0}}$$

E-field is normal to the charged surface

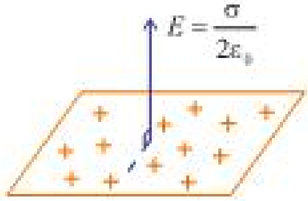
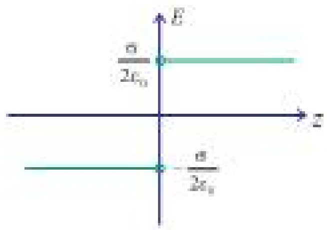


Figure 2.2: E-field due to an infinite sheet of charge, charge density $= \sigma$

Q: What's the E-field belows the charged sheet?

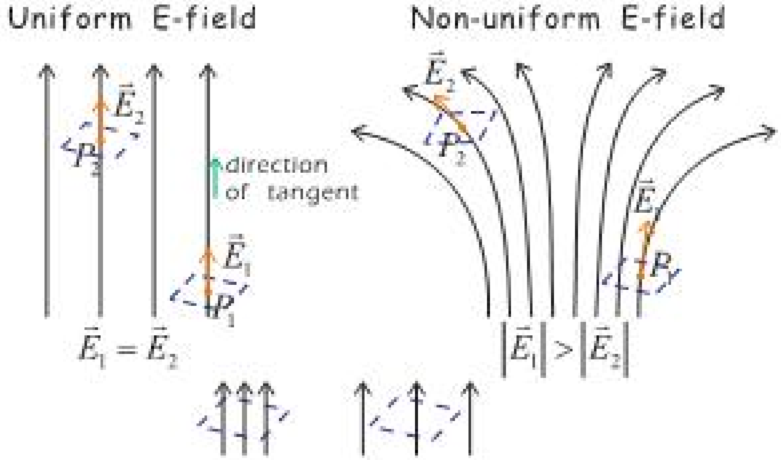


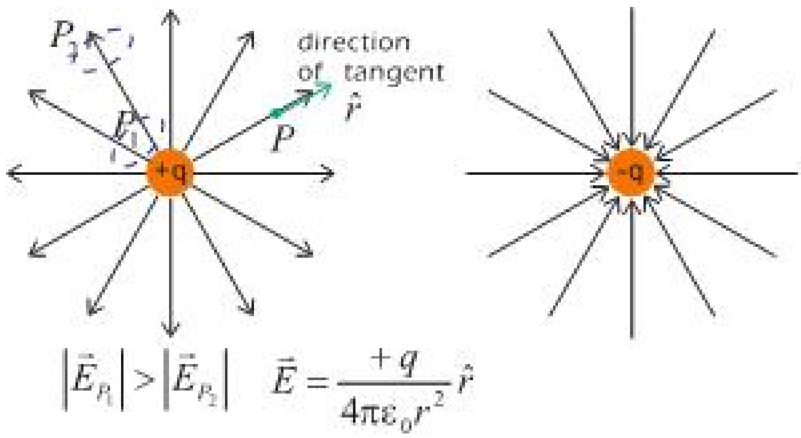
2.4 Electric Field Lines

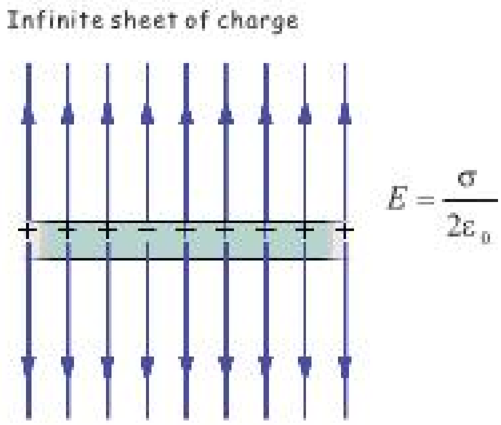
To visualize the electric field, we can use a graphical tool called the **electric field lines**.

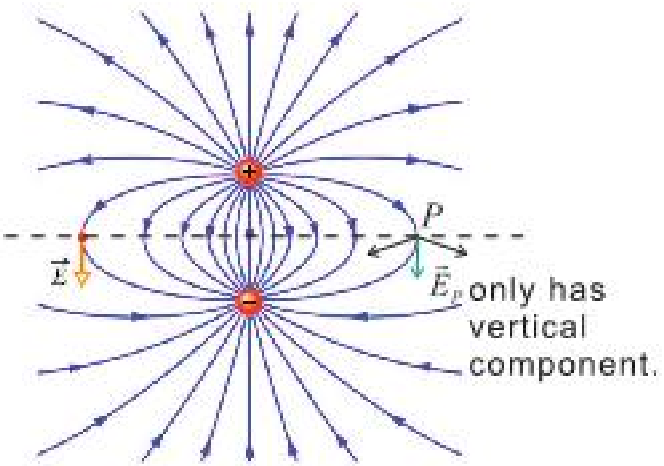
Conventions:

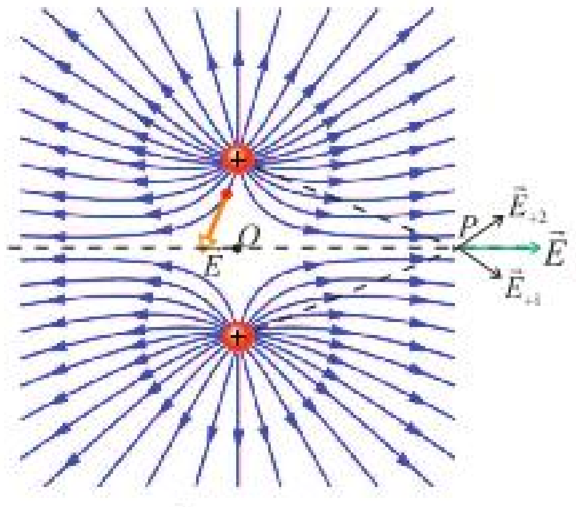
1. The start on position charges and end on negative charges.
2. *Direction* of E-field at any point is given by *tangent* of E-field line.
3. *Magnitude* of E-field at any point is proportional to *number of E-field lines per unit area perpendicular to the lines.*


Uniform E-field
Non-uniform E-field
direction
of tangent
$\vec{E}_1 = \vec{E}_2$
$|\vec{E}_1| > |\vec{E}_2|$


direction
of tangent
+q
-q
$|\vec{E}_{P_1}| > |\vec{E}_{P_2}|$
$\vec{E} = \frac{+q}{4\pi\varepsilon_0 r^2}\hat{r}$


Infinite sheet of charge
$E = \frac{\sigma}{2\varepsilon_0}$





$$\vec{E}_{\text{at point } O} = 0$$

2.5 Point Charge in E-field

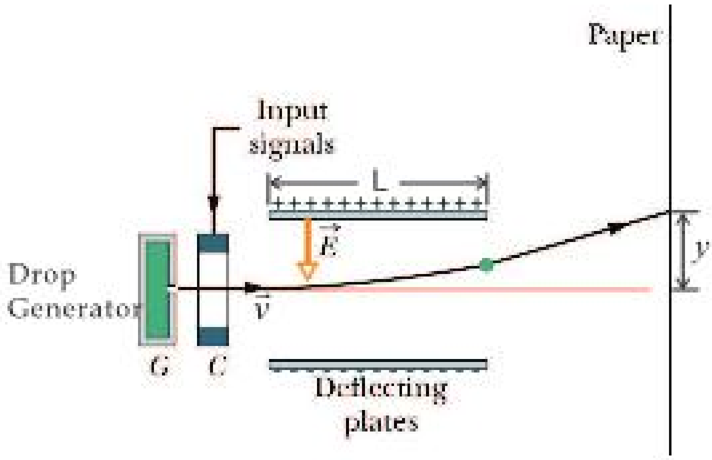
When we place a charge q in an E-field $\vec{E}$, the force experienced by the charge is

$$\boxed{\vec{F} = q\vec{E} = m\vec{a}}$$

Applications: *Ink-jet printer, TV cathoderay tube.*

<u>Example:</u>

Ink particle has mass m, charge q ($q < 0$ here)



Assume that mass of inkdrop is small, what's the deflection y of the charge?

Solution:

First, the charge carried by the inkdrop is *negtive*, `i.e.` $q < 0$.

Note: $q\vec{E}$ points in opposite direction of $\vec{E}$.

Horizontal motion: Net force = 0

$$\therefore \ L = vt \tag{2.1}$$

Vertical motion: $|q\vec{E}| \gg |m\vec{g}|$, q is negative,

$$\therefore \text{ Net force} = -qE = ma \qquad \text{(Newton's 2nd Law)}$$

$$\therefore\ a = -\frac{qE}{m} \tag{2.2}$$

Vertical distance travelled:

$$y = \frac{1}{2}\,at^2$$

2.6 Dipole in E-field

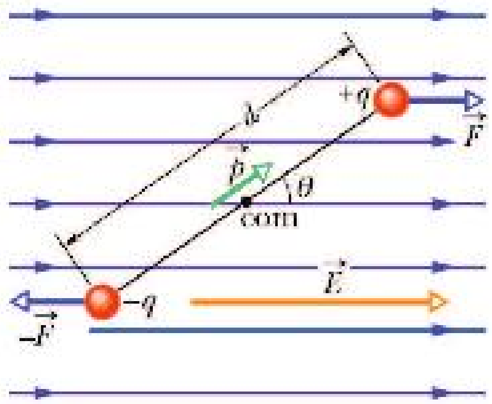
Consider the force exerted on the dipole in an *external* E-field:

Assumption: E-field from dipole doesn't affect the external E-field.

- Dipole moment:

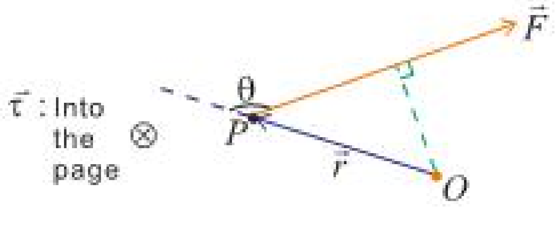
  $$\vec{p} = q\vec{d}$$

- Force due to the E-field on $+ve$ and $-ve$ charge are *equal and opposite in direction*. Total external force on dipole = 0.



BUT:	There is an external **torque** on the center of the dipole.

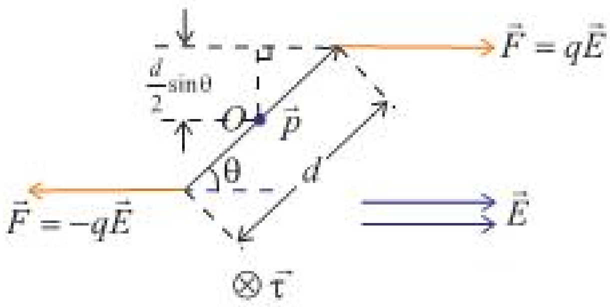
<u>Reminder</u>:



Force $\vec{F}$ exerts at point P. The force exerts a **torque** $\vec{\tau} = \vec{r} \times \vec{F}$ on point P with respect to point O.

Direction of the **torque vector $\vec{\tau}$** is determined from the **right-hand rule**.

<u>Reference</u>: Halliday Vol.1 Chap 9.1 (Pg.175) *torque*
Chap 11.7 (Pg.243) *work done*



Net torque $\vec{\tau}$

- direction: clockwise torque
- magnitude:

$$
\begin{aligned}
\tau &= \tau_{+ve} + \tau_{-ve} \\
&= F \cdot \frac{d}{2}\sin\theta + F \cdot \frac{d}{2}\sin\theta \\
&= qE \cdot d\sin\theta \\
&= pE\sin\theta
\end{aligned}
$$

$$\boxed{\vec{\tau} = \vec{p} \times \vec{E}}$$

Energy Consideration:

When the dipole $\vec{p}$ rotates $d\theta$, the E-field does work.

Work done by external E-field on the dipole:

$$dW = -\tau\, d\theta$$

Negative sign here because torque by E-field acts to *decrease* θ.

BUT: Because E-field is a **conservative force field** [1] [2] , we can define a **potential energy** (U) for the system, so that

$$\boxed{dU = -dW}$$

$\therefore$ For the dipole in external E-field:

$$dU = -dW = pE\sin\theta\, d\theta$$

$$
\begin{aligned}
\therefore\ U(\theta) = \int dU &= \int pE\sin\theta\, d\theta \\
&= -pE\cos\theta + U_0
\end{aligned}
$$

[1] more to come in Chap.4 of notes
[2] ref. Halliday Vol.1 Pg.257, Chap 12.1

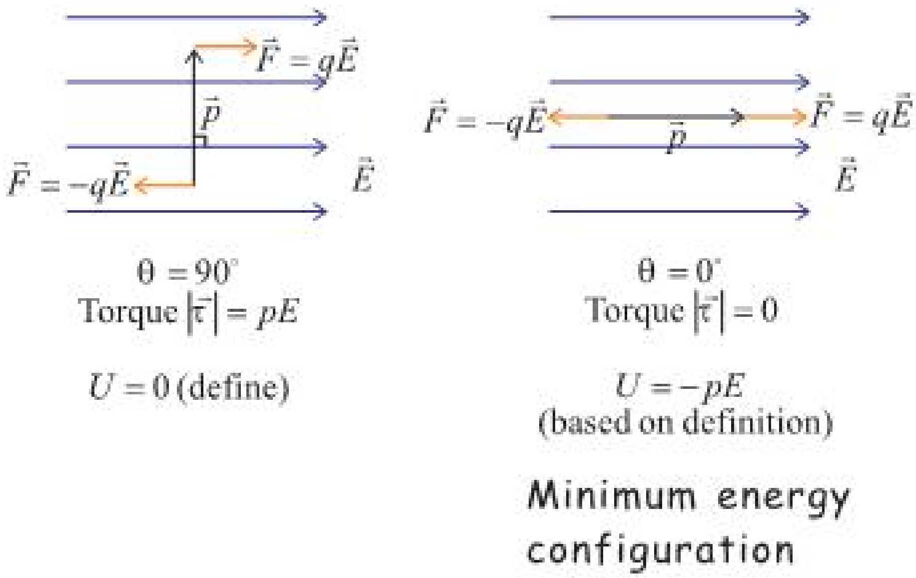
set $U(\theta = 90^\circ) = 0$,

$$\therefore\ 0 = -pE\cos 90^\circ + U_0$$

$$\therefore\ U_0 = 0$$

$\therefore$ Potential energy:

$$\boxed{U = -pE\cos\theta = -\vec{p}\cdot\vec{E}}$$



Chapter 3

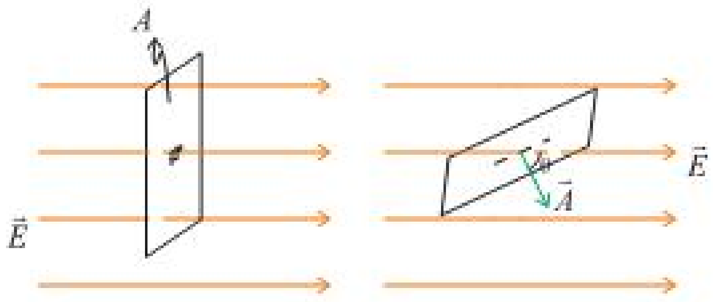
Electric Flux and Gauss' Law

3.1 Electric Flux

Latin: flux = "to flow"

Graphically: Electric flux Φ_E represents the number of E-field lines crossing a surface.

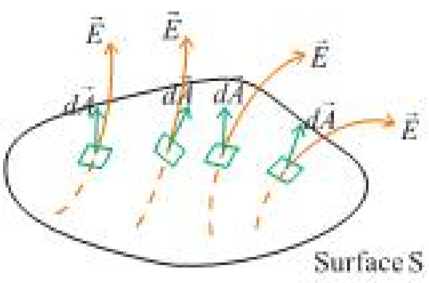
Mathematically:



$\Phi_E = EA$ $\qquad$ $\Phi_E = \vec{E} \cdot \vec{A}$

Reminder: Vector of the area $\vec{A}$ is perpendicular to the area A.

For non-uniform E-field & surface, direction of the area vector $\vec{A}$ is not uniform.



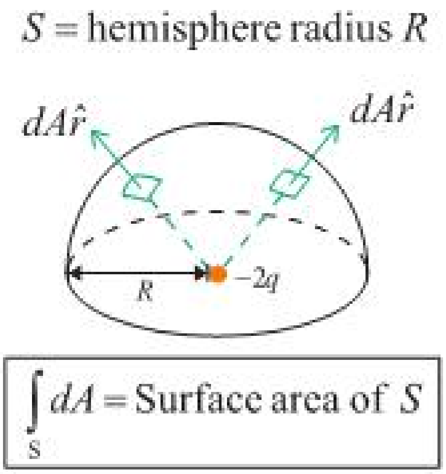
$d\vec{A}$ = Area vector for small area element dA

$$\therefore \text{ Electric flux} \qquad d\Phi_E = \vec{E}\cdot d\vec{A}$$

$$\text{Electric flux of } \vec{E} \text{ through surface S:} \quad \boxed{\Phi_E = \int_S \vec{E}\cdot d\vec{A}}$$

$$\begin{aligned} \int_S &= \text{Surface integral over surface S} \\ &= \text{Integration of integral over all area elements on surface S} \end{aligned}$$

Example:

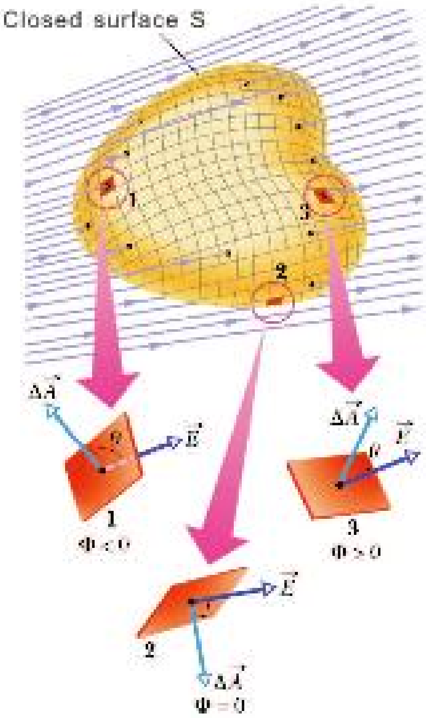


$$\vec{E} = \frac{1}{4\pi\epsilon_0}\cdot\frac{-2q}{r^2}\,\hat{r} = \frac{-q}{2\pi\epsilon_0 R^2}\,\hat{r}$$

For a hemisphere, $d\vec{A} = dA\,\hat{r}$

$$\begin{aligned} \Phi_E &= \int_S \frac{-q}{2\pi\epsilon_0 R^2}\,\hat{r}\cdot(dA\,\hat{r}) \qquad (\because \hat{r}\cdot\hat{r}=1) \\ &= -\frac{q}{2\pi\epsilon_0 R^2}\underbrace{\int_S dA}_{2\pi R^2} \\ &= \frac{-q}{\epsilon_0} \end{aligned}$$

For a closed surface:

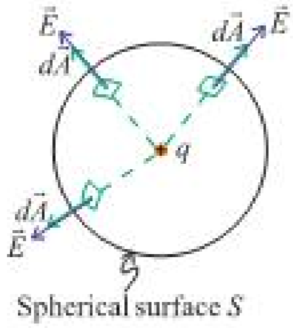


Recall: Direction of area vector $d\vec{A}$ goes *from inside to outside* of closed surface S.

Electric flux over closed surface S: $\boxed{\Phi_E = \oint_S \vec{E} \cdot d\vec{A}}$

$$\oint_S = \text{Surface integral over closed surface S}$$

Example:



Electric flux of charge q over closed spherical surface of radius R.

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \cdot \frac{q}{r^2}\, \hat{r} = \frac{q}{4\pi\epsilon_0 R^2}\, \hat{r} \qquad \text{at the surface}$$

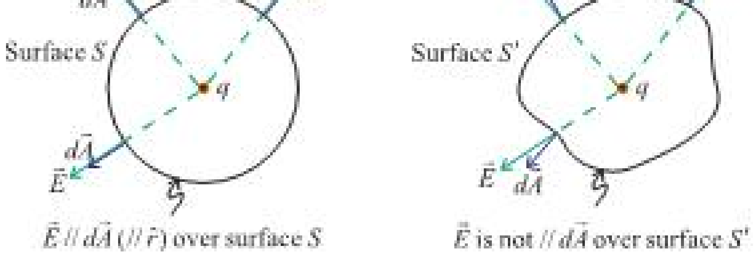
Again, $d\vec{A} = dA \cdot \hat{r}$

$$\begin{aligned}
\therefore \Phi_E &= \oint_S \overbrace{\frac{q}{4\pi\epsilon_0 R^2}\, \hat{r}}^{\vec{E}} \cdot \overbrace{dA\, \hat{r}}^{d\vec{A}} \\
&= \frac{q}{4\pi\epsilon_0 R^2} \underbrace{\oint_S dA}_{\text{Total surface area of S} = 4\pi R^2} \\
\Phi_E &= \frac{q}{\epsilon_0}
\end{aligned}$$

IMPORTANT POINT:

If we remove the spherical symmetry of closed surface S, *the total number of E-field lines crossing the surface remains the same.*

$\therefore$ The electric flux Φ_E



$$\Phi_E = \oint_S \vec{E} \cdot d\vec{A} = \oint_{S'} \vec{E} \cdot d\vec{A} = \frac{q}{\epsilon_0}$$

3.2 Gauss' Law

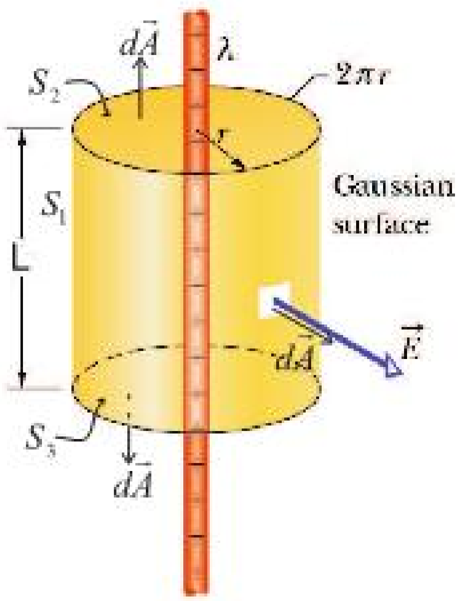
$$\boxed{\Phi_E = \oint_S \vec{E} \cdot d\vec{A} = \frac{q}{\epsilon_0}} \quad \text{for any closed surface S}$$

And q is the net electric charge enclosed in closed surface S.

- Gauss' Law is valid for *all charge distributions and all closed surfaces. (Gaussian surfaces)*
- Coulomb's Law can be derived from Gauss' Law.
- For system with high order of *symmetry*, E-field can be easily determined if we construct *Gaussian surfaces with the same symmetry* and applies Gauss' Law

3.3 E-field Calculation with Gauss' Law

(A) Infinite line of charge



Linear charge density: λ
Cylindrical symmetry.
E-field directs radially outward from the rod.
Construct a Gaussian surface S in the shape of a **cylinder**, making up of a curved surface S_1, and the top and bottom circles S_2, S_3.

Gauss' Law: $$\oint_S \vec{E} \cdot d\vec{A} = \frac{\text{Total charge}}{\epsilon_0} = \frac{\lambda L}{\epsilon_0}$$

$$\oint_S \vec{E}\cdot d\vec{A} = \underbrace{\int_{S_1} \vec{E}\cdot d\vec{A}}_{\vec{E}\parallel d\vec{A}} + \underbrace{\int_{S_2} \vec{E}\cdot d\vec{A} + \int_{S_3} \vec{E}\cdot d\vec{A}}_{=0\ \because\ \vec{E}\perp d\vec{A}}$$

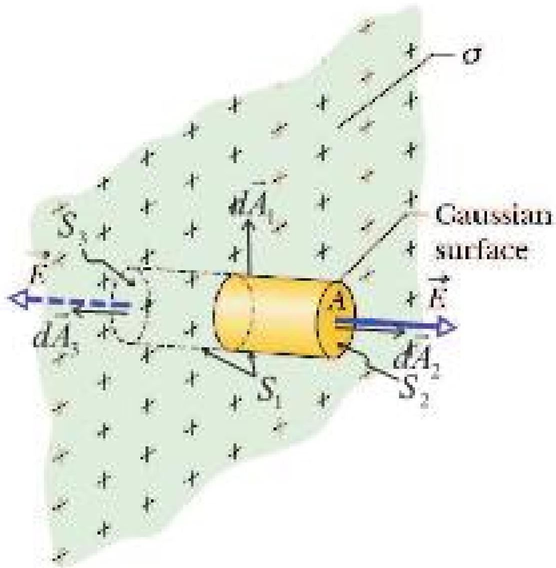
$$\therefore \quad E\underbrace{\int_{S_1} dA}_{\text{Total area of surface } S_1} = \frac{\lambda L}{\epsilon_0}$$

$$E(2\pi r L) = \frac{\lambda L}{\epsilon_0}$$

$$\therefore \boxed{E = \frac{\lambda}{2\pi\epsilon_0 r}} \qquad \text{(Compare with Chapter 2 note)}$$

$$\vec{E} = \frac{\lambda}{2\pi\epsilon_0 r}\,\hat{r}$$

(B) Infinite sheet of charge



Uniform surface charge density: σ
Planar symmetry.
E-field directs perpendicular to the sheet of charge.
Construct Gaussian surface S in the shape of a **cylinder (pill box)** of cross-sectional area A.

Gauss' Law: $$\oint_S \vec{E}\cdot d\vec{A} = \frac{A\sigma}{\epsilon_0}$$

$$\int_{S_1} \vec{E}\cdot d\vec{A} = 0 \quad \because \vec{E}\perp d\vec{A} \text{ over whole surface } S_1$$

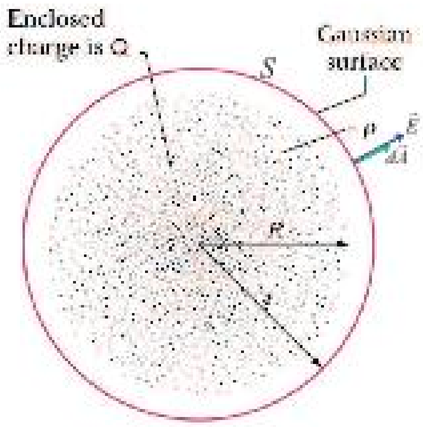
$$\int_{S_2} \vec{E}\cdot d\vec{A} + \int_{S_3} \vec{E}\cdot d\vec{A} = 2EA \quad (\vec{E}\parallel d\vec{A}_2, \vec{E}\parallel d\vec{A}_3)$$

Note: For S_2, both $\vec{E}$ and $d\vec{A}_2$ point up
For S_3, both $\vec{E}$ and $d\vec{A}_3$ point down

$$\therefore \quad 2EA = \frac{A\sigma}{\epsilon_0} \quad \Rightarrow \quad \boxed{E = \frac{\sigma}{2\epsilon_0}} \quad \text{(Compare with Chapter 2 note)}$$

(C) Uniformly charged sphere
Total charge $= Q$
Spherical symmetry.

(a) For $r > R$:



Consider a spherical Gaussian surface S of radius r:

$$\vec{E} \parallel d\vec{A} \parallel \hat{r}$$

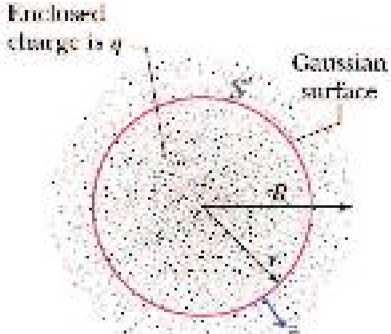
Gauss' Law: $$\oint_S \vec{E} \cdot d\vec{A} = \frac{Q}{\epsilon_0}$$

$$\oint_S E \cdot dA = \frac{Q}{\epsilon_0}$$

$$E \underbrace{\oint_S dA}_{\text{surface area of } S \,=\, 4\pi r^2} = \frac{Q}{\epsilon_0}$$

$$\therefore \quad \vec{E} = \frac{Q}{4\pi\epsilon_0 r^2}\,\hat{r}\,; \qquad \text{for } r > R$$

(b) For $r < R$:



Consider a spherical Gaussian surface S' of radius $r < R$, then total charge included q is *proportional to the volume included by* S'

$$\therefore \quad \frac{q}{Q} = \frac{\text{Volume enclosed by } S'}{\text{Total volume of sphere}}$$

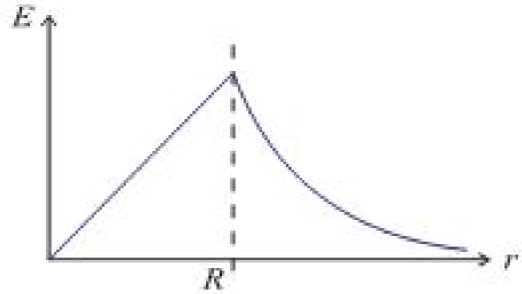
$$\frac{q}{Q} = \frac{4/3\,\pi r^3}{4/3\,\pi R^3} \quad \Rightarrow \quad q = \frac{r^3}{R^3}\,Q$$

Gauss' Law: $\oint_{S'} \vec{E}\cdot d\vec{A} = \frac{q}{\epsilon_0}$

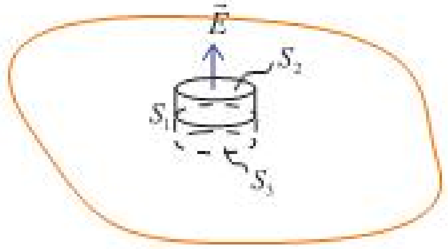
$$E\underbrace{\oint_{S'} dA} = \frac{r^3}{R^3}\frac{1}{\epsilon_0}\cdot Q$$

surface area of $S' = 4\pi r^2$

$$\therefore \quad \vec{E} = \frac{1}{4\pi\epsilon_0}\cdot\frac{Q}{R^3}\,r\,\hat{r}\,; \qquad \text{for } r \le R$$



3.4 Gauss' Law and Conductors



Cross-sectional area A

For *isolated* conductors, charges are free to move until *all charges lie outside the surface of the conductor*. Also, the *E-field at the surface of a conductor is perpendicular to its surface.* (`Why?`)

Consider Gaussian surface S of shape of cylinder:

$$\oint_S \vec{E}\cdot d\vec{A} = \frac{\sigma A}{\epsilon_0}$$

BUT $\int_{S_1} \vec{E} \cdot d\vec{A} = 0 \quad (\because\ \vec{E} \perp d\vec{A}\,)$

$\int_{S_3} \vec{E} \cdot d\vec{A} = 0 \quad (\because\ \vec{E} = 0 \text{ inside conductor })$

$$\begin{aligned}\int_{S_2} \vec{E} \cdot d\vec{A} &= E \underbrace{\int_{S_2} dA}_{\text{Area of } S_2} \quad (\because\ \vec{E} \parallel d\vec{A}\,) \\ &= EA\end{aligned}$$

$$\therefore \text{ Gauss' Law} \quad \Rightarrow \quad EA = \frac{\sigma A}{\epsilon_0}$$

$$\therefore \boxed{\text{On conductor's surface} \quad E = \frac{\sigma}{\epsilon_0}}$$

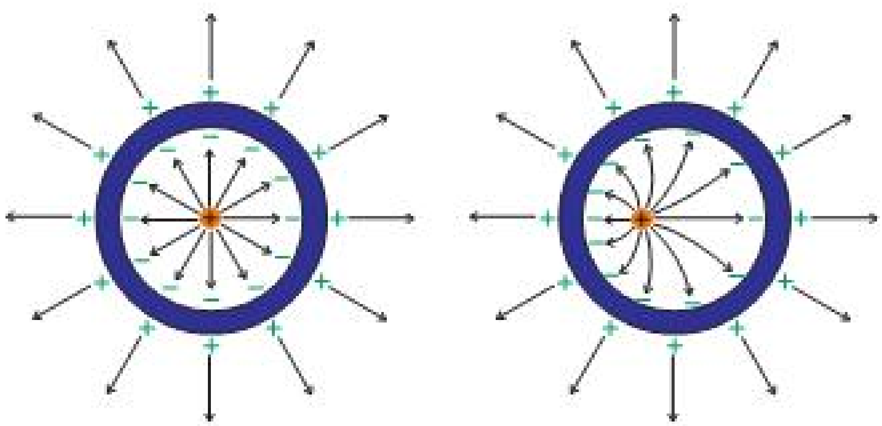
BUT, there's no charge inside conductors.

$$\therefore \boxed{\text{Inside conductors} \quad E = 0} \ \textit{Always!}$$

Notice: Surface charge density on a conductor's surface is *not uniform.*

Example: Conductor with a charge inside

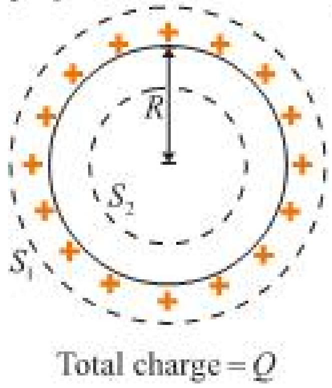
Note: This is <u>not</u> an isolated system (because of the charge inside).



Note : In BOTH cases, $\vec{E} = \frac{1}{4\pi\varepsilon_0} \cdot \frac{q}{r^2}\hat{r}$ outside

Example:

I. Charge sprayed on a conductor sphere:



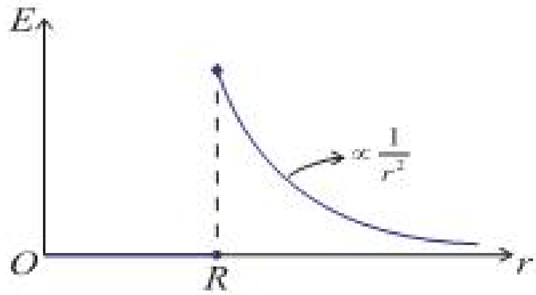
First, we know that charges all move to the *surface* of conductors.

(i) For $r < R$:
Consider Gaussian surface S_2

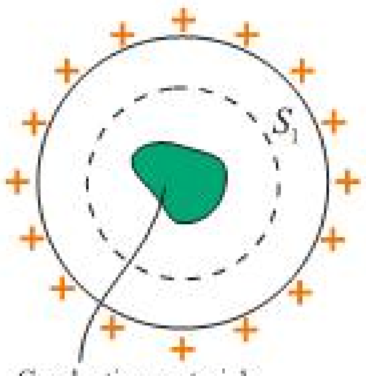
$$\oint_{S_2} \vec{E} \cdot d\vec{A} = 0 \quad (\because \text{ no charge inside })$$
$$\Rightarrow \quad E = 0 \quad \text{everywhere.}$$

(ii) For $r \geq R$:
Consider Gaussian surface S_1:

$$\begin{aligned} \oint_{S_1} \vec{E} \cdot d\vec{A} &= \frac{Q}{\epsilon_0} \\ E \underbrace{\oint_{S_1} d\vec{A}}_{4\pi r^2} &= \frac{Q}{\epsilon_0} \quad (\overbrace{\vec{E} \parallel d\vec{A}}^{\text{For a conductor}} \underbrace{\parallel \hat{r}}_{\text{Spherically symmetric}}) \\ E &= \frac{Q}{4\pi\epsilon_0 r^2} \end{aligned}$$



II. Conductor sphere with hole inside:

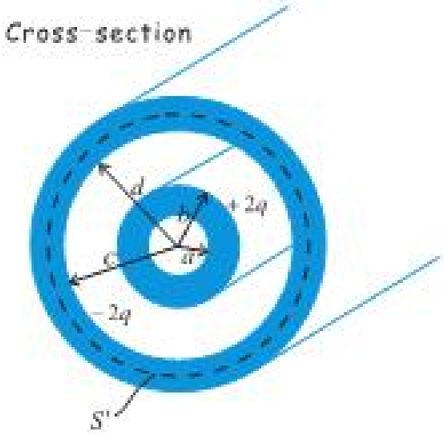


Consider Gaussian surface S_1: Total charge included = 0

$$\therefore \text{ E-field} = 0 \text{ inside}$$

The E-field is identical to the case of a solid conductor!!

III. A long hollow cylindrical conductor:



Example:
Inside hollow cylinder ($+2q$)

$$\begin{cases} \text{Inner radius} & \text{a} \\ \text{Outer radius} & \text{b} \end{cases}$$

Outside hollow cylinder ($-3q$)

$$\begin{cases} \text{Inner radius} & \text{c} \\ \text{Outer radius} & \text{d} \end{cases}$$

Question: Find the charge on each surface of the conductor.

For the inside hollow cylinder, charges distribute only on the surface.
$\therefore$ Inner radius a surface, charge = 0
and Outer radius b surface, charge = $+2q$

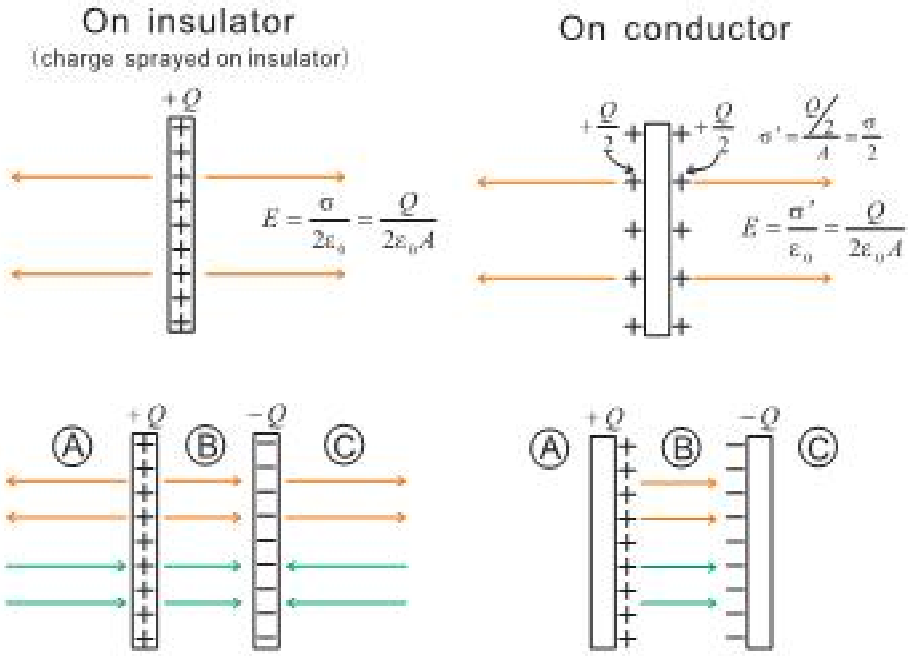
For the outside hollow cylinder, charges do <u>not</u> distribute only on outside.
$\because$ It's not an isolated system. (There are charges inside!)

Consider Gaussian surface S' inside the conductor:
E-field always = 0
$\therefore$ Need charge $-2q$ on radius c surface to balance the charge of inner cylinder.
So charge on radius d surface = $-q$. (`Why?`)

IV. Large sheets of charge:
Total charge Q on sheet of area A.

$\therefore$ Surface charge density $\sigma = \dfrac{Q}{A}$

By principle of superposition



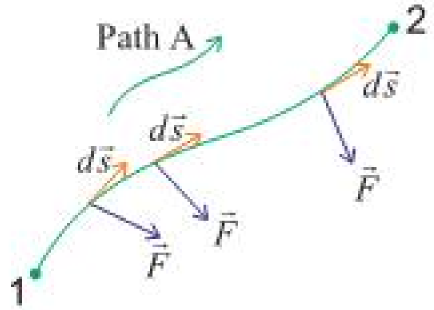
Region A:	$E = 0$	$E = 0$
Region B:	$E = \dfrac{Q}{\epsilon_0 A}$	$E = \dfrac{Q}{\epsilon_0 A}$
Region C:	$E = 0$	$E = 0$

Chapter 4

Electric Potential

4.1 Potential Energy and Conservative Forces

(Read Halliday Vol.1 Chap.12)
Electric force is a **conservative force**



Work done by the electric force $\vec{F}$ as a charge moves an infinitesimal distance $d\vec{s}$ along *Path A* $= dW$

Note: $d\vec{s}$ is in the *tangent* direction of the curve of *Path A*.

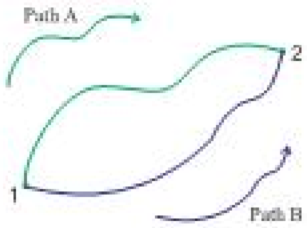
$$dW = \vec{F} \cdot d\vec{s}$$

$\therefore$ Total work done W by force $\vec{F}$ in moving the particle from Point 1 to Point 2

$$\boxed{W = \underset{\text{Path A}}{\int_1^2} \vec{F} \cdot d\vec{s}}$$

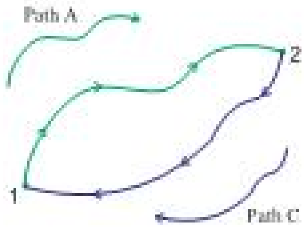
$$\begin{aligned} \underset{\text{Path A}}{\int_1^2} &= \textit{Path Integral} \\ &= \text{Integration over Path A from Point 1 to Point 2.} \end{aligned}$$

DEFINITION: A force is **conservative** *if* the work done on a particle by the force is *independent of the path taken.*



$\therefore$ For conservative forces,

$$\underset{\text{Path A}}{\int_1^2 \vec{F}\cdot d\vec{s}} = \underset{\text{Path B}}{\int_1^2 \vec{F}\cdot d\vec{s}}$$



Let's consider a path starting at point 1 to 2 through *Path A* and from 2 to 1 through *Path C*

$$\begin{aligned} \text{Work done} &= \underset{\text{Path A}}{\int_1^2 \vec{F}\cdot d\vec{s}} + \underset{\text{Path C}}{\int_2^1 \vec{F}\cdot d\vec{s}} \\ &= \underset{\text{Path A}}{\int_1^2 \vec{F}\cdot d\vec{s}} - \underset{\textbf{Path B}}{\int_{\mathbf{1}}^{\mathbf{2}} \vec{F}\cdot d\vec{s}} \end{aligned}$$

DEFINITION: The work done by a **conservative force** on a particle when it *moves around a closed path returning to its initial position* is *zero.*

MATHEMATICALLY, $\vec{\nabla}\times\vec{F}=0$ everywhere for conservative force $\vec{F}$

Conclusion: Since the work done by a conservative force $\vec{F}$ is *path-independent*, we can define a quantity, **potential energy**, that depends only on the *position* of the particle.

Convention: We define **potential energy** U such that

$$\boxed{dU = -W = -\int \vec{F}\cdot d\vec{s}}$$

$\therefore$ For particle moving from 1 to 2

$$\int_1^2 dU = U_2 - U_1 = -\int_1^2 \vec{F}\cdot d\vec{s}$$

where U_1, U_2 are **potential energy** at position 1, 2.

Example:

Suppose charge q_2 moves from point 1 to 2.

$$\begin{aligned}
\text{From definition:}\quad U_2 - U_1 &= -\int_1^2 \vec{F}\cdot d\vec{r} \\
&= -\int_{r_1}^{r_2} F\,dr \quad (\because\ \vec{F} \parallel d\vec{r}\,) \\
&= -\int_{r_1}^{r_2} \frac{1}{4\pi\epsilon_0}\,\frac{q_1 q_2}{r^2}\,dr \\
(\because \int \frac{dr}{r^2} = -\frac{1}{r} + C\ ) \qquad &= \frac{1}{4\pi\epsilon_0}\,\frac{q_1 q_2}{r}\bigg|_{r_1}^{r_2} \\
-\Delta W = \Delta U &= \frac{1}{4\pi\epsilon_0}\,q_1 q_2\left(\frac{1}{r_2} - \frac{1}{r_1}\right)
\end{aligned}$$

Note:

(1) This result is generally true for 2-Dimension or 3-D motion.

(2) If q_2 moves away from q_1,
then $r_2 > r_1$, we have

- If q_1, q_2 are of *same* sign,
  then $\Delta U < 0$, $\Delta W > 0$
  (ΔW = Work done by electric *repulsive* force)
- If q_1, q_2 are of *different* sign,
  then $\Delta U > 0$, $\Delta W < 0$
  (ΔW = Work done by electric *attractive* force)

(3) If q_2 moves towards q_1,
then $r_2 < r_1$, we have

- If q_1, q_2 are of *same* sign,
  then ΔU 0, ΔW 0
- If q_1, q_2 are of *different* sign,
  then ΔU 0, ΔW 0

(4) Note: It is the *difference* in potential energy that is important.

$$\text{REFERENCE POINT:} \quad U(r=\infty)=0$$

$$\therefore\ U_\infty - U_1 = \frac{1}{4\pi\epsilon_0} q_1 q_2 \left(\frac{1}{\underset{\infty}{\underset{\downarrow}{r_2}}} - \frac{1}{r_1}\right)$$

$$\boxed{U(r) = \frac{1}{4\pi\epsilon_0}\cdot\frac{q_1 q_2}{r}}$$

If q_1, q_2 *same sign*, then $U(r) > 0$ for all r
If q_1, q_2 *opposite sign*, then $U(r) < 0$ for all r

(5) Conservation of Mechanical Energy:
For a system of charges with no external force,

$$E = \underset{\text{(Kinetic Energy)}}{K} + \underset{\text{(Potential Energy)}}{U} = \text{Constant}$$

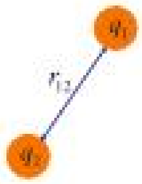
$$\text{or} \quad \boxed{\Delta E = \Delta K + \Delta U = 0}$$

Potential Energy of A System of Charges

Example: P.E. of 3 charges q_1, q_2, q_3

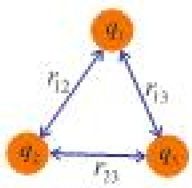
Start: q_1, q_2, q_3 all at $r=\infty, U=0$

Step1:  Move q_1 from ∞ to its position $\Rightarrow$ $U=0$

Step2: 

Move q_2 from ∞ to new position $\Rightarrow$

$$U = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r_{12}}$$

Step3: 

Move q_3 from ∞ to new position $\Rightarrow$ Total P.E.

$$U = \frac{1}{4\pi\epsilon_0}\left[\frac{q_1 q_2}{r_{12}} + \frac{q_1 q_3}{r_{13}} + \frac{q_2 q_3}{r_{23}}\right]$$

Step4: What if there are 4 charges?

4.2 Electric Potential

Consider a charge q at center, we consider its effect on test charge q_0

DEFINITION: We define electric potential V so that

$$\boxed{\Delta V = \frac{\Delta U}{q_0} = \frac{-\Delta W}{q_0}}$$

($\therefore$ V is the P.E. per unit charge)

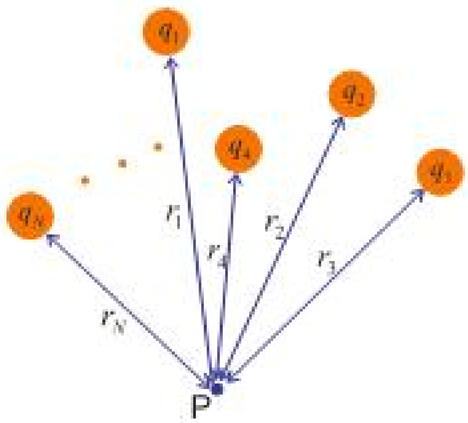
- Similarly, we take $V(r = \infty) = 0$.
- Electric Potential is a **scalar**.
- Unit: $Volt(V) = Joules/Coulomb$
- For a single point charge:

$$\boxed{V(r) = \frac{1}{4\pi\epsilon_0} \cdot \frac{q}{r}}$$

- Energy Unit: $\Delta U = q\Delta V$

$$electron - Volt(eV) = \underbrace{1.6 \times 10^{-19}}_{\text{charge of electron}} J$$

<u>Potential For A System of Charges</u>



For a total of N point charges, the potential V at any point P can be derived from the **principle of superposition**.

Recall that potential due to q_1 at point P: $V_1 = \frac{1}{4\pi\epsilon_0} \cdot \frac{q_1}{r_1}$

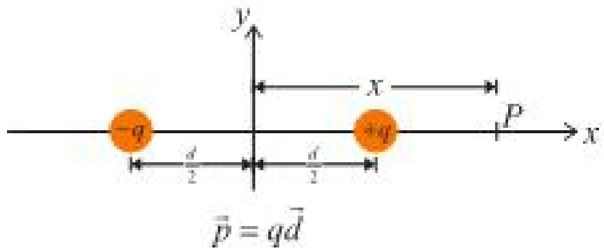
$\therefore$ Total potential at point P due to N charges:

$$\begin{aligned} V &= V_1 + V_2 + \cdots + V_N \quad \text{(principle of superposition)} \\ &= \frac{1}{4\pi\epsilon_0}\left[\frac{q_1}{r_1} + \frac{q_2}{r_2} + \cdots + \frac{q_N}{r_N}\right] \end{aligned}$$

$$\boxed{V = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{N}\frac{q_i}{r_i}}$$

Note: For $\vec{E}, \vec{F}$, we have a sum of vectors
For V, U, we have a sum of scalars

Example: Potential of an electric dipole



Consider the potential of point P at distance $x > \frac{d}{2}$ from dipole.

$$V = \frac{1}{4\pi\epsilon_0}\left[\frac{+q}{x-\frac{d}{2}} + \frac{-q}{x+\frac{d}{2}}\right]$$

Special Limiting Case: $x \gg d$

$$\frac{1}{x \mp \frac{d}{2}} = \frac{1}{x}\cdot\frac{1}{1 \mp \frac{d}{2x}} \simeq \frac{1}{x}\left[1 \pm \frac{d}{2x}\right]$$

$$\therefore \quad V = \frac{1}{4\pi\epsilon_0}\cdot\frac{q}{x}\left[1 + \frac{d}{2x} - (1 - \frac{d}{2x})\right]$$

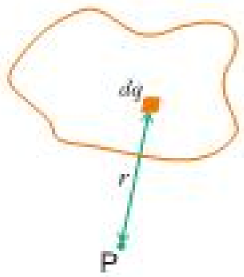
$$V = \frac{p}{4\pi\epsilon_0 x^2} \qquad \text{(Recall } p = qd\text{)}$$

For a point charge $\quad E \propto \frac{1}{r^2} \quad V \propto \frac{1}{r}$

For a dipole $\quad E \propto \frac{1}{r^3} \quad V \propto \frac{1}{r^2}$

For a quadrupole $\quad E \propto \frac{1}{r^4} \quad V \propto \frac{1}{r^3}$

Electric Potential of Continuous Charge Distribution



For any charge distribution, we write the electrical potential dV due to infinitesimal charge dq:

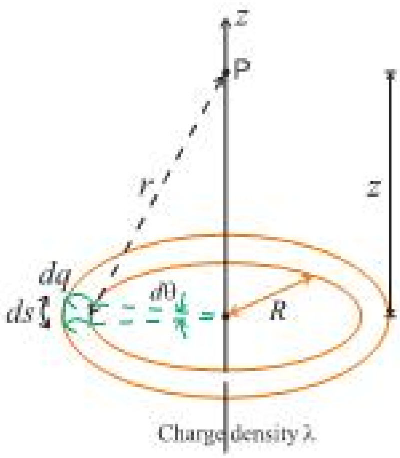
$$dV = \frac{1}{4\pi\epsilon_0}\cdot\frac{dq}{r}$$

$$\therefore \quad \boxed{V = \int\limits_{\substack{\text{charge}\\ \text{distribution}}} \frac{1}{4\pi\epsilon_0} \cdot \frac{dq}{r}}$$

Similar to the previous examples on E-field, for the case of *uniform* charge distribution:

1-D	$\Rightarrow$	long rod	$\Rightarrow$	$dq = \lambda\, dx$
2-D	$\Rightarrow$	charge sheet	$\Rightarrow$	$dq = \sigma\, dA$
3-D	$\Rightarrow$	uniformly charged body	$\Rightarrow$	$dq = \rho\, dV$

Example (1): Uniformly-charged ring



Length of the infinitesimal ring element $= ds = R d\theta$

$$\therefore \quad \text{charge } dq = \lambda\, ds = \lambda R\, d\theta$$

$$dV = \frac{1}{4\pi\epsilon_0} \cdot \frac{dq}{r} = \frac{1}{4\pi\epsilon_0} \cdot \frac{\lambda R\, d\theta}{\sqrt{R^2 + z^2}}$$

The integration is around the entire ring.

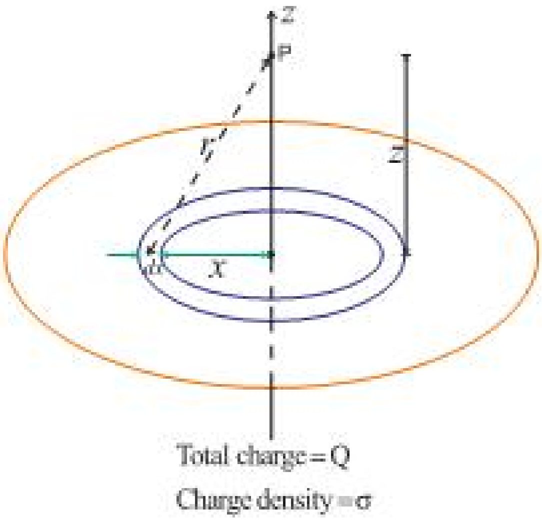
$$\begin{aligned} \therefore \quad V &= \int\limits_{\text{ring}} dV \\ &= \int_0^{2\pi} \frac{1}{4\pi\epsilon_0} \cdot \frac{\lambda R\, d\theta}{\sqrt{R^2 + z^2}} \\ &= \frac{\lambda R}{4\pi\epsilon_0 \sqrt{R^2 + z^2}} \underbrace{\int_0^{2\pi} d\theta}_{2\pi} \end{aligned}$$

Total charge on the ring $= \lambda \cdot (2\pi R)$

$$V = \frac{Q}{4\pi\epsilon_0 \sqrt{R^2 + z^2}}$$

LIMITING CASE: $z \gg R \Rightarrow V = \dfrac{Q}{4\pi\epsilon_0 \sqrt{z^2}} = \dfrac{Q}{4\pi\epsilon_0 |z|}$

Example (2): Uniformly-charged disk



Using the **principle of superposition**, we will find the potential of a disk of uniform charge density by integrating the potential of *concentric rings*.

$$\therefore \quad dV = \frac{1}{4\pi\epsilon_0}\int_{\text{disk}} \frac{dq}{r}$$

Ring of radius x: $\quad dq = \sigma\, dA = \sigma\,(2\pi x dx)$

$$\begin{aligned}
\therefore \quad V &= \int_0^R \frac{1}{4\pi\epsilon_0}\cdot\frac{\sigma 2\pi x\, dx}{\sqrt{x^2+z^2}} \\
&= \frac{\sigma}{4\epsilon_0}\int_0^R \frac{d(x^2+z^2)}{(x^2+z^2)^{1/2}} \\
V &= \frac{\sigma}{2\epsilon_0}\left(\sqrt{z^2+R^2}-\sqrt{z^2}\right) \\
&= \frac{\sigma}{2\epsilon_0}\left(\sqrt{z^2+R^2}-|z|\right)
\end{aligned}$$

Recall:
$|x| = \begin{cases} +x; & x \geq 0 \\ -x; & x < 0 \end{cases}$

Limiting Case:

(1) If $|z| \gg R$

$$\begin{aligned}
\sqrt{z^2+R^2} &= \sqrt{z^2\left(1+\frac{R^2}{z^2}\right)} \\
&= |z|\cdot\left(1+\frac{R^2}{z^2}\right)^{\frac{1}{2}} \qquad (\,(1+x)^n \approx 1+nx \text{ if } x \ll 1\,) \\
&\simeq |z|\cdot\left(1+\frac{R^2}{2z^2}\right) \qquad (\,\frac{|z|}{z^2} = \frac{1}{|z|}\,)
\end{aligned}$$

$\therefore$ At large z, $V \simeq \dfrac{\sigma}{2\epsilon_0}\cdot\dfrac{R^2}{2|z|} = \dfrac{Q}{4\pi\epsilon_0|z|}$ (like a point charge)
where Q = total charge on disk $= \sigma\cdot\pi R^2$

(2) If $|z| \ll R$

$$\begin{aligned}\sqrt{z^2+R^2} &= R\cdot\Big(1+\frac{z^2}{R^2}\Big)^{\frac{1}{2}} \\ &\simeq R\Big(1+\frac{z^2}{2R^2}\Big)\end{aligned}$$

$$\therefore \quad V \simeq \frac{\sigma}{2\epsilon_0}\Big[R-|z|+\frac{z^2}{2R}\Big]$$

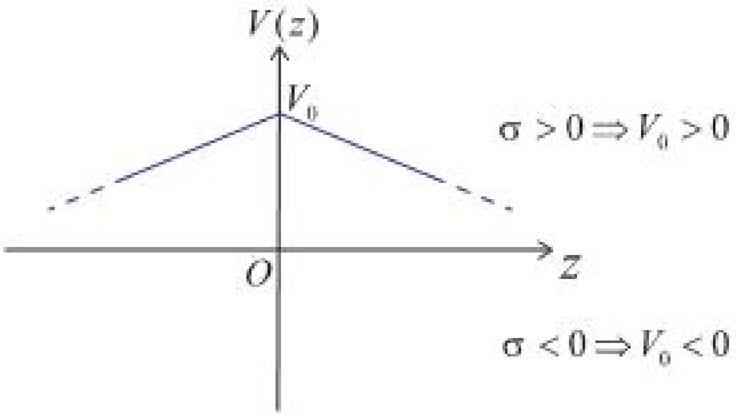
At $z=0$, $V=\dfrac{\sigma R}{2\epsilon_0}$; Let's call this V_0

$$\begin{aligned}\therefore \quad V(z) &= \frac{\sigma R}{2\epsilon_0}\Big[1-\frac{|z|}{R}+\frac{z^2}{2R^2}\Big] \\ V(z) &= V_0\Big[1-\frac{|z|}{R}+\frac{z^2}{2R^2}\Big]\end{aligned}$$

The *key* here is that it is the difference between potentials of two points that is important.
$\Rightarrow$ A convenience reference point to compare in this example is the potential of the charged disk.
$\therefore$ The important quantity here is

$$V(z)-V_0 = -\frac{|z|}{R}V_0 + \underbrace{\cancel{\frac{z^2}{2R^2}V_0}}_{\text{neglected as } z \ll R}$$

$$\boxed{V(z)-V_0 = -\frac{V_0}{R}|z|}$$



4.3 Relation Between Electric Field E and Electric Potential V

(A) To get V from $\vec{E}$:

Recall our definition of the potential V:

$$\Delta V = \frac{\Delta U}{q_0} = -\frac{W_{12}}{q_0}$$

where ΔU is the change in P.E.; W_{12} is the work done in bringing charge q_0 from point 1 to 2.

$$\therefore \quad \Delta V = V_2 - V_1 = \frac{-\int_1^2 \vec{F} \cdot d\vec{s}}{q_0}$$

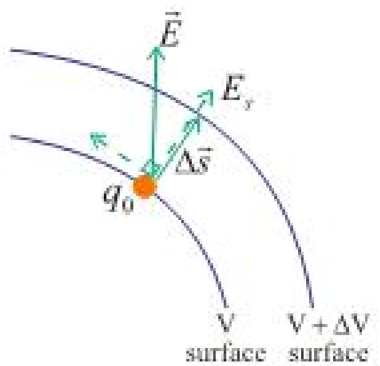
However, the definition of E-field: $\quad \vec{F} = q_0 \vec{E}$

$$\therefore \quad \boxed{\Delta V = V_2 - V_1 = -\int_1^2 \vec{E} \cdot d\vec{s}}$$

Note: The integral on the right hand side of the above can be calculated *along any path from point 1 to 2.* (**Path-Independent**)

Convention: $V_\infty = 0 \quad \Rightarrow \quad \boxed{V_P = -\int_\infty^P \vec{E} \cdot d\vec{s}}$

(B) To get $\vec{E}$ from V:



(i.e. Potential = V on the surface)

Again, use the definition of V:

$$\Delta U = q_0 \Delta V = \underbrace{-W}_{\text{Work done}}$$

However,

$$W = \underbrace{q_0 \vec{E}}_{\text{Electric force}} \cdot \Delta\vec{s}$$
$$= q_0 \, E_s \, \Delta s$$

where E_s is the E-field component along the path $\Delta\vec{s}$.

$$\therefore \quad q_0 \Delta V = -q_0 E_s \Delta s$$

$$\therefore \quad E_s = -\frac{\Delta V}{\Delta s}$$

For infinitesimal Δs,

$$\therefore \quad \boxed{E_s = -\frac{dV}{ds}}$$

Note: (1) Therefore the E-field component along *any direction* is the negtive derivative of the potential *along the same direction.*

(2) If $d\vec{s} \perp \vec{E}$, then $\Delta V = 0$

(3) ΔV is biggest/smallest if $d\vec{s} \parallel \vec{E}$

Generally, for a potential $V(x, y, z)$, the relation between $\vec{E}(x, y, z)$ and V is

$$E_x = -\frac{\partial V}{\partial x} \qquad E_y = -\frac{\partial V}{\partial y} \qquad E_z = -\frac{\partial V}{\partial z}$$

$\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$ are **partial derivatives**

For $\frac{\partial}{\partial x} V(x, y, z)$, everything y, z are treated like a *constant* and we only take derivative with respect to x.

Example: If $V(x, y, z) = x^2 y - z$

$$\frac{\partial V}{\partial x} =$$

$$\frac{\partial V}{\partial y} =$$

$$\frac{\partial V}{\partial z} =$$

For other co-ordinate systems

(1) Cylindrical:

$$V(r, \theta, z) \qquad \begin{cases} E_r = -\dfrac{\partial V}{\partial r} \\[2ex] E_\theta = -\dfrac{1}{r} \cdot \dfrac{\partial V}{\partial \theta} \\[2ex] E_z = -\dfrac{\partial V}{\partial z} \end{cases}$$

(2) Spherical:

$$V(r,\theta,\phi) \qquad \begin{cases} E_r = -\dfrac{\partial V}{\partial r} \\ E_\theta = -\dfrac{1}{r}\cdot\dfrac{\partial V}{\partial \theta} \\ E_\phi = -\dfrac{1}{r\sin\theta}\cdot\dfrac{\partial V}{\partial \phi} \end{cases}$$

Note: Calculating V involves summation of *scalars*, which is easier than adding *vectors* for calculating E-field.

$\therefore$ To find the E-field of a general charge system, we first calculate $\boldsymbol{V}$, and then derive $\vec{\boldsymbol{E}}$ from the partial derivative.

Example: Uniformly charged disk

From potential calculations:

$$V = \frac{\sigma}{2\epsilon_0}\left(\sqrt{R^2+z^2} - |z|\,\right)$$ for a point along the z-axis

For $z > 0, \quad |z| = z$

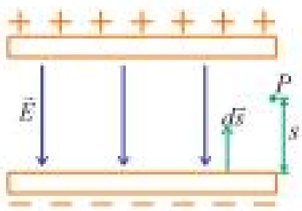
$$\therefore \quad E_z = -\frac{\partial V}{\partial z} = \frac{\sigma}{2\epsilon_0}\Big[1 - \frac{z}{\sqrt{R^2+z^2}}\Big]$$ (Compare with Chap.2 notes)

Example: Uniform electric field

(e.g. Uniformly charged $+ve$ and $-ve$ plates)

Consider a path going from the $-ve$ plate to the $+ve$ plate

Potential at point P, V_P can be deduced from definition.



$$\begin{aligned} \text{i.e.} \qquad V_P - V_- &= -\int_0^s \vec{E}\cdot d\vec{s} \\ &= -\int_0^s (-E\,ds) \\ &= E\int_0^s ds = Es \end{aligned}$$

(V_- = Potential of $-ve$ plate)

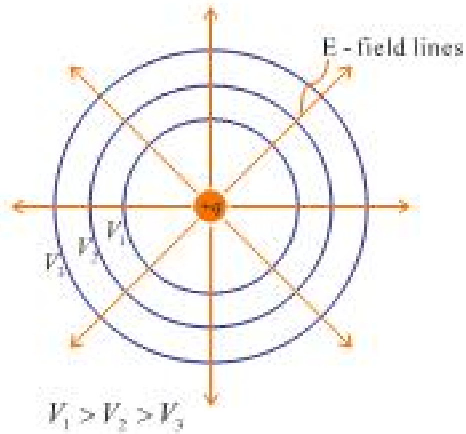
$\because$ $\vec{E}, d\vec{s}$ pointing opposite directions

Convenient reference: $\quad V_- = 0$

$$\therefore \quad \boxed{V_P = E\cdot s}$$

4.4 Equipotential Surfaces

Equipotential surface is a surface on which the *potential is constant.*
$\Rightarrow \quad (\Delta V = 0)$



$$V(r) = \frac{1}{4\pi\epsilon_0} \cdot \frac{+q}{r} = const$$

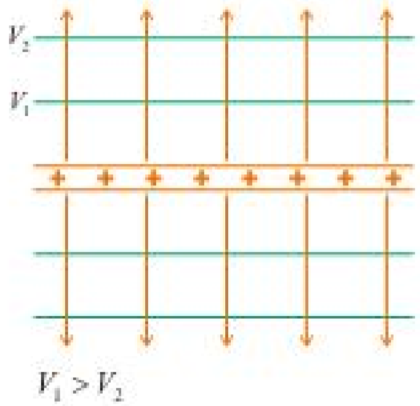
$\Rightarrow \quad r = const$

$\Rightarrow$ Equipotential surfaces are *circles/spherical surfaces*

Note: (1) A charge can move freely on an equipotential surface without any work done.

(2) The **electric field lines** must be *perpendicular* to the **equipotential surfaces.** (`Why?`)
On an equipotential surface, $V = constant$
$\Rightarrow \ \Delta V = 0 \ \Rightarrow \ \vec{E} \cdot d\vec{l} = 0$, where $d\vec{l}$ is *tangent* to equipotential surface
$\therefore \quad \vec{E}$ must be *perpendicular* to equipotential surfaces.

Example: Uniformly charged surface (infinite)



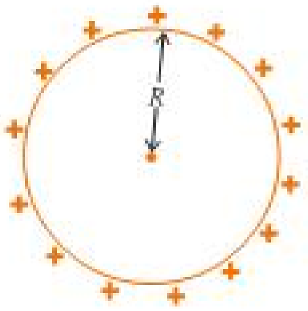
Recall $\quad V = V_0 - \dfrac{\sigma}{2\epsilon_0}|z|$

$\uparrow$
Potential at $z = 0$

Equipotential surface means

$$V = const \ \Rightarrow \ V_0 - \frac{\sigma}{2\epsilon_0}|z| = C$$
$$\Rightarrow \ |z| = constant$$

Example: Isolated spherical charged conductors



Recall:

(1) E-field inside = 0

(2) charge distributed on the *outside* of conductors.

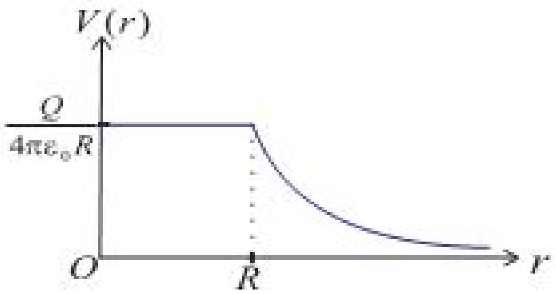
(i) Inside conductor:

$$
\begin{aligned}
E = 0 \quad &\Rightarrow \quad \Delta V = 0 \text{ everywhere in conductor} \\
&\Rightarrow \quad V = \textit{constant} \text{ everywhere in conductor} \\
&\Rightarrow \quad \text{The entire conductor is at the same potential}
\end{aligned}
$$

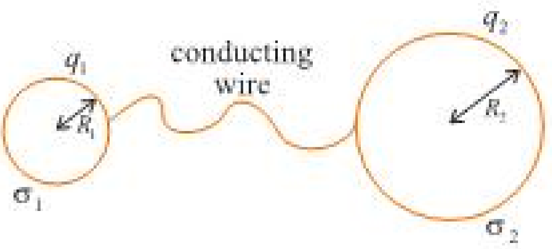
(ii) Outside conductor:

$$V = \frac{Q}{4\pi\epsilon_0 r}$$

$\because$ Spherically symmetric (Just like a point charge.)
<u>BUT</u> not true for conductors of arbitrary shape.



Example: Connected conducting spheres



Two conductors connected can be seen as a *single conductor*

$\therefore$ Potential everywhere is identical.

Potential of radius R_1 sphere $\quad V_1 = \dfrac{q_1}{4\pi\epsilon_0 R_1}$

Potential of radius R_2 sphere $\quad V_2 = \dfrac{q_2}{4\pi\epsilon_0 R_2}$

$$\boldsymbol{V_1 = V_2}$$

$$\Rightarrow \quad \frac{q_1}{R_1} = \frac{q_2}{R_2} \quad \Rightarrow \quad \frac{q_1}{q_2} = \frac{R_1}{R_2}$$

Surface charge density

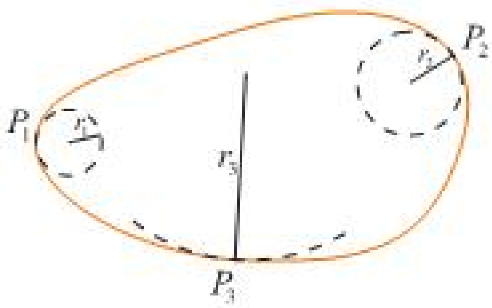
$$\sigma_1 = \underbrace{\frac{q_1}{4\pi R_1^2}}_{\text{Surface area of radius } R_1 \text{ sphere}}$$

$$\therefore \quad \frac{\sigma_1}{\sigma_2} = \frac{q_1}{q_2} \cdot \frac{R_2^2}{R_1^2} = \frac{R_2}{R_1}$$

$$\therefore \quad \text{If } R_1 < R_2, \text{ then } \sigma_1 > \sigma_2$$

And the surface electric field $E_1 > E_2$

For arbitrary shape conductor:



$$E_3 < E_2 < E_1$$

At every point on the conductor, we fit a *circle*. The radius of this circle is the *radius of curvature*.

Note: Charge distribution on a conductor does **not** have to be uniform.