Non-Linear Vibration

If any term of the equation of motion is not linear (second or more degree), then vibration is nonlinear

Examples:

Simple pendulum:

$$ml^2\ddot{\theta} + mgl\sin\theta = 0$$

When θ is small, then $\sin\theta$ is approximated as = θ

Better approximation near 0 is given by $\theta - \theta^3/6$

$$ml^2\ddot{\theta} + mgl\left(\theta - \frac{\theta^3}{6}\right) = 0$$

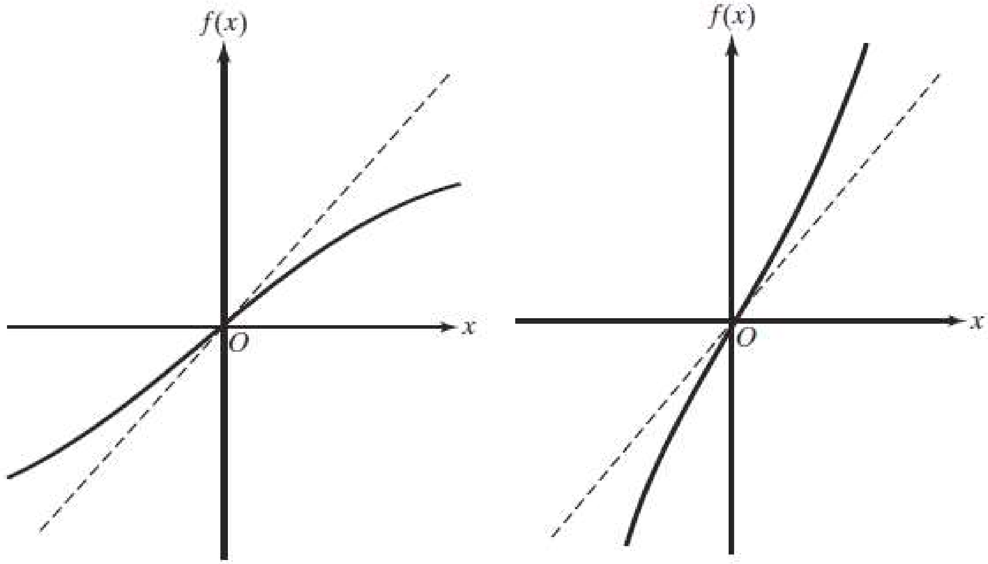
or

$$\ddot{\theta} + \omega_0^2(\theta - \tfrac{1}{6}\theta^3) = 0$$

Another example, is nonlinear spring

$F = k_1x + k_2x^2$, so the equation of motion is given by:

$m\ddot{x} + k_1x + k_2x^2 = 0$



(a) Soft spring (b) Hard spring

Solution Methods

1. Exact Method: Only possible for few cases
2. Approximate Analytical Methods
3. Approximate Numerical Methods

Approximate Analytical Methods

Poincare Method

Given that the nonlinear terms in the equation are associated with a parameter α, the solution may be expressed by:

$$\vec{x}(t) = \vec{x}_0(t) + \alpha\vec{x}_1(t) + \alpha^2\vec{x}_2(t) + \alpha^3\vec{x}_3(t) + \cdots$$

1) As α approaches zero, the solution reduced to linear solution
2) When α is small, the higher terms converge rapidly to zero such that few terms yield reasonable accuracy

Solution of Pendulum problem

$$\ddot{x} + \omega_0^2 x + \alpha x^3 = 0 \quad \text{with} \quad \alpha = -\omega_0^2/6$$

By using only two terms of the solution:

$$(\ddot{x}_0 + \alpha\ddot{x}_1) + \omega_0^2(x_0 + \alpha x_1) + \alpha(x_0 + \alpha x_1)^3 = 0$$

$$(\ddot{x}_0 + \omega_0^2 x_0) + \alpha(\ddot{x}_1 + \omega_0^2 x_1 + x_0^3) + \alpha^2(3x_0^2 x_1)$$
$$+ \alpha^3(3x_0 x_1^2) + \alpha^4 x_1^3 = 0$$

If the terms containing α^2, α^3, and α^4 are neglected, then:

$$\ddot{x}_0 + \omega_0^2 x_0 = 0$$
$$\ddot{x}_1 + \omega_0^2 x_1 = -x_0^3$$

The solution of the first equation is given by: $x_0(t) = A_0 \sin(\omega_0 t + \phi)$

Substitute the solution in the second equation yields:

$$\ddot{x}_1 + \omega_0^2 x_1 = -A_0^3 \sin^3(\omega_0 t + \phi)$$
$$= -A_0^3\left[\tfrac{3}{4}\sin(\omega_0 t + \phi) - \tfrac{1}{4}\sin 3(\omega_0 t + \phi)\right]$$

With

$$x_1(t) = \frac{3}{8\omega_0} t A_0^3 \cos(\omega_0 t + \phi) - \frac{A_0^3}{32\omega_0^2}\sin 3(\omega_0 t + \phi)$$

Hence:

$$x(t) = x_0(t) + \alpha x_1(t)$$
$$= A_0 \sin(\omega_0 t + \phi) + \frac{3\alpha t}{8\omega_0} A_0^3 \cos(\omega_0 t + \phi) - \frac{A_0^3 \alpha}{32\omega_0^2}\sin 3(\omega_0 t + \phi)$$

Lindstedt's Perturbation Method

Assumes the angular frequency is a function of amplitude A_0.

$$x(t) = x_0(t) + \alpha x_1(t) + \alpha^2 x_2(t) + \cdots$$

$$\omega^2 = \omega_0^2 + \alpha\omega_1(A_0) + \alpha^2\omega_2(A_0) + \cdots$$

To solve pendulum problem, $\ddot{x} + \omega_0^2 x + \alpha x^3 = 0$, assume only two terms:

$$x(t) = x_0(t) + \alpha x_1(t)$$

$$\omega^2 = \omega_0^2 + \alpha\omega_1(A_0)$$

Substitutin in the equation of pendulum, we obtain:

$$\ddot{x}_0 + \alpha\ddot{x}_1 + [\omega^2 - \alpha\omega_1(A_0)][x_0 + \alpha x_1] + \alpha[x_0 + \alpha x_1]^3 = 0$$

Or:

$$\ddot{x}_0 + \omega^2 x_0 + \alpha(\omega^2 x_1 + x_0^3 - \omega_1 x_0 + \ddot{x}_1)$$
$$+ \alpha^2(3x_1x_0^2 - \omega_1 x_1) + \alpha^3(3x_1^2 x_0) + \alpha^4(x_1^3) = 0$$

Neglecting terms containing α^2, $\alpha^{3,}$ α^4 yeilds

$$\ddot{x}_0 + \omega^2 x_0 = 0$$

$$\ddot{x}_1 + \omega^2 x_1 = -x_0^3 + \omega_1 x_0$$

The solutions are:

$$x_0(t) = A_0 \sin(\omega t + \phi)$$

$$\ddot{x}_1 + \omega^2 x_1 = -[A_0 \sin(\omega t + \phi)]^3 + \omega_1[A_0 \sin(\omega t + \phi)]$$
$$= -\tfrac{3}{4}A_0^3 \sin(\omega t + \phi) + \tfrac{1}{4}A_0^3 \sin 3(\omega t + \phi)$$
$$+ \omega_1 A_0 \sin(\omega t + \phi)$$

The first and third term of the above equation cause secular motion (resonance condition) which violate the truth that the motion is bounded, so the terms can be eliminated by imposing:

$$\omega_1 = \tfrac{3}{4}A_0^2$$

Hence the solution for x_1 is given by:

$$x_1(t) = A_1 \sin(\omega t + \phi_1) - \frac{A_0^3}{32\omega^2} \sin 3(\omega t + \phi)$$

Let the initial conditions be $x(0) = A,\ and\ \dot{x}(0) = 0$

We force $x_0(t)$ to satisfy the initial conditions, so that

$$x(0) = A = A_0 \sin \phi, \qquad \dot{x}(0) = 0 = A_0 \omega \cos \phi$$

Which yields:

$$A_0 = A \qquad \text{and} \qquad \phi = \frac{\pi}{2}$$

Since the initial conditions are satisfied by $x_0(t)$, then $x_1(t)$ and its derivative are initially zero:

$$x_1(0) = 0 = A_1 \sin \phi_1 - \frac{A_0^3}{32\omega^2} \sin 3\phi$$

$$\dot{x}_1(0) = 0 = A_1 \omega \cos \phi_1 - \frac{A_0^3}{32\omega^2}(3\omega) \cos 3\phi$$

In view of the known relations $A_0 = A$ and $\phi = \pi/2$, the above equations yield

$$A_1 = -\left(\frac{A^3}{32\omega^2}\right) \qquad \text{and} \qquad \phi_1 = \frac{\pi}{2}$$

The total solution will be:

$$x(t) = A_0 \sin(\omega t + \phi) - \frac{\alpha A_0^3}{32\omega^2} \sin 3(\omega t + \phi)$$

with

$$\omega^2 = \omega_0^2 + \alpha \tfrac{3}{4} A_0^2$$

Ritz-Galrkin Method

The solution is found by satisfying the nonlinear equation in average. Let the equation be given as

$$E[x] = 0$$

An approximate solution is represented by

$$\underset{\sim}{x}(t) = a_1\phi_1(t) + a_2\phi_2(t) + \cdots + a_n\phi_n(t)$$

Where φ_n are prescribed time functions and constants a_n are weighting factors. When the approximate solution is substituted in the nonlinear equation *E[x],* we obtain an approximate equation which must be minimized.

The coefficients are found by solving the equations resulting from minimization the following integral:

$$\int_0^{\tau} \underset{\sim}{E}^2[t]\, dt$$

, where τ is the period of motion.

Minimization gives a set of simultaneous equations:

$$\frac{\partial}{\partial a_i}\left(\int_0^{\tau} \underset{\sim}{E}^2[t]\, dt\right) = 2\int_0^{\tau} \underset{\sim}{E}[t] \frac{\partial \underset{\sim}{E}[t]}{\partial a_i}\, dt = 0,$$

$$i = 1, 2, \ldots, n$$

Example: pendulum equation

Using a one-term approximation, find the solution of the pendulum equation

$$E[x] = \ddot{x} + \omega_0^2 x - \frac{\omega_0^2}{6} x^3 = 0$$

By using a one-term approximation for $x(t)$ as

$$\underset{\sim}{x}(t) = A_0 \sin \omega t$$

$$E[\underset{\sim}{x}(t)] = -\omega^2 A_0 \sin \omega t + \omega_0^2\left[A_0 \sin \omega t - \frac{1}{6} \sin^3 \omega t\right]$$

$$= \left(\omega_0^2 - \omega^2 - \frac{1}{8}\omega_0^2 A_0^2\right) A_0 \sin \omega t + \frac{\omega_0^2}{24} A_0^3 \sin 3\omega t$$

The Ritz-Galerkin method requires the minimization of

$$\int_0^{\tau} E^2[\underset{\sim}{x}(t)]\, dt$$

$$\int_0^{\tau} \underset{\sim}{E}\frac{\partial \underset{\sim}{E}}{\partial A_0}\, dt = \int_0^{\tau}\left[\left(\omega_0^2 - \omega^2 - \frac{1}{8}\omega_0^2 A_0^2\right)A_0 \sin \omega t + \frac{\omega_0^2}{24}A_0^3 \sin 3\omega t\right]$$
$$\times \left[\left(\omega_0^2 - \omega^2 - \frac{3}{8}\omega_0^2 A_0^2\right)\sin \omega t + \frac{1}{8}\omega_0^2 A_0^2 \sin 3\omega t\right] dt = 0$$

That's lead to:

$$A_0\left(\omega_0^2 - \omega^2 - \frac{1}{8}\omega_0^2 A_0^2\right)\left(\omega_0^2 - \omega^2 - \frac{3}{8}\omega_0^2 A_0^2\right)\int_0^{\tau} \sin^2 \omega t\, dt$$
$$+ \frac{\omega_0^2 A_0^3}{24}\left(\omega_0^2 - \omega^2 - \frac{3}{8}\omega_0^2 A_0^2\right)\int_0^{\tau} \sin \omega t \sin 3\omega t\, dt$$
$$+ \frac{1}{8}\omega_0^2 A_0^2\left(\omega_0^2 - \omega^2 - \frac{1}{8}\omega_0^2 A_0^2\right)\int_0^{\tau} \sin \omega t \sin 3\omega t\, dt$$
$$+ \frac{\omega_0^4 A_0^5}{192}\int_0^{\tau} \sin^2 3\omega t\, dt = 0$$

The nonzero integrals are obtained from the first and last terms only:

$$A_0\left[\left(\omega_0^2 - \omega^2 - \frac{1}{8}\omega_0^2 A_0^2\right)\left(\omega_0^2 - \omega^2 - \frac{3}{8}\omega_0^2 A_0^2\right) + \frac{\omega_0^4 A_0^4}{192}\right] = 0$$

The nontrival solution is given by:

$$\omega^4 + \omega^2\omega_0^2\left(\tfrac{1}{2}A_0^2 - 2\right) + \omega_0^4\left(1 - \tfrac{1}{2}A_0^2 + \tfrac{5}{96}A_0^4\right) = 0$$

The solution of the above equation is given by:

$$\omega^2 = \omega_0^2(1 - 0.147938A_0^2)$$

$$\omega^2 = \omega_0^2(1 - 0.352062A_0^2)$$

Only the first value satisfies minimization of E[x] while the second one causes maximization of E[x].