FUZZY SYSTEMS

- In two-valued logic, an element either belongs to a set or not.
- It is not possible to solve all problems by mapping the domain into two-valued variables.
- Most real-world problems are characterized by the ability of a representation language to process incomplete, imprecise, vague or uncertain information.
- With fuzzy logic, domains are characterized by linguistic terms, rather than by numbers.
- For example, in the phrases *"it is partly cloudy"*, or *"Stephan is very tall"*, both *partly* and *very* are linguistic terms describing the *magnitude* of the fuzzy (or linguistic) variables *cloudy* and *tall.*
- The human brain has the ability to understand these terms, and infer from them that it will most probably not rain, and that Stephan might just be a good basket ball player.
- In 1965, that LotfiZadeh produced the foundations of infinite-valued logic with his mathematics of fuzzy set theory.

Fuzzy Sets

The elements of a fuzzy set have membership degrees to that set. The degree of membership to a fuzzy set indicates the certainty (or uncertainty) that the element belongs to that set.

Suppose X is the domain, or universe of discourse, and $x \in X$is a specific element of the domain X. Then, the fuzzy set A is characterized by a *membership mapping function:*

$$\mu_A : X \to [0, 1]$$

Membership Functions

- The membership function is the essence of fuzzy sets.
- The function is used to associate a degree of membership of each of the elements of the domain to the corresponding fuzzy set.
- Two-valued sets are also characterized by a membership function.
- For example, consider the domain X of all floating-point numbers in the range [0, 100]. Define the crisp set $A \subset X$ of all floating-point numbers in the range [10, 50].

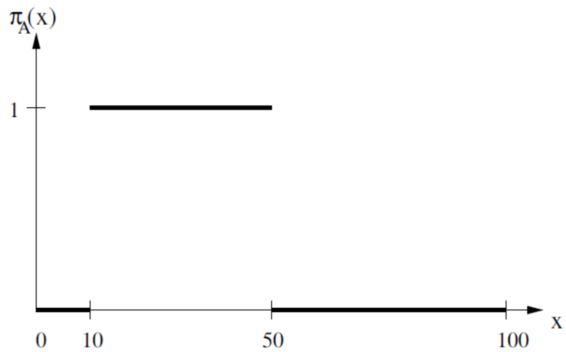


Figure 20.1 Illustration of Membership Function for Two-Valued Sets

All membership functions must satisfy the following constraints:

- ☒ A membership function must be bounded from below by 0 and from above by 1.

- The range of a membership function must therefore be [0, 1].
- For each $x \in X$, $\mu A(x)$ must be unique. That is, the same element cannot map to different degrees of membership for the same fuzzy set.

Example: for the *tall* fuzzy set, a possible membership function can be defined as:

$$tall(x) = \begin{cases} 0 & \text{if } length(x) < 1.5m \\ (length(x) - 1.5m) \times 2.0m & \text{if } 1.5m \le length(x) \le 2.0m \\ 1 & \text{if } length(x) > 2.0m \end{cases}$$

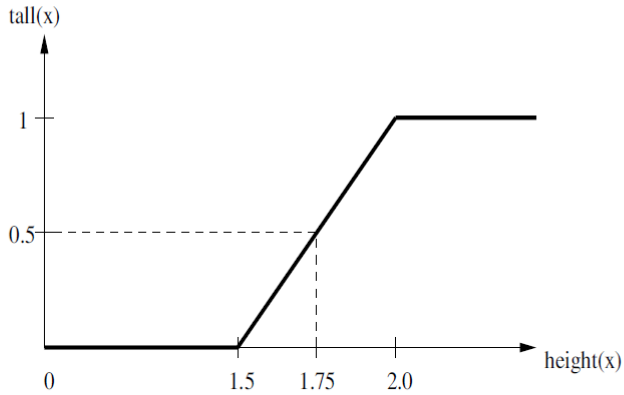


Figure 20.2 Illustration of *tall* Membership Function

In what follow, we list more examples of the popular functions.

Triangular functions, defined as:

$$\mu_A(x) = \begin{cases} 0 & \text{if } x \le \alpha_{min} \\ \frac{x-\alpha_{min}}{\beta-\alpha_{min}} & \text{if } x \in (\alpha_{min}, \beta] \\ \frac{\alpha_{max}-x}{\alpha_{max}-\beta} & \text{if } x \in (\beta, \alpha_{max}) \\ 0 & \text{if } x \ge \alpha_{max} \end{cases}$$

Trapezoidal functions, defined as:

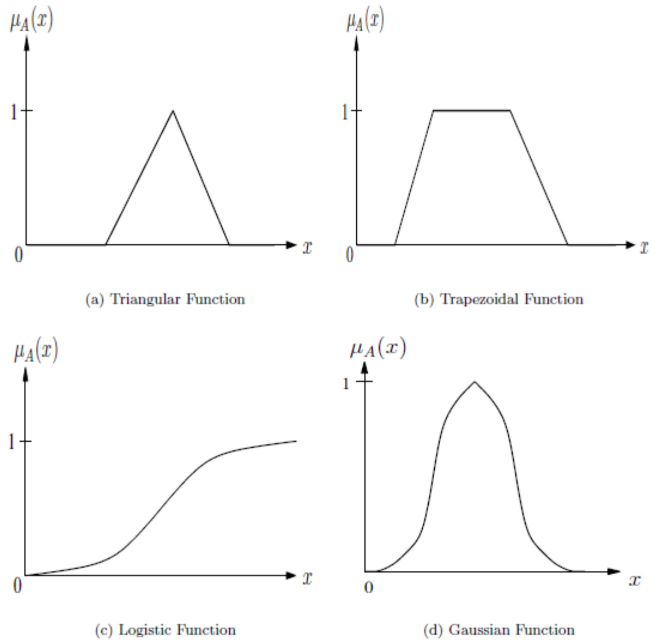
$$\mu_A(x) = \begin{cases} 0 & \text{if } x \le \alpha_{min} \\ \frac{x-\alpha_{min}}{\beta_1-\alpha_{min}} & \text{if } x \in [\alpha_{min}, \beta_1) \\ \frac{\alpha_{max}-x}{\alpha_{max}-\beta_2} & \text{if } x \in (\beta_2, \alpha_{max}) \\ 0 & \text{if } x \ge \alpha_{max} \end{cases}$$

Logistic function, defined as:

$$\mu_A(x) = \frac{1}{1 + e^{-\gamma x}}$$

Gaussian function, defined as:

$$\mu_A(x) = e^{-\gamma(x-\beta)^2}$$



Fuzzy Operators

Let X be the domain, or universe, and A and B are fuzzy sets defined over the domain X.

Equality of fuzzy sets:

Two fuzzy sets A and B are equal if and only if the sets have the same domain, and $\mu A(x) = \mu B(x)$ for all $x \in X$. That is, $A = B$.

Containment of fuzzy sets:

Fuzzy set A is a subset of fuzzy set B if and only if $\mu A(x) \leq \mu B(x)$ for all $x \in X$. That is, $A \subset B$.

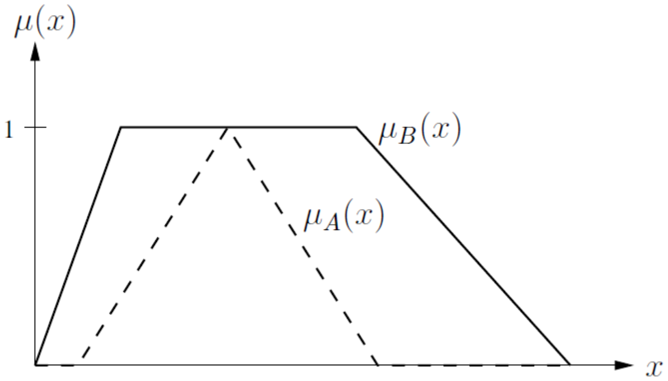
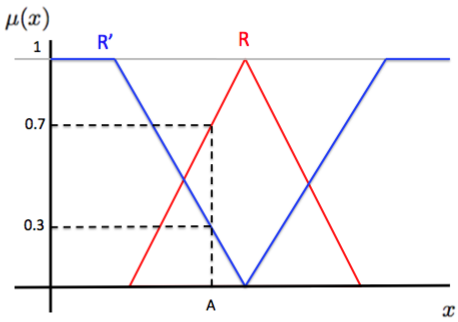


Figure 20.4 Illustration of Fuzzy Set Containment

Complement of a fuzzy set (NOT):

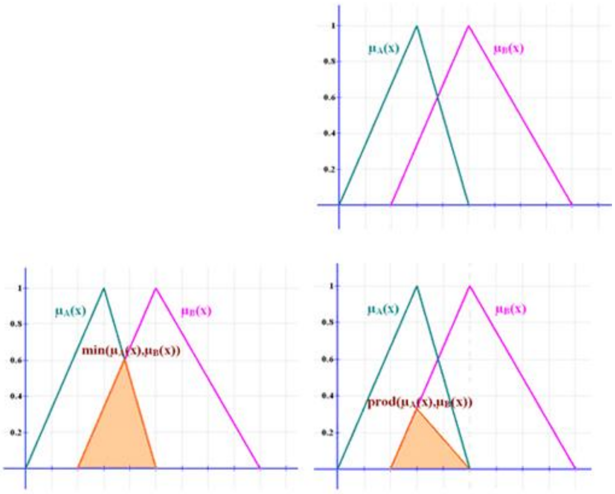
For fuzzy sets, the complement of the set A consists of all the elements of set A, but the membership degrees differ. Let A denote the complement of set A. Then, for all $x \in X$, $\mu A(x) = 1 - \mu A(x)$.



Intersection of fuzzy sets (AND): if A and B are two fuzzy sets, the intersection of A and B can be defined by:

- **Min-operator**: $\mu_{A\cap B}(x) = \min\{\mu_A(x), \mu_B(x)\},\ \forall x \in X$
- **Product operator**: $\mu_{A\cap B}(x) = \mu_A(x)\mu_B(x),\ \forall x \in X$

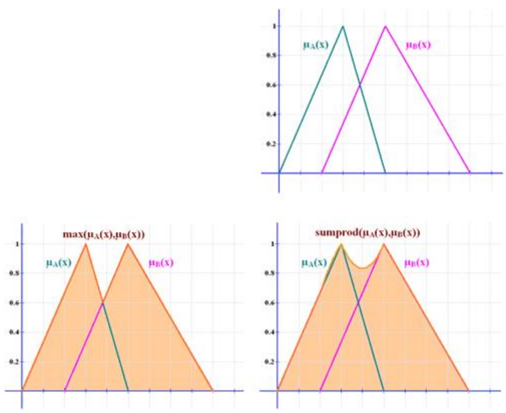
Taking the product of membership degrees is a much stronger operator than taking the minimum, resulting in lower membership degrees for the intersection.



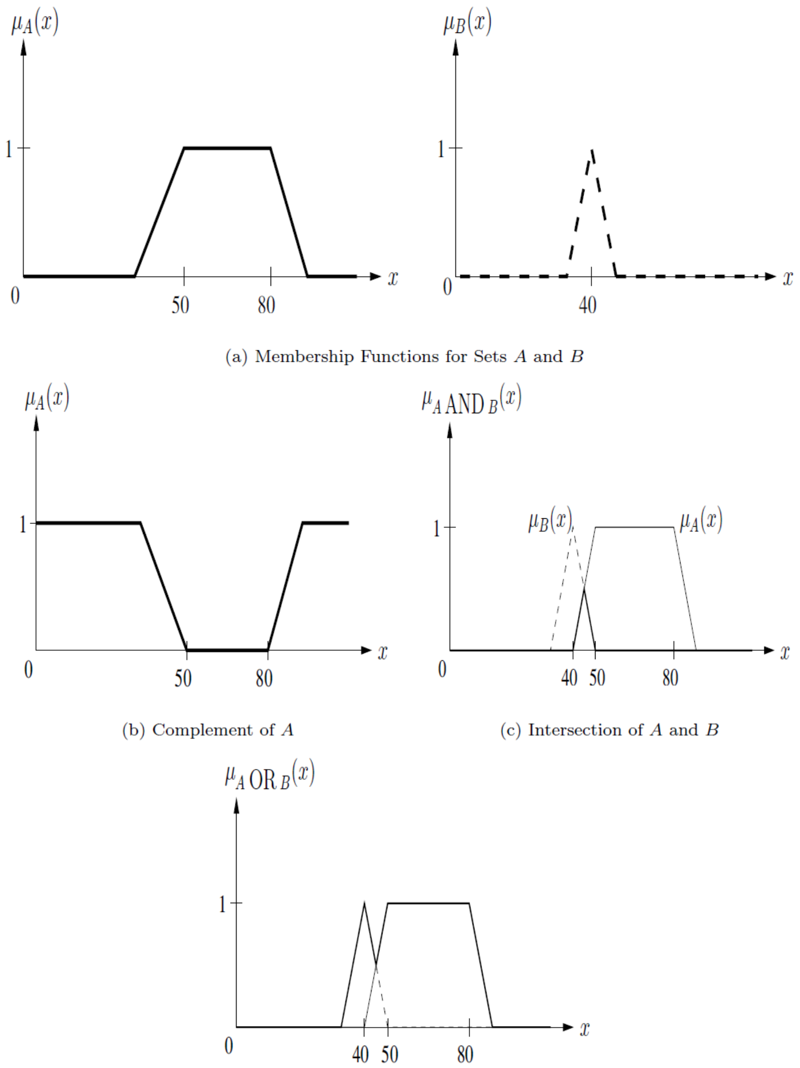
Union of fuzzy sets (OR):

Union of fuzzy sets can be implemented as the following:

- **Max-operator**: $\mu_{A\cup B}(x) = \max\{\mu_A(x), \mu_B(x)\},\ \forall x \in X$, or
- **Summation operator**: $\mu_{A\cup B}(x) = \mu_A(x) + \mu_B(x) - \mu_A(x)\mu_B(x),\ \forall x \in X$



Example:



(a) Membership Functions for Sets A and B

(b) Complement of A

(c) Intersection of A and B

(d) Union of A

Fuzzy Set Characteristics

The main characteristics of membership functions include:*normality, height, support, core, cut, unimodality,* and *cardinality*.

Normality: A fuzzy set Ais normal if that set has an element that belongs to thesetwithdegree1.Thatis,

$$\exists x \in A \bullet \mu_A(x) = 1$$

Height: The height of a fuzzy set is defined as the *supremum* of the membershipfunction, i.e.

$$height(A) = \sup_x \mu_A(x)$$

Where, sup(): is the maximum.

Support: The support of fuzzy set Ais the set of all elements in the universe ofdiscourse,X, that belongs to Awith non-zero membership. That is,

$$support(A) = \{x \in X | \mu_A(x) > 0\}$$

Core: The core of fuzzy set Ais the set of all elements in the domain that belongs toAwith membership degree 1. That is,

$$core(A) = \{x \in X | \mu_A(x) = 1\}$$

α-cut:ThesetofelementsofAwith membership degree greater thanαis referredto as theα-cut ofA:

$$A_\alpha = \{x \in X | \mu_A(x) \geq \alpha\}$$

Unimodality: A fuzzy set is *unimodal* if its membership function is a unimodalfunction, i.e. the function has just one maximum.

Cardinality:The cardinality of fuzzy setA, for a finite domain, X, is defined as

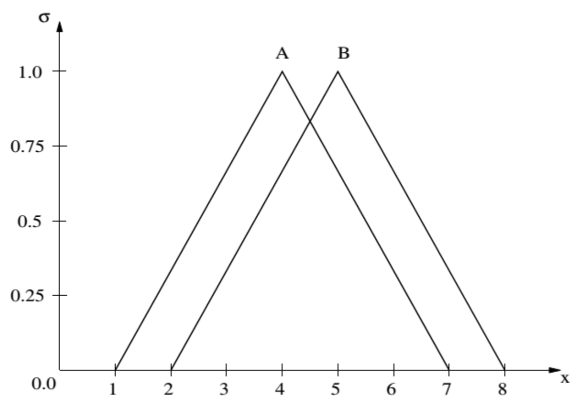
$$card(A) = \sum_{x \in X} \mu_A(x)$$

Normalization: A fuzzy set is normalized by dividing the membership function bythe height of the fuzzy set. That is,

$$normalized(A) = \frac{\mu_A(x)}{height(x)}$$

Assignment

1- Given the following figure

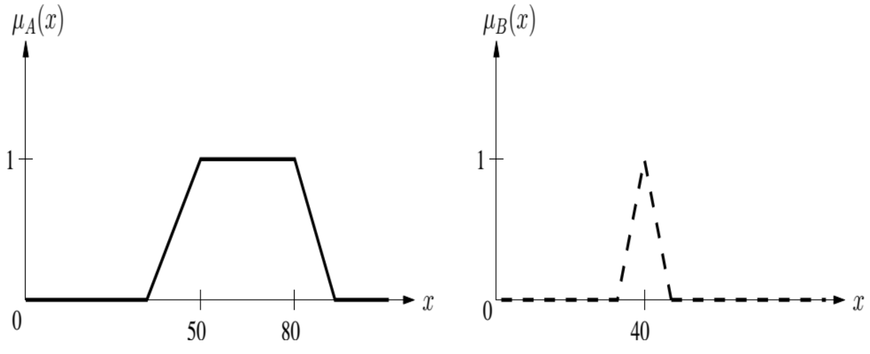


a- Draw the membership function for the fuzzy set C=A∩B,usingthemin-operator.
b- ComputeμC(5).
c- IsCnormal? Justify your answer.

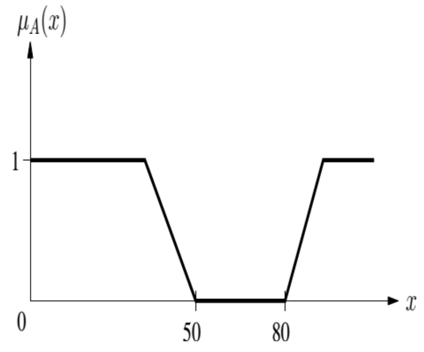
2- Consider two fuzzy subsets of the set X, X = {a, b, c, d, e } referred to as A and B
A = {1/a, 0.3/b, 0.2/c 0.8/d, 0/e} and B = {0.6/a, 0.9/b, 0.1/c, 0.3/d, 0.2/e}

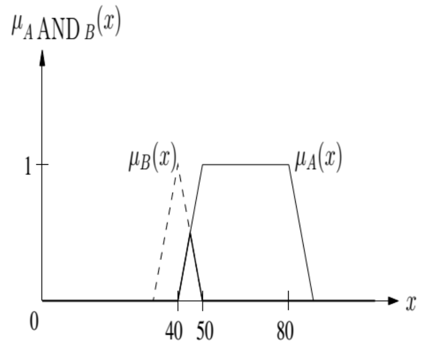
Find 1) support 2) core 3) cardinality 4) complement, 5)union, 6)intersection 7) α -cut for each set where α =0.5, and α =0.3

3- Give the height, support, core and normalization of the fuzzy sets in the following figure:



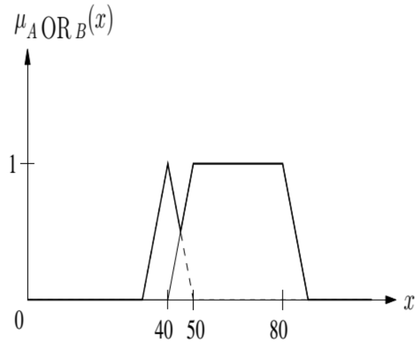
(a) Membership Functions for Sets A and B





(b) Complement of A

(c) Intersection of A and B



(d) Union of A

4- Compute the union, intersection, containment of the sets: A = {(x, 0.5), (y, 0.4), (z, 0.9), (w, 0.1)} B = {(x, 0.4), (y, 0.8), (z, 0.1), (w, 1)}