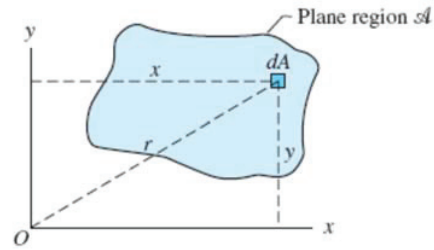
MOMENTS OF INERTIA

5.1 Moment of Inertia of Area

The ***moments of inertia of the area*** about the x- and y-axes, respectively, are defined by

$$I_x = \int_A y^2 dA \qquad I_y = \int_A x^2 dA \qquad \text{...(5-1)}$$



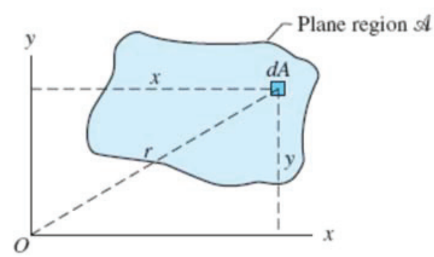
Because the distances x and y are squared, I_x and I_y are sometimes called the ***second moments of the area.***

The dimension for moment of inertia of area is $[L^4]$. Therefore, the units are in^4, mm^4, and so for the other units. Although the first moment of an area can be positive, negative, or zero, its moment of inertia is always positive, because both x and y in Eqs. (5.1) are squared.

5.2 Polar Moment of Inertia

The polar moment of inertia of the area about point O (strictly speaking, about an axis through O, perpendicular to the plane of the area) is defined by:

$$J_o = \int_A r^2 dA \qquad \text{...(5-1)}$$



where r : is the distance from O to the differential area element dA.

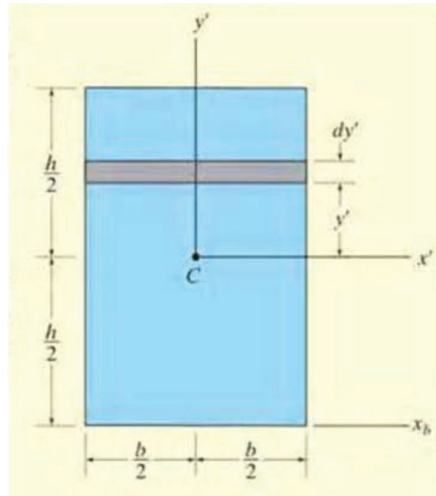
Note that the polar moment of an area is always positive and its dimension is $[L^4]$

From Figure, we note that $r^2 = y^2 + x^2$, which gives the following relationship between polar moment of inertia and moment of inertia:

$$J_o = \int_A r^2 dA = \int_A (x^2 + y^2) dA = \int_A x^2 dA + \int_A y^2 dA$$

$$\therefore J_o = I_x + I_y \qquad \text{...(5-2)}$$

Example 2: Determine the moment of inertia for the rectangular area shown in Figure (a) with respect to



(a) the centroidal x'-axis,

(b) the x_b passing through the base of the rectangle, and

(c) the pole or z'-axis perpendicular to the x'-y' plane and passing through the centroid C.

Solution: Part (a). The horizontal differential element is chosen for integration. Here it is necessary to integrate from $y' = -h/2$ to $y' = h/2$ since $dA = bdy'$, then

$$\bar{I}_{x'} = \int_A y'^2 .dA = \int_{-h/2}^{h/2} y'^2 (bdy') = b\left[\frac{y'^3}{3}\right]_{-h/2}^{h/2}$$

$$= \frac{b}{3}\left[\left(\frac{h}{2}\right)^3 - \left(\frac{-h}{2}\right)^3\right] = \frac{b}{3}\left[\frac{h^3}{8} + \frac{h^3}{8}\right] = \frac{2bh^3}{(3)(8)} = \frac{bh^3}{12}$$

Part (b): Applying the parallel-axis theorem.

$$I_{x_b} = \bar{I}_{x'} + Ad_y^2$$

$$= \frac{bh^3}{12} + bh\left(\frac{h}{2}\right)^2 = \frac{bh^3}{3}$$

Part (c): To obtain the polar moment of inertia about point C, we must first obtain $\bar{I}_{y'}$, which may be found by interchanging the dimensions b and h in the result of part (a), i.e.,

$$\bar{I}_{y'} = \frac{hb^3}{12}$$

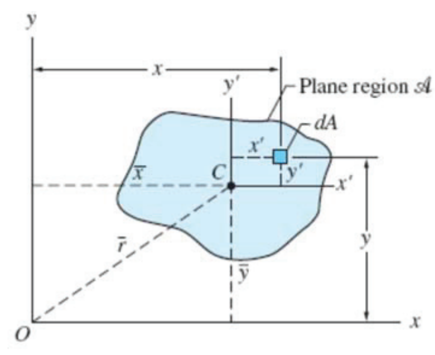
So the polar moment of inertia about C is therefore

$$\bar{J}_C = \bar{I}_{x'} + \bar{I}_{y'} = \frac{bh}{12}(h^2 + b^2)$$

This relationship states that the polar moment of inertia of an area about a point O equals the sum of the moments of inertia of the area about two perpendicular axes that intersect at O.

5.3 Parallel-Axis Theorems:

Observe that the y-coordinate of the differential area dA can be written as $y = \bar{y} + y`$ where $\bar{y}$ (the centroidal coordinate of the area) is the distance between the two axes.



(a)

$$I_x = \int_A y^2 dA = \int_A (\bar{y} + y`)^2 dA = \bar{y}^2 \int_A dA + 2\bar{y}\int_A y`dA + \int_A (y`)^2 dA$$

Where $\int_A dA = A$ the area of the region,

$\int_A y`dA = 0$ the first moment of the area about a centroidal axis vanishes,

$\int_A (y`)^2 dA = \bar{I}_x$ the second moment of the area about the $x`$-axis).

Above equation simplifies to

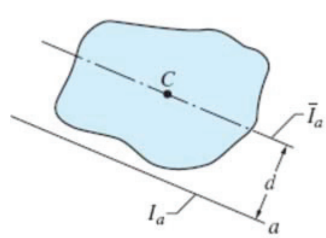
$$I_x = \bar{I}_x + A\bar{y}^2 \qquad \textbf{…(5-3a)}$$

This relationship is known as the ***parallel-axis theorem*** for moment of inertia of an area. The distance $\bar{y}$ is sometimes called the ***transfer distance*** (the distance through which the moment of inertia is to be "transferred").

Note: It is important to remember that the theorem is valid only if $\bar{I}_x$ is the moment of inertia about the centroidal x-axis. If this is not the case, the term $\int_A y`dA$ in Eq. (a) would not vanish, giving rise to an additional term in Eq. (5-3a).

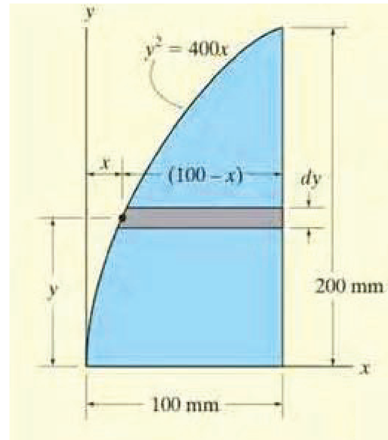
In general, the parallel-axis theorem can be written as:

$$I_a = \bar{I}_a + Ad^2 \qquad \textbf{…(5-3b)}$$



(b)

Example 3: Determine the moment of inertia for the shaded area shown in Figure (a) about the *x*-axis:



(a)

Solution I: A differential element of the area that is *parallel* to the *x*-axis is chosen for integration. Its area is

$$dA = (100 - x)dy$$

Integrating with respect to y, from $y = 0$ to $y = 200$ mm, yields

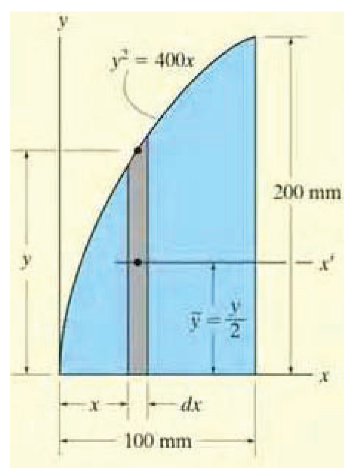
$$I_x = \int_A y^2.dA = \int_0^{200} y^2(100-x).dy = \int_0^{200} y^2\left(100-\frac{y^2}{400}\right).dy$$

$$= \int_0^{200}\left(100y^2-\frac{y^4}{400}\right).dy = \left[\frac{100y^3}{3}-\frac{y^5}{(400)(5)}\right]_0^{200}$$

$$= \left[\left(\frac{100(200)^3}{3}-\frac{(200)^5}{2000}\right)-\left(\frac{100(0)^3}{3}-\frac{(0)^5}{2000}\right)\right]$$

$$= 107 \times 10^6 \text{ mm}^4$$

Solution II: A differential element *parallel* to the *y*-axis as shown in Figure (b), is chosen for integration. For a differential element chosen in Figure (b),



(b)

$b = dx$ and $h = y$,

and thus $d\bar{I}_{x'} = \frac{(dx)(y)^3}{12}$.

Since the centroid of the element $\bar{y} = y/2$ from the x-axis, the moment of inertia of the element about this axis is

$$dI_x = d\bar{I}_{x'} + dA\bar{y}^2 = \frac{y^3}{12}.dx + (y.dx)\left(\frac{y}{2}\right)^2$$

$$=\left(\frac{y^3}{12}+\frac{y^3}{4}\right)dx=\left(\frac{1+3}{12}\right)y^3.dx=\frac{1}{3}y^3.dx$$ (This result can also concluded from part (b) of Example 2)

Integrating with respect to x, from $x = 0$ to $x = 100$ mm, yields

$$I_x=\int_A dI_x=\int_0^{100}\frac{1}{3}y^3.dx=\frac{1}{3}\int_0^{100}(400x)^{3/2}.dx$$

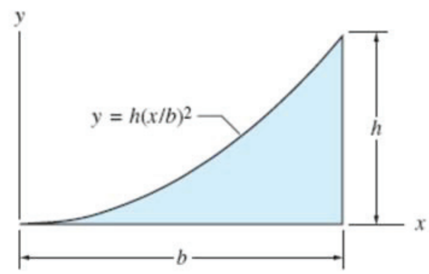
$$=\frac{1}{(3)(400)}\left[\frac{(400x)^{5/2}}{5/2}\right]_0^{100}=\frac{1}{3000}\left[(400(100))^{5/2}-(400(0))^{5/2}\right]$$

$$=107\times10^6\ \text{mm}$$

Example 4: By integration, calculate the moment of inertia about the y-axis of the area shown in Fig. (a) by the following methods:

(1) using a vertical differential area element; and

(2) using a horizontal differential area element.

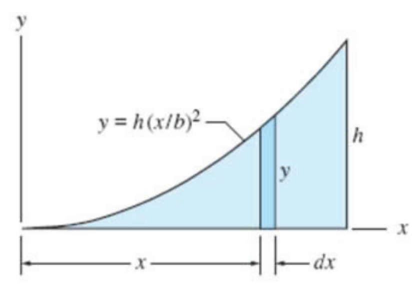


(a)

Solution: Part (1): The vertical differential area element is shown in Fig. (b).

$$dA = y\,dx = h(x/b)^2\,dx,$$

we have



(b)

$$I_y=\int_A x^2dA=\int_0^b x^2.[h(x/b)^2dx]=\frac{h}{b^2}\int_0^b x^4.dx$$

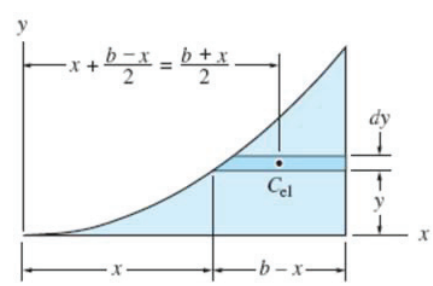
$$=\frac{h}{b^2}\left[\frac{x^5}{5}\right]_0^b=\frac{h}{b^2}\left[\frac{b^5}{5}-\frac{(0)^5}{5}\right]=\frac{b^3h}{5}$$

Part 2: The horizontal differential area element is shown in Fig. (c).

$$d\bar{I}_y=\frac{dy(b-x)^3}{12}.$$

According to the parallel-axis theorem,

$$dI_y=d\bar{I}_{y'}+dA(d_x^2),$$



(c)

where d_x is the distance between the *y*-axis and the vertical centroidal axis of the element. Using $d_x = \frac{(b+x)}{2}$ as shown in Fig. (c), and integrating, we obtain I_y for the entire area:

$$I_y = \int_A dI_y = \int_0^h \left[\frac{dy(b-x)^3}{12} + [(b-x)dy]\left(\frac{b+x}{2}\right)^2 \right] dy$$

Substituting $x = b(y/h)^{1/2}$ and completing the integration gives

$$I_y = \int_0^h \left[\frac{\left(b - b\sqrt{(y/h)}\right)^3}{12} + (b - b\sqrt{(y/h)})\left(\frac{b + b\sqrt{(y/h)}}{2}\right)^2 \right].dy$$

$$= b^3 \int_0^h \left[\frac{\left(1 - \sqrt{(y/h)}\right)^3}{12} + (1 - \sqrt{(y/h)})\left(\frac{1 + \sqrt{(y/h)}}{2}\right)^2 \right].dy$$

Let $z^2 = \frac{y}{h}$ $\qquad 2z.dz = \frac{dy}{h}$ $\qquad$ or $\qquad dy = 2hz.dz$

Note that when $\quad y = 0 \qquad z = 0$

And when $\quad y = h \qquad z = 1$

$$= b^3 \int_0^1 \left[\frac{(1-z)^3}{12} + (1-z)\left(\frac{1+z}{2}\right)^2 \right].2hz.dz$$

$$= 2b^3 h \int_0^1 \left[\frac{1}{12}(1-z)(1-2z+z^2) + \frac{1}{4}(1-z)(1+2z+z^2) \right].z.dz$$

$$= 2b^3 h \int_0^1 \left[\frac{1}{12}(1-2z+z^2-z+2z^2-z^3) + \frac{1}{4}(1+2z+z^2-z-2z^2-z^3) \right].z.dz$$

$$= 2b^3 h \int_0^1 \left[\frac{1}{12}(1-3z+3z^2-z^3) + \frac{1}{4}(1+z-z^2-z^3) \right].z.dz$$

$$= \frac{2b^3 h}{12} \int_0^1 \left[1-3z+3z^2-z^3+3+3z-3z^2-3z^3\right].z.dz$$

$$= \frac{b^3 h}{6} \int_0^1 \left[4-4z^3\right].z.dz$$

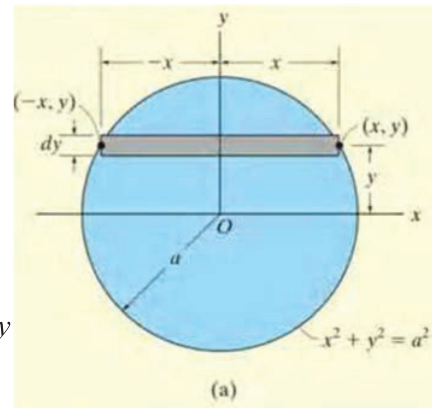
$$= \frac{4b^3 h}{6} \int_0^1 \left[z-z^4\right].dz$$

$$= \frac{2b^3 h}{3} \left[\frac{z^2}{2} - \frac{z^5}{5} \right]_0^1 = \frac{2b^3 h}{3} \left[\left(\frac{(1)^2}{2} - \frac{(1)^5}{5} \right) - \left(\frac{(0)^2}{2} - \frac{(0)^5}{5} \right) \right]$$

$$= \frac{2b^3h}{3}\left(\frac{5-2}{10}\right) = \frac{2b^3h}{3}\left(\frac{3}{10}\right) = \frac{b^3h}{5}$$

***Note*:** Obviously, the horizontal differential area element is not as convenient as the other choices in this problem.

Example 5: Determine the moment of inertia with respect to the *x*-axis for the circular area shown in Figure (a)



(a)

Solution I: Using horizontal differential element, since

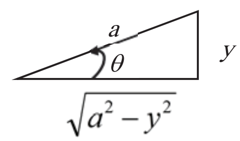
$$dA = 2x.dy$$

we have

$$I_x = \int_A y^2.dA = \int_{-a}^{a} y^2(2x).dy = 2\int_{-a}^{a} y^2\sqrt{a^2-y^2}.dy = 4\int_0^a y^2\sqrt{a^2-y^2}.dy$$

Using trigonometric substitutions:

Let $y = a\sin\theta$ $\quad dy = a\cos\theta.d\theta$



$$\int y^2\sqrt{a^2-y^2}.dy = \int a^2\sin^2\theta\sqrt{a^2-a^2\sin^2\theta}(a\cos\theta).d\theta$$

$$= a^4\int \sin^2\theta\cos^2\theta.d\theta = a^4\int(\sin\theta\cos\theta)^2.d\theta = a^4\int\left(\frac{\sin 2\theta}{2}\right)^2.d\theta$$

$$= \frac{a^4}{4}\int\left(\frac{1-\cos 4\theta}{2}\right)d\theta = \frac{a^4}{8}\left[\theta - \frac{\sin 4\theta}{4}\right] + C$$

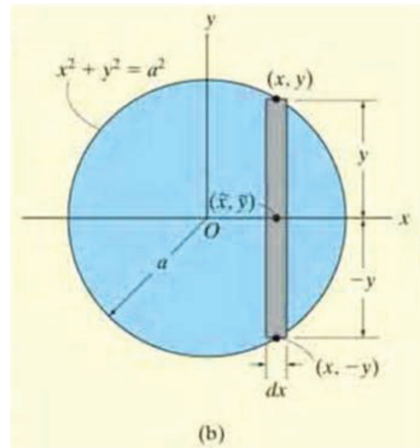
$$= \frac{a^4}{8}\left[\sin^{-1}\left(\frac{y}{a}\right) - \frac{\sin 4\left(\sin^{-1}\left(\frac{y}{a}\right)\right)}{4}\right] + C$$

$$I_x = 4\int_0^a y^2\sqrt{a^2-y^2}.dy = \frac{4a^4}{8}\left[\sin^{-1}\left(\frac{y}{a}\right) - \frac{\sin 4\left(\sin^{-1}\left(\frac{y}{a}\right)\right)}{4}\right]_0^a$$

$$= \frac{4a^4}{8}\left[\left(\sin^{-1}(1) - \frac{\sin 4(\sin^{-1}(1))}{4}\right) - \left(\sin^{-1}(0) - \frac{\sin 4(\sin^{-1}(0))}{4}\right)\right]$$

$$= \frac{a^4}{2}\left[\left(\frac{\pi}{2} - \frac{\sin 4\left(\frac{\pi}{2}\right)}{4}\right) - (0-0)\right] = \frac{a^4}{2}\left(\frac{\pi}{2} - 0\right) = \frac{\pi a^4}{4}$$

Solution II: When vertical differential element as shown in Figure (c) is chosen, the centroid for the element happens to the lie on the x-axis, and since $\bar{I}_{x'} = \frac{bh^3}{12}$ for a rectangle, we have

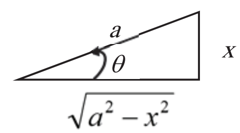


$$dI_x = \frac{(dx)(2y)^3}{12} = \frac{2}{3} y^3 dx = \frac{2}{3}\left(a^2 - x^2\right)^{3/2}.dx$$

Integrating with respect to x yields

$$I_x = \int_{-a}^{a} \frac{2}{3}\left(a^2 - x^2\right)^{3/2}.dx = 2\int_0^a \frac{2}{3}\left(a^2 - x^2\right)^{3/2}.dx = \frac{4}{3}\int_0^a \left(a^2 - x^2\right)^{3/2}.dx$$

By trigonometric substitutions



Let $x = a\sin\theta \qquad dx = a\cos\theta.d\theta$

$$\int \left(a^2 - x^2\right)^{3/2} dx = \int \left(a^2 - a^2\sin^2\theta\right)^{3/2} a\cos\theta.d\theta = a^4 \int \cos^4\theta.d\theta$$

$$= a^4 \int \left(\frac{1+\cos 2\theta}{2}\right)^2 d\theta = \frac{a^4}{4}\int \left(1 + 2\cos 2\theta + \cos^2 2\theta\right) d\theta$$

$$= \frac{a^4}{4}\int \left(1 + 2\cos 2\theta + \frac{1+\cos 4\theta}{2}\right) d\theta = \frac{a^4}{4}\int \left(\frac{3}{2} + 2\cos 2\theta + \frac{\cos 4\theta}{2}\right) d\theta$$

$$= \frac{a^4}{4}\left[\frac{3}{2}\theta + \frac{2\sin 2\theta}{2} + \frac{\sin 4\theta}{(2)(4)}\right] + C = \frac{a^4}{4}\left[\frac{3}{2}\theta + 2\sin\theta\cos\theta + \frac{\sin 4\theta}{8}\right] + C$$

$$= \frac{a^4}{4}\left[\frac{3}{2}\sin^{-1}\left(\frac{x}{a}\right) + 2\left(\frac{x}{a}\right)\left(\frac{\sqrt{a^2 - x^2}}{a}\right) + \frac{\sin 4\left(\sin^{-1}\left(\frac{x}{a}\right)\right)}{8}\right] + C$$

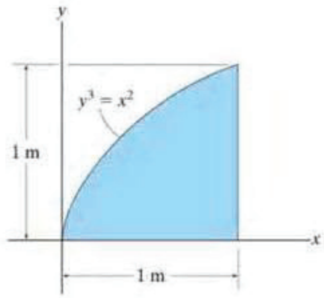
$$I_x = \frac{4}{3}\int_0^a \left(a^2 - x^2\right)^{3/2}.dx = \left(\frac{4}{3}\right)\left(\frac{a^4}{4}\right)\left[\frac{3}{2}\sin^{-1}\left(\frac{x}{a}\right) + \left(\frac{2x\sqrt{a^2 - x^2}}{a^2}\right) + \frac{\sin 4\left(\sin^{-1}\left(\frac{x}{a}\right)\right)}{8}\right]_0^a$$

$$= \frac{a^4}{3}\left[\left(\frac{3}{2}\sin^{-1}(1) + (0) + \frac{\sin 4\left(\sin^{-1}(1)\right)}{8}\right) - \left(\frac{3}{2}\sin^{-1}(0) + (0) + \frac{\sin 4\left(\sin^{-1}(0)\right)}{8}\right)\right]$$

$$= \frac{a^4}{3}\left[\left(\frac{3}{2}\right)\left(\frac{\pi}{2}\right)\right] = \frac{\pi a^4}{4}$$

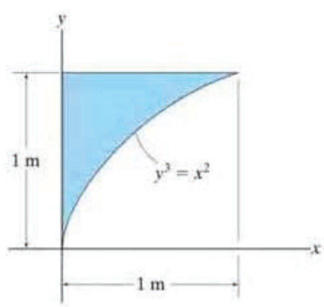
***Note*: By comparison, Solution I requires much less computations. Therefore, if an integral using a particular element appears difficult to evaluate, try solving the problem using an element oriented in the other direction.

F10–1. Determine the moment of inertia of the shaded area about the x axis.



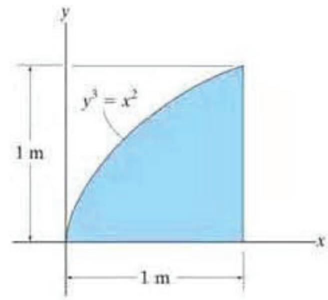
F10–1

F10–2. Determine the moment of inertia of the shaded area about the x axis.



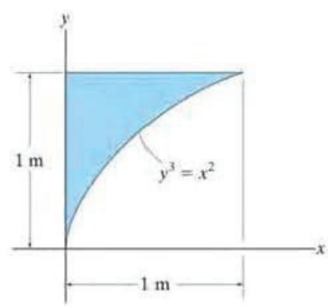
F10–2

F10–3. Determine the moment of inertia of the shaded area about the y axis.



F10–3

F10–4. Determine the moment of inertia of the shaded area about the y axis.



F10–4

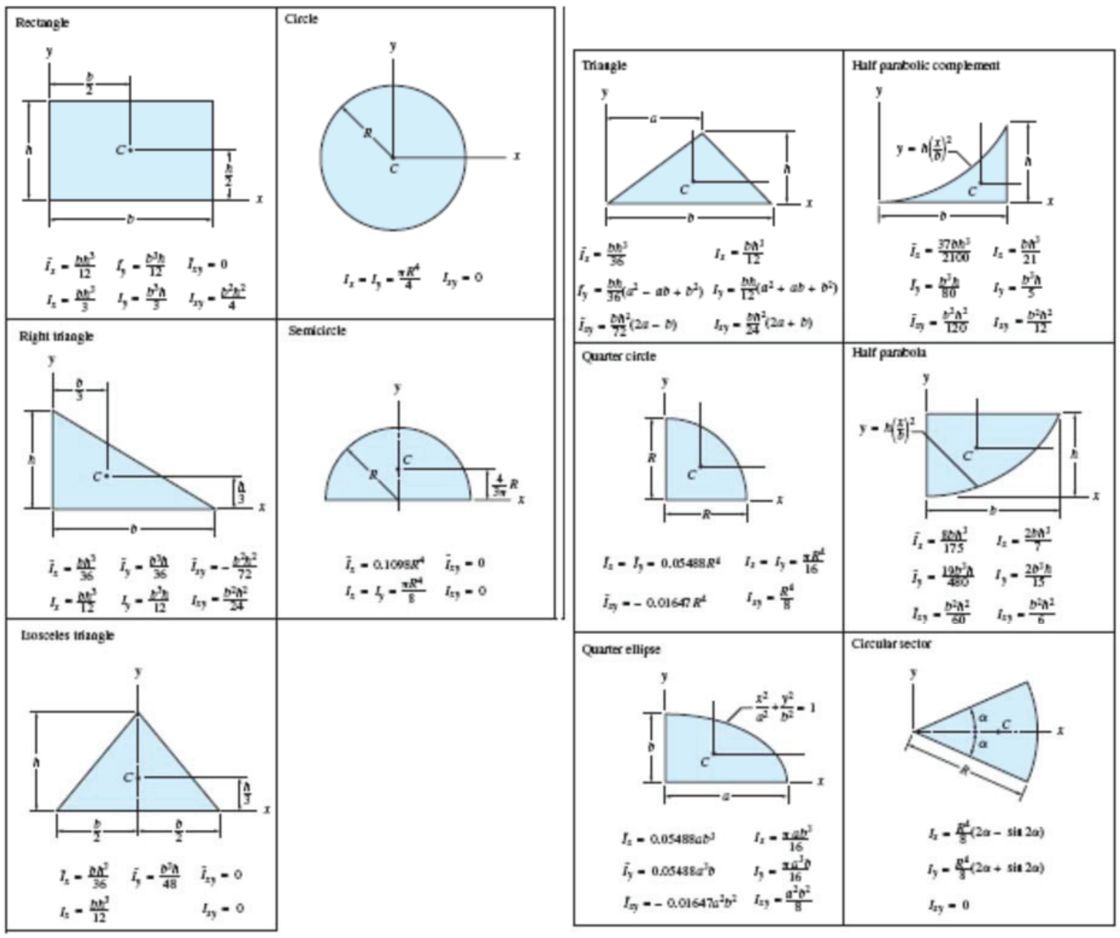
5.6 Moment of Inertia for Composite Areas

A composite area consists of a series of connected "simpler" parts or shapes, such as rectangles, triangles and circles. Provided the moment of inertia of each of these parts is known or can be determined about a common axis, then the moment of inertia for the composite area about this axis equals the ***algebraic sum*** of the moments of inertia of all its part.

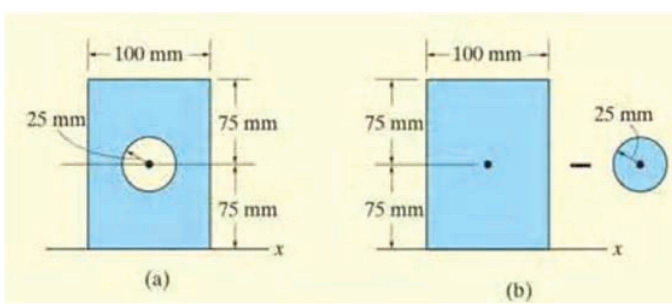
Procedure for Analysis:

The moment of inertia for a composite area about a reference axis can be determined using the following procedure.

- Using a sketch, divide the area into its composite parts and indicate the perpendicular distance from the centroid of each part to the reference axis.
- If the centroid axis for each part does not coincide with the reference axis, the parallel-axis theorem, $[I = \bar{I} + Ad^2]$ should be used to determine the moment of inertia of the part about the reference axis.
- The moment of inertia of the entire area about the reference axis is determined by summing the results of its composite parts about this axis.
- If a composite part has a "hole" its moment of inertia is found by "subtracting" the moment of inertia of the hole from the moment of inertia of the entire part including the hole.



Example 6: Determine the moment of inertia of the area shown in Figure (a) about the *x*-axis



Solution: The area can be obtained by *subtracting* the circle from the rectangle shown in Figure (b). The centroid of each area is located in the Figure.

<u>Circle</u>

$$I_x = \bar{I}_{x'} + Ad_y^2 = \frac{\pi(25)^4}{4} + \pi(25)^2(75)^2 \quad = 11.4 \times 10^6 \text{ mm}$$

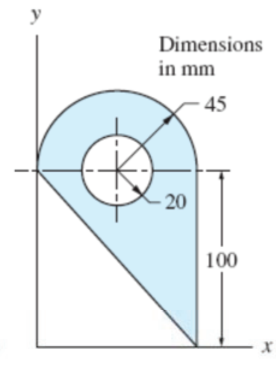
<u>Rectangle</u>

$$I_x = \bar{I}_{x'} + Ad_y^2 = \frac{(100)(150)^3}{12} + (100)(150)(75)^2 = 112.5 \times 10^6 \text{ mm}$$

The moment of inertia for the area is therefore

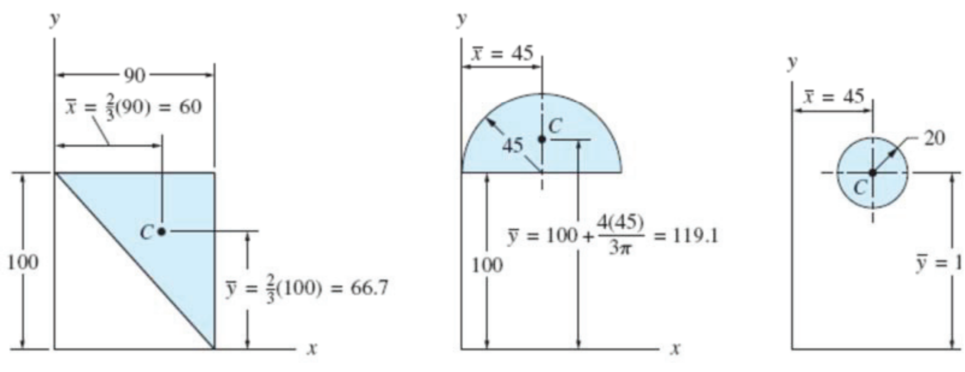
$$I_x = 112.5 \times 10^6 - 11.4 \times 10^6 = 101 \times 10^6 \text{ mm}$$

Example 7: For the area shown in Fig. (a), calculate the radii of gyration about the *x*- and *y*-axes



(a)

Solution: We consider the area to be composed of the three parts shown in Figs. (b)–(d):



(b) (c) (d)

<u>*Triangle:*</u>

$$A = \frac{bh}{2} = \frac{90(100)}{2} = 4500\, mm^2$$

$$\bar{I}_x = \frac{bh^3}{36} = \frac{90(100)^3}{36} = 2.50 \times 10^6\, mm^4$$

$$I_x = \bar{I}_x + A(d_y)^2 = (2.50 \times 10^6) + (4500)(66.7)^2 = 22.52 \times 10^6\, mm^4$$

$$\bar{I}_y = \frac{hb^3}{36} = \frac{100(90)^3}{36} = 2.025 \times 10^6\, mm^4$$

$$I_y = \bar{I}_y + A(d_x)^2 = (2.025 \times 10^6) + (4500)(60)^2 = 18.23 \times 10^6\, mm^4$$

Semicircle:

$$A = \frac{\pi R^2}{2} = \frac{\pi (45)^2}{2} = 3181\, mm^2$$

$$\bar{I}_x = 0.1098 R^4 = 0.1098(45)^4 = 0.450 \times 10^6\, mm^4$$

$$I_x = \bar{I}_x + A(d_y)^2 = (0.450 \times 10^6) + (3181)(119.1)^2 = 45.57 \times 10^6\, mm^4$$

$$\bar{I}_y = \frac{\pi R^4}{8} = \frac{\pi (45)^4}{8} = 1.61 \times 10^6\, mm^4$$

$$I_y = \bar{I}_y + A(d_x)^2 = (1.61 \times 10^6) + (3181)(45)^2 = 8.05 \times 10^6\, mm^4$$

Circle:

$$A = \pi R^2 = \pi (20)^2 = 1257\, mm^2$$

$$\bar{I}_x = \frac{\pi R^4}{4} = \frac{\pi (20)^4}{4} = 0.1257 \times 10^6\, mm^4$$

$$I_x = \bar{I}_x + A(d_y)^2 = (0.1257 \times 10^6) + (1257)(100)^2 = 12.70 \times 10^6\, mm^4$$

$$\bar{I}_y = \frac{\pi R^4}{4} = \frac{\pi (20)^4}{4} = 0.1257 \times 10^6\, mm^4$$

$$I_y = \bar{I}_y + A(d_x)^2 = (0.1257 \times 10^6) + (1257)(45)^2 = 2.67 \times 10^6\, mm^4$$

Composite Area

$$A = \Sigma A = 4500 + 3181 - 1257 = 6424 \text{ mm}^2$$

$$I_x = \Sigma I_x = (22.52 + 45.57 - 12.70) \times 10^6 = 55.39 \times 10^6 \text{ mm}^4$$

$$I_y = \Sigma I_y = (18.23 + 8.05 - 2.67) \times 10^6 = 23.61 \times 10^6 \text{ mm}^4$$

Therefore, for the radii of gyration we have

$$k_x = \sqrt{\frac{I_x}{A}} = \sqrt{\frac{55.39 \times 10^6}{6424}} = 92.9\, mm$$

$$k_y = \sqrt{\frac{I_y}{A}} = \sqrt{\frac{23.61 \times 10^6}{6424}} = 60.6\, mm$$