Chapter 6-Dimensionless Analysis

Dimensional analysis is a mathematical technique used to predict physical parameters that influence the flow in fluid mechanics, heat transfer in thermodynamics, and so forth. The analysis involves the fundamental units of dimensions M\(\cdot\)L\(\cdot\)T\(\cdot\)mass\(\cdot\)length\(\cdot\)time. It is helpful in experimental work because it provides a guide to factors that significantly affect the studied phenomena.

Dimensionless analysis is commonly used to determine the relationships between several variables, i.e., to find the force as a function of other variables when an exact functional relationship is unknown. Based on understanding of the problem, we assume a certain functional form.

Before beginning, it is important to define some dimensionless parameters:

1- **Reynolds number**: the ratio of inertia force to viscous force. It is important in all types of fluid dynamics problems.

\[ Re = \frac{\rho Vl}{\mu} \]

2- **Mach number**: the ratio of inertia force to compressibility force. It is important in flows in which the compressibility of the fluid is important.

\[ M = \frac{V}{C} \]

Here \(C\) is the speed of sound \(C = \sqrt{\gamma RT}\) for gases and \(C = \sqrt{\frac{E}{\rho}}\) for liquids. Here \(\gamma\) is the bulk modulus of compression and \(\gamma\) is the specific heat ratio.
3- **Froude number** is the ratio of inertia force to gravitational force. It is important in flows with a free surface.

\[ Fr = \frac{V}{\sqrt{gl}} \]

4- **Weber number** is the ratio of inertia force to surface tension force. It is important in problems in which surface tension is important.

\[ We = \frac{\rho V l}{\sigma} \]

**Notes:**

\( l \) is a characteristic length for the system.

5- **Euler number** is the ratio of the pressure force to inertia force. It is important in problems in which pressure or pressure differences are of interest.

\[ Eu = \frac{P}{\rho V^2} \]

**Units/Dimensions**

The defined units are based on the modern MLT system: mass, length, time. All other quantities can be expressed in terms of these basic units.

For example:

<table>
<thead>
<tr>
<th>Quantity</th>
<th>Unit</th>
<th>Dimension</th>
</tr>
</thead>
<tbody>
<tr>
<td>velocity</td>
<td>m/s</td>
<td>LT</td>
</tr>
<tr>
<td>acceleration</td>
<td>m/s²</td>
<td>LT²</td>
</tr>
<tr>
<td>force</td>
<td>kN/m²</td>
<td>MT</td>
</tr>
</tbody>
</table>
Another system for dimensionless analysis is the \( ML^2T \) system, which makes the units of mass as \( FT^2/L \), since acceleration has units of \( LT \).

**Rayleigh Method**

An elementary method for finding a functional relationship with respect to a parameter in interest is the Rayleigh Method, and will be illustrated with an example using the \( MLT \) system.

Assume that we are interested in the drag, \( F_D \), which is a force on a ship. What exactly is the drag a function of? These variables need to be chosen correctly, though selection of such variables depends largely on one's experience in the topic. It is known that drag depends on

<table>
<thead>
<tr>
<th>Quantity</th>
<th>Symbol</th>
<th>Dimension</th>
</tr>
</thead>
<tbody>
<tr>
<td>Size</td>
<td>( l )</td>
<td>( L )</td>
</tr>
<tr>
<td>Viscosity</td>
<td>( \mu )</td>
<td>( mL/T )</td>
</tr>
<tr>
<td>Density</td>
<td>( \rho )</td>
<td>( mL^3 )</td>
</tr>
<tr>
<td>Velocity</td>
<td>( V )</td>
<td>( L/T )</td>
</tr>
<tr>
<td>Gravity</td>
<td>( g )</td>
<td>( L/T^2 )</td>
</tr>
</tbody>
</table>

This means that \( F_D = f(l, \rho, \mu, V, g) \). Here \( f \) is some function. With the Rayleigh Method, we assume that \( F_D = Cl^a \rho^b \mu^c V^d g^e \), here \( C \) is a dimensionless constant and \( a, b, c, d, \) and \( e \) are exponents whose values are not yet known. Note that the dimensions of the left side force must equal those on the right side. Here we use only the three independent dimensions for the variables on the right side: \( M, L \), and \( T \).

**Step 1: Setting up the equation**

Write the equation in terms of dimensions only, i.e. replace the quantities with their respective units. The equation then becomes:

\( F_D = Cl^a \rho^b \mu^c V^d g^e \mathbb{L}^a \mathbb{M}^b \mathbb{L}^c \mathbb{T}^d \mathbb{L}^{-d} \mathbb{T}^e \)

\( F_D = Cl^a \rho^b \mu^c V^d g^e \mathbb{L}^a \mathbb{M}^b \mathbb{T}^d \mathbb{T}^e \)
\[
\frac{ML}{T^2} = (L)^a \left( \frac{M}{L^3} \right)^b \left( \frac{M}{LT} \right)^c \left( \frac{L}{T} \right)^d \left( \frac{L}{T^2} \right)^e
\]

In the left side we have \( M'L'T^2 \) which is equal to the dimensions on the right side. Therefore the exponents of the right side must be such that the units are \( M'L'T^2 \).

**Step 2: Solving for the exponents**

Equate the exponents to each other in terms of their respective fundamental units:

\[
\begin{align*}
\text{M: } 1 & = b + c \\
\text{L: } 1 & = a - 3b - c + d + e \\
\text{T: } -2 & = -c - d - 2e
\end{align*}
\]

It is seen that there are three equations but five unknown variables. This means that a complete solution cannot be obtained. Thus we choose to solve \( a, b, \) and \( d \) in terms of \( c \) and \( e \). These choices are based on experience. Therefore

\[
\begin{align*}
\text{from M: } b & = 1 - c \quad (i) \\
\text{from T: } d & = 2 - c - 2e \quad (ii) \\
\text{from L: } a & = 1 + 3b + c - d - e \quad (iii)
\end{align*}
\]

Solving (i), (ii), and (iii) simultaneously, we obtain

\[
a = 2 - c + e
\]

Substituting the exponents back into the original equation, we obtain

\[
FD = \frac{C l^2 e}{\rho^{1-c} \mu^c V^{2-c-2e}}
\]

Collecting like exponents together,

\[
\begin{align*}
\text{Terms with exponent of 1: } C \\
\text{Terms with exponent of 2: } \frac{1}{l g} \\
\text{Terms with exponent of } e: \frac{l}{V^2} = \frac{1}{l g} e = e
\end{align*}
\]

Which means

\[
\begin{align*}
\text{Terms with exponent of 1: } C p \\
\text{Terms with exponent of 2: } \frac{l}{V} \\
\text{Terms with exponent of } e: \frac{1}{l g} e = e
\end{align*}
\]

The right sides of (iv) and (v) are known as the dimensionless groups.
Step 3: Determining the dimensionless groups

Note that $e$ and $c$ are unknown. Consider the following cases:

If $e = 1$ then (iv) becomes

$$\frac{\ell}{V^2}$$

If $e = -1$ then (iv) becomes

$$\frac{V^2}{l\ell}$$

If $c = 1$ then (v) becomes

$$\frac{\mu}{l \rho V}$$

If $c = -1$ then (v) becomes

$$\frac{\mu}{l \rho V} = \frac{V}{\nu}$$

Where $\nu$ is the kinematic viscosity of the fluid. And so on for different exponents. It turns out that

- $e$-nolds number $Re = \frac{V\ell}{\nu}$
- Vroude number $Fr = \left(\frac{V^2}{l\ell}\right)^{\frac{1}{2}} = \frac{V}{\sqrt{l\ell}}$

Here $Re$ and $Fr$ are the usual notations for the $e$-nolds and $Vroude$ numbers respectively. Such dimensionless groups keep reoccurring throughout fluid mechanics and other fields.

Choosing exponents of $-1$ for $c$ and $-\frac{1}{2}$ for $e$ which result in the $e$-nolds and $Vroude$ numbers respectively, we obtain

$$F_D = g(Fr, Re)\rho l^2 V^2$$

Here $F_D$ is a dimensionless function.

This can also be written as $\frac{F_D}{\rho l^2 V^2} = g(Fr, Re)$

Which is a dimensionless quantity and a function of only two variables instead of five. This dimensionless quantity turns out to be the drag coefficient $C_D$.

$$C_D = \frac{F_D}{\rho l^2 V^2}$$

Notes

The Rayleigh Method has limitations because of the premise that an exponential relationship exists between the variables.

The Buckingham $\pi$ Theorem/Method

This method will be illustrated the same example as that for the Rayleigh Method, the drag on a ship. Say that we have $n$ number of quantities which are $D$, $l$, $\rho$, $\mu$, $V$, and $g$ and $m$ number of dimensions which are $M$, $L$, and $T$. These quantities can be reduced to $(n - m)$ independent dimensionless groups such as $Re$ and $Fr$. Say that

$A1 = f(A2, A3, A4, ... , An)$
Here \( A_x \) are quantities such as drag, length, and so forth as mentioned under the \( n \) number of quantities and \( f \) implies the functional relationship between \( A_I \) and the other quantities.

Then rearranging we obtain

\[
0 = f(A_2, A_3, A_4, \ldots , A_n) - A_1
\]

which can be further reduced using the Buckingham \( \pi \) Theorem, to obtain

\[
0 = f(\pi_1, \pi_2, \ldots , \pi_{n-m})
\]

Forming \( \pi \) Groups

For each \( \pi \) group, take \( m \) of the quantities \( A_x \) known as \( m \) repeating variables and one of the other remaining variables. Note that experience dictates which quantities make the best repeating variables. The \( \pi \) groups, in general form, could then be

\[
\pi_1 = A_1^{x_1} A_2^{y_1} A_3^{z_1} A_4
\]

\[
\pi_2 = A_1^{x_2} A_2^{y_2} A_3^{z_2} A_5
\]

\[
\vdots
\]

\[
\pi_{n-m} = A_1^{x_{n-m}} A_2^{y_{n-m}} A_3^{z_{n-m}} A_n
\]

which are all dimensionless quantities.

Step 1: Setup \( \pi \) groups

For the MLT system \( m = 3 \) so choose \( A_1, A_2, A_3 \) as the repeating variables. Using the Buckingham \( \pi \) Theorem on the Drag Equation:

\[
f(F_D, l, \rho, \mu, V, g) = 0
\]

Here \( m = 3 \) \( n = 6 \) so there will be \( n - m = 3 \) \( \pi \) groups. We will select \( \rho, V \) and \( l \) as the repeating variables leaving the remaining quantities as \( F_D, \mu \) and \( g \). Note that if the analysis does not work out we could always go back and repeat using new \( \pi \)s. Thus

\[
\pi_1 = \rho^{x_1} V^{y_1} l^{z_1} F_D
\]

\[
\pi_2 = \rho^{x_2} V^{y_2} \mu
\]

\[
\pi_3 = \rho^{x_3} V^{y_3} l^{z_3} g
\]

which are all dimensionless quantities i.e. having units of \( M^0 L^0 T^0 \)
Step 2: Determine π groups

For the first π group,

\[ \pi_1 \quad M^0L^0T^0 = \left( \frac{M}{L^3} \right)^{x_1} \left( \frac{L}{T} \right)^{y_1} (L)^{z_1} \left( \frac{ML}{T^2} \right) \]

Expanding and collecting like units, we can solve for the exponents:

For \( M \):

\[ 0 = x_1 + 1 \quad \Rightarrow \quad x_1 = -1 \]

For \( T \):

\[ 0 = -y_1 - 2 \quad \Rightarrow \quad y_1 = -2 \]

For \( L \):

\[ 0 = -3x_1 + y_1 + z_1 + 1 \quad \Rightarrow \quad z_1 = 3(-1) - 2 - 1 = -2 \]

Therefore, we find that the exponents \( x_1, y_1, \) and \( z_1 \) are \(-1, -2, \) and \(-2\) respectively.

This means that the first dimensionless π group, \( \pi_1 \), is

\[ \pi_1 = \frac{F_D}{\rho V^2 l^2} \]

For the second π group,

\[ \pi_2 \quad M^0L^0T^0 = \left( \frac{M}{L^3} \right)^{x_2} \left( \frac{L}{T} \right)^{y_2} (L)^{z_2} \left( \frac{ML}{T^2} \right) \]

Expanding for the exponents:

For \( M \):

\[ x_2 + 1 = 0 \quad \Rightarrow \quad x_2 = -1 \]

For \( T \):

\[ -y_2 - 1 = 0 \quad \Rightarrow \quad y_2 = -1 \]

For \( L \):

\[ -3x_2 + y_2 + z_2 - 1 = 0 \quad \Rightarrow \quad z_2 = 1 - (-1) + 3(-1) = -1 \]

Thus,

\[ \pi_2 = \rho^{-1} V^{-1} l^{-1} \mu = \frac{\mu}{\rho V l} = \frac{\nu}{V l} \]

However, we will now invert \( \pi_2 \) so that \( \pi_2 = \frac{V l}{\nu} = \text{Reynolds Number} \)

For the third π group,

\[ \pi_3 \quad M^0L^0T^0 = \left( \frac{M}{L^3} \right)^{x_3} \left( \frac{L}{T} \right)^{y_3} (L)^{z_3} \left( \frac{L}{T^2} \right) \]

Expanding for the exponents:
For M: $x_3 = 0$ 
$\Rightarrow x_3 = 0$

For T: $-y_3 - 2 = 0$ 
$\Rightarrow y_3 = -2$

For L: $-3x_3 + y_3 + z_3 + 1 = 0$  
$\Rightarrow z_3 = -1 - (-2) = 1$

Thus, the three $\pi$ groups can be written together as:

$$f\left(\frac{F_D}{\rho V^2 l^2}, Re, Fr\right) = 0$$

$$\frac{F_D}{\rho V^2 l^2} = f(Re, Fr)$$

Thus, raising it to the power of $-\frac{1}{2}$, we get

$$\sqrt[\frac{1}{\pi_3}] = \frac{V}{\sqrt{l g}} = \text{Froude Number}$$

Thus, the three $\pi$ groups can be written together as:

$$f\left(\frac{F_D}{\rho V^2 l^2}, Re, Fr\right) = 0$$

Note that this is the same result as obtained with the Rayleigh Method, but with the Buckingham $\pi$ Method, we did not have to assume a functional dependence.

Notes:

1. The repeating variables must represent, as possible, geometry, fluid properties, and external effects.
2. It is permissible to exponentiate any $\pi$ group, e.g. $\pi^0$, $\pi^1$, $\pi^{-2}$, etc., to form a new group, as this does not alter the functional form.
3. If the problem contains dimensionless variables, this variable can be directly considered as $\pi$ parameter.
4. If the analysis does not work out, we could always go back and repeat using new repeating variables.

Thus, $R$
This chapter discusses the analysis of fluid in motion – fluid dynamics. It is useful to introduce some definitions about fluid motion.

**Mass flow rate (\( \dot{m} \))** is the mass per time taken to accumulate this mass: \( \dot{m} = \frac{dm}{dt} \)

**Volume flow rate - Discharge (Q)**: \( Q = \frac{\text{Volume of fluid}}{\text{time}} \)

**Pathline**: A pathline follows the movement of a single fluid particle. It can be produced in the laboratory by marking a fluid particle and taking a time exposure photograph of its motion.

**Streakline**: consists of all particles in a flow that have previously passed through a common point. Streaklines are more of a laboratory tool than an analytical tool. They can be obtained by taking instantaneous photographs of marked particles that all passed through a given location in the flow field at some earlier time. Such a line can be produced by continuously injecting marked fluid (neutral buoyant smoke in air or dye in water) at a given location.

**Streamline**: A streamline is a line that is everywhere tangent to the velocity field. If the flow is steady, nothing at a fixed point, including the velocity direction, changes with time, so the streamlines are fixed lines in space. For unsteady flows, the streamlines may change shape with time. Streamlines are obtained analytically by integrating the equations defining lines tangent to the velocity field. For two-dimensional flows, the slope of the streamline must be equal to the tangent of the angle that the velocity vector makes with the x-axis.

**Uniform flow**: If the flow velocity is the same magnitude and direction at every point in the fluid, it is said to be uniform.

**Non-uniform**: If at a given instant the velocity is not the same at every point, the flow is non-uniform. In practice, this definition means that flows near a solid boundary will be non-uniform, as the fluid at the boundary must take the speed of the boundary, usually zero. However, if the size and shape of the of the cross-section of the stream of fluid is constant the flow is considered uniform.