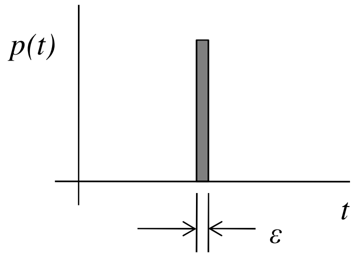
Transient Vibration

Impulse force



$$\hat{F} = \int p(t)dt$$

Delta function

$$\delta(t-\tau) = \begin{cases} non\ zero & t=\tau \\ 0 & t \neq \tau \end{cases}$$

$$\int_0^\infty \delta(t-\tau)dt = 1$$

$$and\ \int_0^\infty p(t)\delta(t-\tau)dt = p(\tau)$$

Solution of undamped system:

$$x(t) = \frac{\dot{x}(0)}{\omega_n}\sin\omega_n t + x(0)\cos\omega_n t$$

Since an impulse force $\hat{F}$ acting at small time may present an initial velocity $\frac{\hat{F}}{m}$, then

$$x(t) = \frac{\hat{F}}{m\omega_n}\sin\omega_n t = \hat{F}h(t)$$

Where *h(t)* is the unit impulse response

$$h(t) = \frac{1}{m\omega_n}\sin\omega_n t$$

For damped system: $x(t) = \frac{\hat{F}e^{-\zeta\omega_n t}}{m\omega_n\sqrt{1-\zeta^2}}\sin\omega_n\sqrt{1-\zeta^2}\,t = \hat{F}h(t)$

For Arbitrary Excitation:

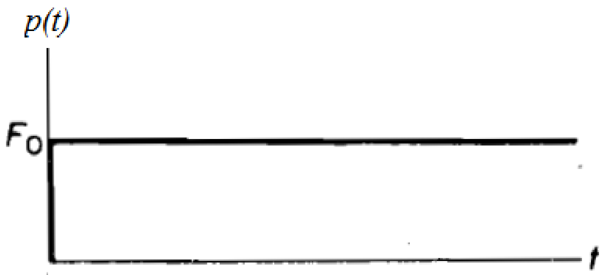
Given the response for unit impulse force is known $h(t)$, the response for an arbitrary force $p(t)$ can be estimated by using superposition integral or convolution integral (Duhamel Integral):

$$x(t) = \int_0^t p(\tau)h(t-\tau)d\tau = \frac{1}{m\omega_n}\int_0^t p(\tau)\sin\omega_n(t-\tau)d\tau$$

The above integral state that the overall response is the sum of responses to forces each force is acting at different time τ and have duration $d\tau$.

Example:

Determine the response of undamped system to the force shown in the figure:



Soultion:

$$h(t) = \frac{1}{m\omega_n}\sin\omega_n t$$

$$x(t) = \int_0^t p(t)h(t-\tau)d\tau = \int_0^t \frac{F_0}{m\omega_n}\sin\omega_n(t-\tau)d\tau$$

$$= \frac{F_0}{k}(1-\cos\omega_n t)$$

Laplace transform Method

And so, we can evaluate the response due to ramp or parabolic function. Laplace transform can also be used to find the transient response.

Numerical Methods

When the integral is not possible in closed form, then numerical methods such as Euler and Runge-Kutta methods can be used.

Numerical Integration

The response to an arbitrary excitation p(t) with zero initial condition is given by

$$
\begin{aligned}
x(t) &= \int_0^t p(\tau)h(t-\tau)d\tau = \frac{1}{m\omega_n}\int_0^t p(\tau)\sin\omega_n(t-\tau)d\tau \\
&= \sin\omega_n t\,\frac{1}{m\omega_n}\int_0^t p(\tau)\cos\omega_n\tau d\tau - \cos\omega_n t\,\frac{1}{m\omega_n}\int_0^t p(\tau)\sin\omega_n\tau d\tau \\
&= A(t)\sin\omega_n t - B(t)\cos\omega_n t
\end{aligned}
$$

Where:

$$A(t) = \frac{1}{m\omega_n}\int_0^t p(\tau)\cos\omega_n\tau d\tau$$

$$B(t) = \frac{1}{m\omega_n}\int_0^t p(\tau)\sin\omega_n\tau d\tau$$

Can be evaluated numerically. Consider A(t):

$$A(t) = \frac{1}{m\omega_n}\int_0^t p(\tau)\cos\omega_n\tau d\tau = \frac{1}{m\omega_n}\int_0^t y(\tau)d\tau = \frac{\Delta\tau}{m\omega_n}\frac{1}{\mu}\mathrm{T}_\mu^t$$

Where

$$
\begin{aligned}
&\mathrm{T}_1^t = y_0 + y_1 + y_2 + \ldots\ldots + y_{N-1} && \textit{Simple Integration} \\
&\mathrm{T}_2^t = y_0 + 2y_1 + 2y_2 + \ldots\ldots + 2y_{N-1} + y_N && \textit{Trapizoidal Integration} \\
&\mathrm{T}_3^t = y_0 + 4y_1 + 2y_2 + \ldots\ldots + 4y_{N-1} + y_N && \textit{Simpson Integration}
\end{aligned}
$$

Using Incremental integration for numerical solution:

$$\mathrm{T}_1^t = \mathrm{T}_1^{t-\Delta\tau} + p(t-\Delta\tau)\cos\omega_n(t-\Delta\tau) \qquad \textit{Simple Integration}$$

$$\mathrm{T}_2^t = \mathrm{T}_2^{t-\Delta\tau} + p(t-\Delta\tau)\cos\omega_n(t-\Delta\tau) + p(t)\cos\omega_n t \qquad \textit{Trapizoidal}$$

$$\mathrm{T}_3^t = \mathrm{T}_3^{t-\Delta\tau} + [\,p(t-2\Delta\tau)\cos\omega_n(t-2\Delta\tau) + 4p(t-\Delta\tau)\cos\omega_n(t-\Delta\tau) + p(t)\cos\omega_n t] \qquad \textit{Simpson}$$

For damped vibration:

$$h(t) = \frac{e^{-\zeta\omega_n t}}{m\omega_d}\sin\omega_d t$$

$$x(t) = A(t)e^{-\zeta\omega_n t}\sin\omega_n t - B(t)e^{-\zeta\omega_n t}\cos\omega_n t$$

$$A(t) = \frac{1}{m\omega_n}\int_0^t p(\tau)e^{\zeta\omega_n\tau}\cos\omega_n\tau d\tau$$

$$B(t) = \frac{1}{m\omega_n}\int_0^t p(\tau)e^{\zeta\omega_n\tau}\sin\omega_n\tau d\tau$$

H.W: