First Order Logic

(Predicate Logic)

Propositional logic assumes world contain *facts*.

First order logic (FOL) is more *expressive*. It assumes world contains:

1- *Objects*: people, houses, numbers, colors, wars, ….
2- *Predicates*: to describe *relations* between objects or describe the *properties* of objects: red, round, prime, is brother of, is bigger than, is inside, is part of, has color, after, own, …
Example Red (Car12), IsBiggerThan(4,2), Man(Ali), Man (x), 2<3, Sister(Ali, Huda)
    - Predicates return either true or false when evaluated.
3- *Functions*: father of, best friend, one more than, …
Example: Sqrt(4), Mul(x,y), father of (Ali)
    - Functions used to map some objects into an object.

Syntax of FOL

- Constant symbols: 2, kingJohn, University of Purdu, ….
- Predicate symbols: brother(x,y) , >, <, >=, <= …
- Function symbols: Sqrt, LeftLegOf, ….
- Variable symbols: x, y, a, b, …
- Connectives: ∧ ∨ ¬ → ↔
- Equality: =
- Quantifiers: ∀,∃
- Punctuation: () ,

Atomic Sentence

- Atomic sentence = predicate ($term_1$, …, $term_n$) | $term_1$=$term_2$
- *term* =function($term_1$, …, $term_n$) | *constant* | *variable*

Example: Brother (Ali, Huda)

Complex Sentences

¬S, S1 ∧ S2, S1 ∨ S2, S1 → S2, S1 ↔ S2

Example: Sibling (Ali, Huda) → sibling (Huda, Ali)

Truth in FOL

The atomic sentence predicate ($term_1$, …, $term_n$) is truth iff the objects $term_1$, …$term_n$ are in the relation predicate.

Example: Brothers (Ali, Huda) is true iff the pair (Ali, Huda) is in the relation brothers.

Universal Quantification

$\forall$ (variables) (Predicate)

$\forall x P$ is true iff P is true in a model m with x being ***each*** possible object in the model.

Example1: everyone at AI course is smart.

$\forall x$ Student (x) $\wedge$ Study(x, AI) $\rightarrow$ Smart (x)

Example2: every man is mortal.

$\forall x$ Man(x) $\rightarrow$ mortal (x)

Exampe3: $\forall x \forall y$ (Dog (x)$\wedge$ Cat (y))$\rightarrow$ Hates(x, y)

Existential Quantification

$\exists x P$ is true in a model m iff P is true with x being ***some*** possible object in the model.

$\exists$ (variables) (Predicate)

Example1: Someone at AI is smart.

$\exists x$ Student (x) $\wedge$ Study(x, AI) $\rightarrow$ Smart(x)

$\exists x$ $\neg$ At (x, AI) $\wedge$ Smart(x) (what does it mean?)

Properties of quantifiers

$\forall x \forall y$ is the same as $\forall y \forall x$

$\exists x \exists y$ is the same as $\exists y \exists x$

$\exists x \forall y$ is not the same $\forall x \exists y$

Example: $\exists x \forall y$ Loves(x ,y) "there is someone who loves everyone in the world"

$\forall x \exists y$ Loves(x, y) "everyone in the world is lovely by at least someone"

- $\forall x \exists y\ \neg$ Loves(x, y) (what does it mean?)

Quantifier duality: each can be expressed using the other

$\forall x$ Likes (x, Ice_cream) the same as $\neg\exists x\ \neg$ Likes(x, Ice_cream)

$\exists x$ Likes(x, Ice_cream) $\neg\forall x\ \neg$ Likes(x, Ice_cream)

Examples of sentences

$\forall x, y$ Brothers (x, y) → Sibling(x, y)

$\forall x, y$ Sibling(x, y) ↔ Sibling(y, x) //Sibling is symmetric

$\forall x, y$ Mother (x, y) ↔ (Female(x) ∧ Parent(x, y) // the mother is the female parent.

$\forall x, y$ FirstCousin (x, y) ↔ $\exists px, py$ Parent (px, x) ∧ Sibling(px, py) ∧ Parent(py, y) // a first cousin is a child of a parent's sibling.

Equality

Term1=term2 is true iff Term1 and Term2 refer to the same object.

Example: $\forall x$ Multiply (Sqrt(x), Sqrt(x)=x

First Order Logic Inference

- The FOL inference is *complete*: if a sentence is entailed by KB, then this can be proved.
- FOL is *semi-decidable*: if the sentence is not entailed by KB, this cannot always be shown (proven).
- Using the truth-table (model Checking) is inefficient because the size is very large.

We will use three methods for FOL inference:

1- *Forward chaining*
2- *Backward chaining*
3- *Resolution*

All methods require *substitution* technique.

Substitution

Is the replacement of variable(s) in a sentence with expressions

SUBST({x/Richard, y/John}, Brother(x,y))=Brother(Richard, John)

Knowledge base

American(x) ∧Weapon (y) ∧ Sell(x, y, z) ∧ Hostile (z) → Criminal(x),

Owns(Nono, M1),

Missile (M1),

Missile(x)→ Weapon(x)

Missile(x) ∧ Owns(Nono,x) → Sell(West, x, Nono),

Enemy(x, America) →Hostile(x),

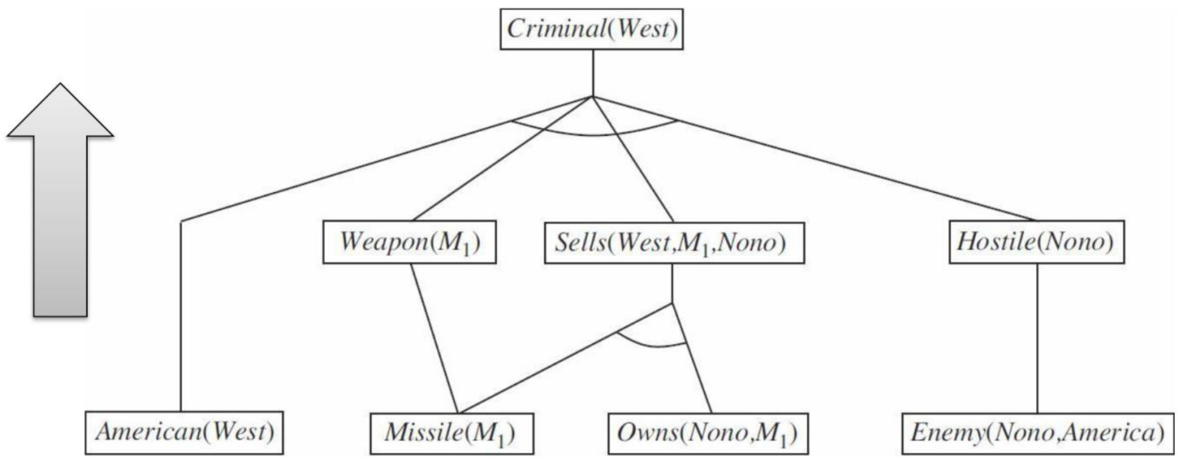
American(West),

Enemy (Nono, American),

* Prove that West is Criminal : Criminal (West).

Forward Chaining –FC

Start with sentences in KB, apply inference rules in forward direction, adding new sentences until goral found or no further inference can be made.



<u>Example:</u>

KB includes:

Parent (x,y) ∧ Male (x) → Father (x, y)

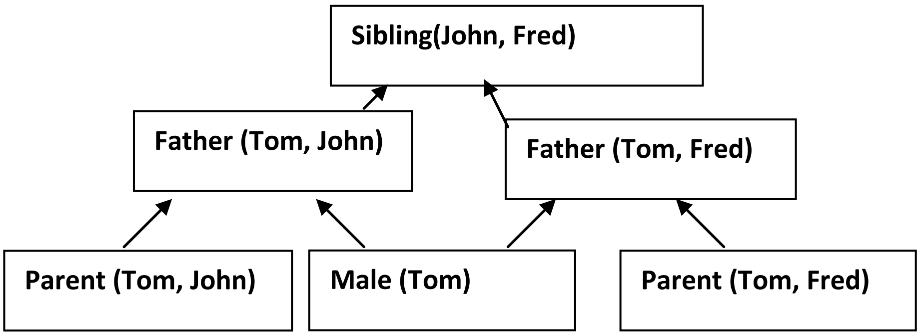
Father (x,y) ∧ Father (x,z) → Sibling(y, z)

Parent (Tom, John)

Parent (Tom, Fred)

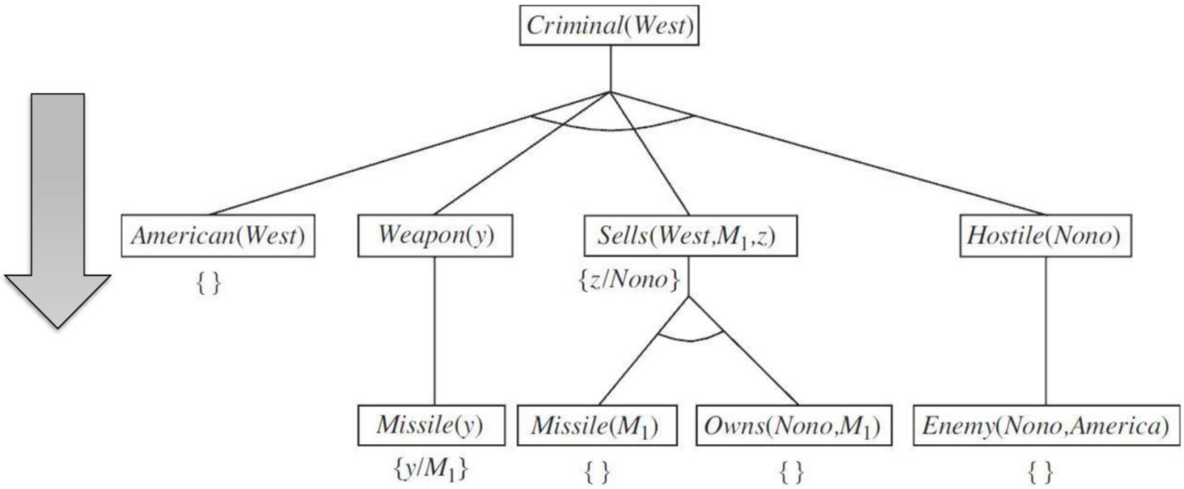
Male (Tom)

Prove: Sibling (John, Fred) by forward chaining.



Backward Chaining – BC

Start with goal sentence; search for rules that support goral, adding new sub-goals until match with KB facts or no further inference can be made.



Resolution – A complete inference procedure

Resolution rule is: $\dfrac{p \vee q, \neg p \vee r}{q \vee r}$

This method assumes that all sentences are written by *conjunctive normal form* (CNF).

- CNF is conjunctive of *clauses*; each clause is a *disjunction* of literals.

<u>Example</u>: (P ∨ A) ∧ (Q∨ R) ∧ M.

- Every sentence in FOL can be converted into CNF.

Conversion to CNF

1. Eliminate implication: $p \Rightarrow q \rightarrow \neg p \vee q$ $p \Leftrightarrow q \rightarrow (\neg p \vee q) \wedge (p \vee \neg q)$

2. Move $\neg$ inwards:

$\neg(p \vee q) \rightarrow \neg p \wedge \neg q$
$\neg(p \wedge q) \rightarrow \neg p \vee \neg q$
$\neg\forall x\, p \rightarrow \exists x\, \neg p$
$\neg\exists x\, p \rightarrow \forall x\, \neg p$
$\neg\neg p \rightarrow p$

4. Standard variables: to avoid use of the same variable name by two different quantifiers.
$\forall x\, P(x) \vee \exists x\, P(x) \rightarrow \forall x_1\, P(x_1) \vee \exists x_2\, P(x_2)$

5. Skolemization: remove of existential quantifiers:

- with *Skolem functions* (within universal quantifier):

$$\forall x_1 \exists x_2\, \mathbf{(}P(x_1) \vee P(x_2)\mathbf{)} \rightarrow \forall x_1 \mathbf{(}P(x_1) \vee P(f_1(x_1))\mathbf{)}$$

- with *Skolem constant*:

$$\exists x\, (P(x) \wedge Q(x)) \rightarrow P(C_1) \wedge Q(C_1)$$

6. Drop universal quantifiers.

7. Distribute AND over V. $(a \wedge b) \vee c \rightarrow (a \vee c) \wedge (b \vee c)$

<u>Example</u> of CNF Conversion

- Example: "Everyone who loves all animals is loved by somebody"

$$\forall x \quad [\forall y \quad Animal(y) \Rightarrow Loves(x,y)] \Rightarrow [\exists y \quad Loves(y,x)]$$

Eliminate implications: $\forall x \quad \neg[\forall y \quad \neg Animal(y) \vee Loves(x,y)] \vee [\exists y \quad Loves(y,x)]$

Move $\neg$ inwards: $\forall x \quad [\exists y \quad Animal(y) \wedge \neg Loves(x,y)] \vee [\exists y \quad Loves(y,x)]$

Standardize variables: $\forall x \quad [\exists y \quad Animal(y) \wedge \neg Loves(x,y)] \vee [\exists z \quad Loves(z,x)]$

Skolemization: $\forall x \quad [Animal(F(x)) \wedge \neg Loves(x,F(x))] \vee [Loves(G(x),x)]$

Drop Universal Quantifiers: $[Animal(F(x)) \wedge \neg Loves(x,F(x))] \vee [Loves(G(x),x)]$

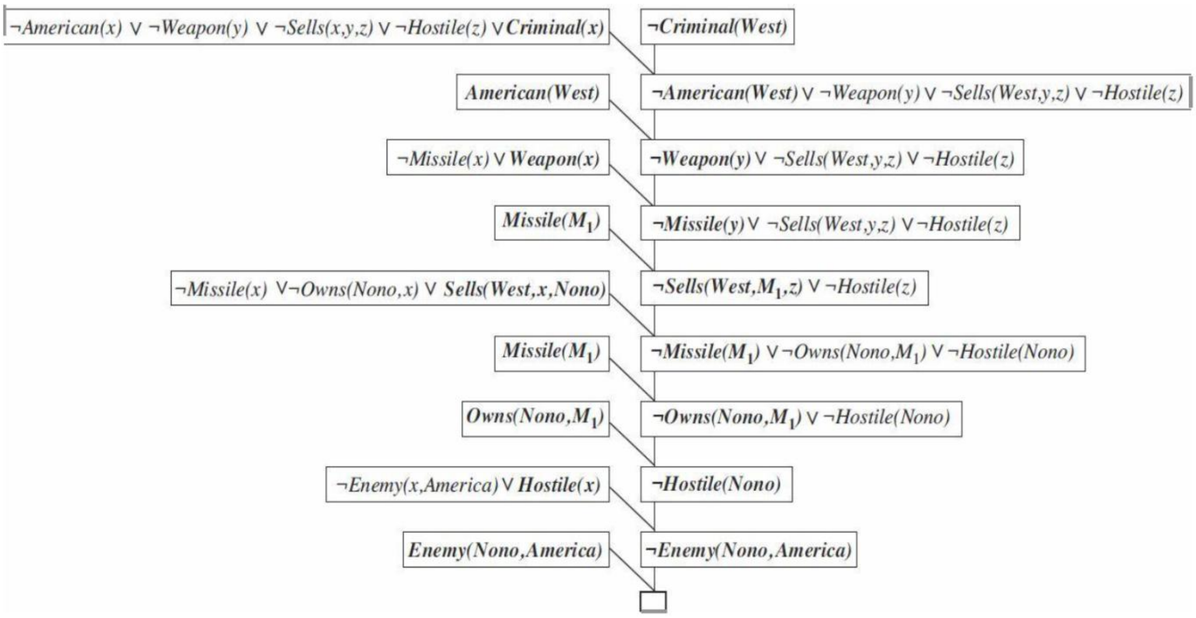
Distribute $\wedge$ over $\vee$:

$$[Animal(F(x)) \vee Loves(G(x),x)] \wedge [\neg Loves(x,F(x)) \vee Loves(G(x),x)]$$

Resolution inference procedure

- Convert KB and Goad into CNF
- Work by refutation, i.e. Add ~goal in KB.
- Apply resolution rule until a contradiction is found.

Example1: prove that West is Criminal.



Example2:

- everyone who loves all animals is loved by someone
- Anyone who kills an animal is loved by no one
- Jack loves all animals
- Either Jack or Curiosity killed the cat, Tuna

$A.\ \forall x\ \ [\forall y\ \ Animal(y) \Rightarrow Loves(x,y)] \Rightarrow [\exists y\ \ Loves(y,x)]$

$B.\ \forall x\ \ [\exists z\ \ Animal(z) \Rightarrow Kill(x,z)] \Rightarrow [\forall y\ \ \neg Loves(y,x)]$

$C.\ \forall x\ \ Animal(x) \Rightarrow Loves(Jack,x)$

$D.\ Kills(Jack,Tuna) \vee Kills(Curiosity,Tuna)$

$E.\ Cat(Tuna)$

$F.\ \forall x\ \ Cat(x) \Rightarrow Animal(x)$

Did Curiosity kill Tuna? Kills(Curiosity, Tuna).

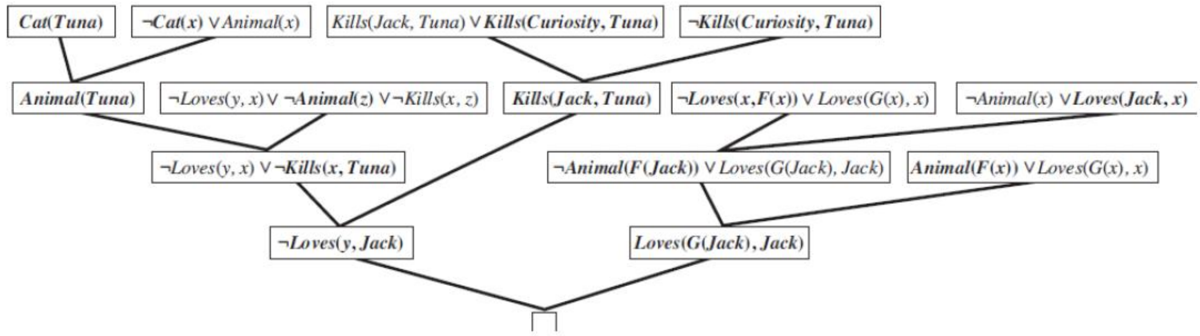
Answer:

1- Add ¬Kills(Curiosity, Tuna).

2- Converting each sentence to CNF yields

$$A_1.\quad Animal(F(x)) \vee Loves(G(x), x)$$
$$A_2.\quad \neg Loves(x, F(x)) \vee Loves(G(x), x)$$
$$B.\quad \neg Loves(y, x) \vee \neg Animal(z) \vee \neg Kill(x, z)$$
$$C.\quad \neg Animal(x) \vee Loves(Jack, x)$$
$$D.\quad Kills(Jack, Tuna) \vee Kills(Curiosity, Tuna)$$
$$E.\quad Cat(Tuna)$$
$$F.\quad \neg Cat(x) \vee Animal(x)$$
$$G.\quad \neg Kills(Curiosity, Tuna)$$

3- Resolution proof that curiosity killed the cat



Proof In English :

Suppose Curiosity did not kill Tuna. We know either Jack or Curiosity did; thus, Jack must have. Further, Tuna is a cat & cats are animals, so Tuna is an animal. Because anybody who kills animals is loved by no one, nobody loves Jack. However, Jack also loves all animals and thus is loved by somebody, which is a contradiction. Thus, Curiosity killed Tuna.

<u>Example 3:</u>

(1) Cats like fish	¬cat (x) ∨ likes (x,fish)
(2) Cats eat everything they like	¬cat (y) ∨ ¬likes (y,z) ∨ eats (y,z)
(3) Josephine is a cat.	cat (jo)
(4) Prove: Josephine eats fish.	eats (jo,fish)

Start : ¬eats (jo, fish)