Chapter 6-Dimensionless Analysis

Dimensional analysis is a mathematical technique used to predict physical parameters that influence the flow in fluid mechanics, heat transfer in thermodynamics, and so forth. The analysis involves the fundamental units of dimensions MLT (mass, length and time). It is helpful in experimental work because it provides a guide to factors that significantly affect the studied phenomena.

Dimensionless analysis is commonly used to determine the relationships between several variables, i.e. to find the force as a function of other variables when an exact functional relationship is unknown. Based on understanding of the problem, we assume a certain functional form.

Before beginning, it is important to define some dimensionless parameters:

1- Reynolds number: the ratio of inertia force to viscous force. It is important in all types of fluid dynamics problems

$$Re = \frac{\rho V l}{\mu}$$

2- Mach number: the ratio of inertia force to compressibility force. It is important in flows in which the compressibility of the fluid is important.

$$M = \frac{V}{C}$$

Where C is the speed of sound, $C = \sqrt{\gamma R T}$ for gases, and $C = \sqrt{\frac{E}{\rho}}$ for liquids, where E is the bulk modulus of compression, γ is the specific heat ratio.

3- Froude number: the ratio of inertia force to gravitational force. It is important in flow with a free surface.

$$Fr = \frac{V}{\sqrt{gl}}$$

4- Weber number: the ratio of inertia force to surface tension force. It is important in problems in which surface tension is important

$$We = \frac{\rho V^2 l}{\sigma}$$

Notes:

l: is a characteristic length for the system.

5- Euler number: the ratio of the pressure force to inertia force. It is important in problems in which pressure (or pressure differences) are of interest.

$$Eu = \frac{P}{\rho V^2}$$

Units/Dimensions

The defined units are based on the modern MLT system (mass, length, time). All other quantities can be express in terms of these basic units.

For example:

velocity	m/s	L/T
acceleration	m/s²	L/T²
force	kg.m/s²	ML/T²

□ here □□T□□□T□□M□□T□□etc. are referred to as the derived units. □nother s□stem for dimensionless anal□sis is the □□T s□stem□the force□len□th□time s□stem. □n this case□ mass ≡F/a, which makes the units of mass as FT^2/L, since acceleration has units of □□T□.

Rayleigh Method

□n elementar□method for findin□a functional relationship □ith respect to a parameter in interest is the □a□lei□h Method□and □ill □e illustrated □ith an e□ample□usin□the M□T s□stem.

□ssume that □e are interested in the dra□□□□□□hich is a force□on a ship. □ hat e□actl□ is the dra□a function of□These varia□les need to □e chosen correctl□□thou□h selection of such varia□les depends lar□el□on one□s e□perience in the topic. □t is kno□n that dra□ depends on

Quantity	Symbol	Dimension
□i□e	l	□
□iscosit□	μ	m□□T
□ensit□	ρ	m□□^{3}
□elocit□	□	□□T
□ravit□	□	□□T□

This means that $F_D = f(l,\rho,\mu,V,g)$ □here *f* is some function. □ ith the □a□lei□h Method□ □e assume that $F_D=Cl^a\rho^b\mu^cV^dg^e$□□here *C* is a dimensionless constant□and *a*□*b*□*c*□*d*□and *e* are e□ponents□□hose values are not □et kno□n. □ote that the dimensions of the left side□ force□must e□ual those on the ri□ht side. □ere□□e use onl□ the threeindependent dimensions for the varia□les on the ri□ht side□M□□□and T.

Step 1: Setting up the equation

□ rite the e□uation in terms of dimensions onl□□i.e. replace the □uantities □ith their respective units. The e□uation then □ecomes□

$$\frac{ML}{T^2} = (L)^a \left(\frac{M}{L^3}\right)^b \left(\frac{M}{LT}\right)^c \left(\frac{L}{T}\right)^d \left(\frac{L}{T^2}\right)^e$$

On the left side, we have $M^1L^1T^{-2}$, which is equal to the dimensions on the right side. Therefore, the exponents of the right side must be such that the units are $M^1L^1T^{-2}$

Step 2: Solving for the exponents

Equate the exponents to each other in terms of their respective fundamental units:

M: $1 = b + c$ since $M = M^b M^c$
L: $1 = a - 3b - c + d + e$ since $L = L^a L^{-3b} L^{-c} L^d L^e$
T: $-2 = -c - d - 2e$ since $T^{-2} = T^{-c} T^{-d} T^{-2e}$

It is seen that there are three equations but 5 unknown variables. This means that a complete solution cannot be obtained. Thus, we choose to solve *a*, *b*, and *d* in terms of *c* and *e*. These choices are based on experience. Therefore:

From M: $b = 1 - c$ (i)
From T: $d = 2 - c - 2e$ (ii)
From L: $a = 1 + 3b + c - d - e$ (iii)

Solving (i), (ii) and (iii) simultaneously, we obtain $a = 2 - c + e$
Substituting the exponents back into the original equation, we obtain
$F_D = C l^{2-c+e} \rho^{1-c} \mu^c V^{2-c-2e} g^e$
Collecting like exponents together: $F_D = C\left(\frac{V^2}{lg}\right)^{-e}\left(\frac{Vl\rho}{\mu}\right)^{-c}\rho l^2 V^2$

Which means
$F_D = C l^2 l^e l^{-c} \rho \rho^{-c} \mu^c V^2 V^{-c} V^{-2e} g^e$
For the different exponents:

Terms with exponent of 1: $C\rho$
Terms with exponent of 2: $l^2 V^2$
Terms with exponent of e: $l^e V^{-2e} g^e = \left(\frac{lg}{V^2}\right)^e = \left(\frac{V^2}{lg}\right)^{-e}$ (iv)

Terms with exponent of c: $l^{-c} \rho^{-c} \mu^c V^{-c} = \left(\frac{l\rho V}{\mu}\right)^{-c}$ (v)

The right sides of (iv) and (v) are known as the dimensionless groups.

Step 3: Determining the dimensionless groups

Note that *e* and *c* are unknown. Consider the following cases:

If e = 1 then (iv) becomes $\left(\frac{lg}{V^2}\right)$

If e = -1 then (iv) becomes $\left(\frac{V^2}{lg}\right)$

If c = 1 then (v) becomes $\left(\frac{\mu}{l\rho V}\right)$

If c = -1 then (v) becomes $\left(\frac{l\rho V}{\mu}\right) = \left(\frac{lV}{\nu}\right)$

Where *ν* is the kinematic viscosity of the fluid. And so on for different exponents. It turns out that:

Reynolds number: $Re = \frac{Vl}{\nu}$

Froude number: $Fr = \left(\frac{V^2}{lg}\right)^{\frac{1}{2}} = \frac{V}{\sqrt{lg}}$

Where *Re* and *Fr* are the usual notations for the Reynolds and Froude Numbers respectively. Such dimensionless groups keep reoccurring throughout Fluid Mechanics and other fields.

Choosing exponents of *-1* for *c* and -½ for *e*, which result in the Reynolds and Froude Numbers respectively, we obtain

$F_D = g(Fr, Re)\rho l^2 V^2$

Where g(Fr, Re) is a dimensionless function.

This can also be written as $\frac{F_D}{\rho l^2 V^2} = g(Fr, Re)$

Which is a dimensionless quantity, and a function of only 2 variables instead of 5. This dimensionless quantity turns out to be the drag coefficient, C_D.

$$C_D \equiv \frac{F_D}{\rho l^2 V^2}$$

Notes

The Rayleigh Method has limitations because of the premise that an exponential relationship exists between the variables.

The Buckingham π Theorem/Method

This method will be illustrated by the same example as that for Rayleigh Method: the drag on a ship. Say that we have *n* number of quantities (e.g. 6 quantities), which are D, l, ρ, μ, V, and g, and *m* number of dimensions (e.g. 3 dimensions), which are M, L, and T. These quantities can be reduced to *(n - m)* independent dimensionless groups, such as *Re* and *Fr*. Say that:

A1 = f(A2, A3, A4, ... , An)

where A_x are quantities such as drag, length, and so forth, as mentioned under the n number of quantities, and f implies the functional relationship between A_1 and the other quantities.

Then rearranging we obtain:

$$0 = f(A2, A3, A4, ... , An) - A1$$
$$0=f(A1,A2,A3,…,An)$$

Which can be further reduced using the Buckingham π Theorem, to obtain:

$$0 = f(\pi 1, \pi 2, ... , \pi n-m)$$

Forming π Groups

For each π group, take m of the quantities A_x known as m repeating variables, and one of the other remaining variables. Note that experience dictates which quantities make the best repeating variables. The π groups, in general form, would then be

$$\pi_1 = A_1^{x_1} A_2^{y_1} A_3^{z_1} A_4$$

$$\pi_2 = A_1^{x_2} A_2^{y_2} A_3^{z_2} A_5$$

.
.
.
.

$$\pi_{n-m} = A_1^{x_{n-m}} A_2^{y_{n-m}} A_3^{z_{n-m}} A_n$$

which are all dimensionless quantities.

Step 1: Setup π groups

For the MLT system, $m = 3$, so choose $A1$, $A2$, and $A3$ as the repeating variables. Using the Buckingham π Theorem on the Drag Equation:

$$f(F_D, l, \rho, \mu, V, g) = 0$$

Where $m = 3$, $n = 6$, so there will be $n - m = 3$ π groups. We will select ρ, V, and l as the repeating variables (RVs), leaving the remaining quantities as F_D, μ, and g. Note that if the analysis does not work out we could always go back and repeat using new RVs. Thus,

$$\pi_1 = \rho^{x1} V^{y1} l^{z1} F_D$$
$$\pi_2 = \rho^{x2} V^{y2} l^{z2} \mu$$
$$\pi_3 = \rho^{x3} V^{y3} l^{z3} g$$

Which are all dimensionless quantities, i.e. having units of $M^0 L^0 T^0$

Step 2: Determine π groups

For the first π group,

$$\pi_1 \quad M^0L^0T^0 = \left(\frac{M}{L^3}\right)^{x_1}\left(\frac{L}{T}\right)^{y_1}(L)^{z_1}\left(\frac{ML}{T^2}\right)$$

Expanding and collecting like units, we can solve for the exponents:

For M: $0 = x_1 + 1 \Rightarrow x_1 = -1$

For T: $0 = -y_1 - 2 \Rightarrow y_1 = -2$

For L: $0 = -3x_1 + y_1 + z_1 + 1 \Rightarrow z_1 = 3(-1) - (-2) - 1 = -2$

Therefore, we find that the exponents x_1, y_1, and z_1 are -1, -2, and -2 respectively. This means that the first dimensionless π group, π_1, is

$$\pi_1 = \rho^{-1}V^{-2}l^{-2}F_D = \frac{F_D}{\rho V^2 l^2}$$

For the second π group,

$$\pi_2 \quad M^0L^0T^0 = \left(\frac{M}{L^3}\right)^{x_2}\left(\frac{L}{T}\right)^{y_2}(L)^{z_2}\left(\frac{M}{LT}\right)$$

Solving for the exponents:

For M: $x_2 + 1 = 0 \Rightarrow x_2 = -1$

For T: $-y_2 - 1 = 0 \Rightarrow y_2 = -1$

For L: $-3x_2 + y_2 + z_2 - 1 = 0 \Rightarrow z_2 = 1 - (-1) + 3(-1) = -1$

Thus:

$$\pi_2 = \rho^{-1}V^{-1}l^{-1}\mu = \frac{\mu}{\rho V l} = \frac{\nu}{Vl}$$

However, we will now invert π_2 so that $\pi_2 = \dfrac{Vl}{\nu} = Reynolds\ Number$

For the third π group,

$$\pi_3 \quad M^0L^0T^0 = \left(\frac{M}{L^3}\right)^{x_3}\left(\frac{L}{T}\right)^{y_3}(L)^{z_3}\left(\frac{L}{T^2}\right)$$

Solving for the exponents:

For M□□3 □0 ⇒ □3 □0

For T□□3 □□ □0 ⇒ □3 □ □□

For □□□3□3 □ □3 □ □3 □1 □0 ⇒ □3 □ □1 □□□□□ 1

Thus, $\pi_3 = \rho^0 V^{-2} l g = \frac{lg}{V^2}$

Raising it to the power of □1□□□, we get $\sqrt{\frac{1}{\pi_3}} = \frac{V}{\sqrt{lg}} = Froude\ Number$

Thus, the three π groups can be written together as :

$$f\left(\frac{F_D}{\rho V^2 l^2}, Re, Fr\right) = 0$$

Finally,

$$\frac{F_D}{\rho V^2 l^2} = f(Re, Fr)$$

Note that this is the same result as obtained with the Rayleigh Method, but with the Buckingham π Method, we did not have to assume a functional dependence.

Notes:

1. The repeating variables must represent as possible as: geometry, fluid properties, external effects.
□. It is permissible to exponentiate any π group, e.g. π^{1}, $\pi^{□}$, $\pi^{□}$ etc. to form a new group, as this does not alter the functional form.
3. If the problem contains dimensionless variable, this variable can be directly considered as π parameter.
□. If the analysis does not work out, we could always go back and repeat using new repeating variables.

Introduction to Fluid Motion

This chapter discusses the anal□sis of fluid in motion □fluid d□namics. □t is useful to introduce some definitions a□out fluid motion.

Mass flow rate (ṁ):is the mass per time taken to accumulate this mass□ $\dot{m} = \frac{dm}{dt}$

Volume flow rate-Discharge (Q), $Q = \frac{Volume\ \ of\ \ fluid}{time}$

Pathline: □ pathline follo□s the movement of a sin□le fluid particle. □t can □e produced in the la□orator□ □□ markin□ a fluid particle □d□in□ a small fluid element□and takin□ a time e□posure photo□raph of its motion.

Streakline: consists of all particles in a flo□ that have previousl□ passed throu□h a common point. □treaklines are more of a la□orator□tool than an anal□tical tool. The□can □e o□tained □□ takin□ instantaneous photo□raphs of marked particles that all passed throu□h a □iven location in the flo□ field at some earlier time. □uch a line can □e produced □□continuousl□in□ectin□marked fluid □neutrall□ □uo□ant smoke in air□or d□e in □ater□at a □iven location.

Streamline□□ *streamline* is a line that is ever□□here tan□ent to the velocit□field. □f the flo□ is stead□□nothin□at a fi□ed point □includin□the velocit□direction□chan□es □ith time□so the streamlines are fi□ed lines in space. □or unstead□flo□s the streamlines ma□ chan□e shape □ith time. □treamlines are o□tained anal□ticall□ □□ inte□ratin□ the e□uations definin□lines tan□ent to the velocit□field. □or t□o□dimensional flo□s the slope of the streamline□must □e e□ual to the tan□ent of the an□le that the velocit□vector makes □ith the *x* a□is.

Uniform flow□□f the flo□ velocit□is the same ma□nitude and direction at ever□point in the fluid it is said to □e uniform.

Non-uniform□□f at a □iven instant□the velocit□is not the same at ever□point the flo□ is non□uniform.□□n practice□□□this definition□ever□fluid that flo□s near a solid □oundar□ □ill □e non□uniform □as the fluid at the □oundar□must take the speed of the □oundar□□ usuall□ □ero. □o□ever if the si□e and shape of the of the cross□section of the stream of fluid is constant the flo□ is considered uniform.□