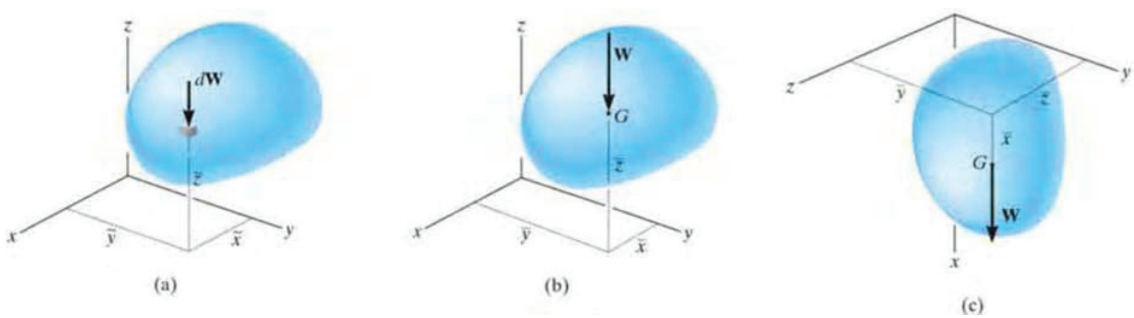
CENTROIDS AND CENTERS OF GRAVITY

1. Center of gravity

A body is composed of an infinite number of particles of differential size, and so if the body is located within a gravitational field, then each of these particles will have a weight *dW*. These weights will form an approximately parallel force system, and the resultant of this system is the total weight of the body, which pass through point called the ***center of gravity***.



The sum of the weights of all particles is,

$$+\downarrow F_R=\Sigma F_z \qquad \Longrightarrow \qquad W=\int dW$$

The location of center of gravity, measured from *y*-axis, is determined by equating the moment of *W* about *y*-axis to the sum of the moments of the weights of the particles about this same axis. If *dW* is located at point (*x*`, *y*`, *z*`) then,

$$(M_R)_y=\Sigma M_y; \qquad \bar{x}W=\int x`.dW$$

Similarly, if moments are summed about the *x*-axis,

$$(M_R)_x=\Sigma M_x; \qquad \bar{y}W=\int y`.dW$$

Finally, imagine that the body is fixed within the coordinate system and this system is rotated 90° about *y*-axis (Figure (c) above). Then the sum of the moment about *y*-axis gives,

$$(M_R)_y=\Sigma M_y; \qquad \bar{z}W=\int z`.dW$$

Therefore the location of the center of gravity *G* with respect to the *x,y,z* axis becomes,

$$\bar{x} = \frac{\int x`dW}{\int dW}, \qquad \bar{y} = \frac{\int y`dW}{\int dW}, \qquad \bar{z} = \frac{\int z`dW}{\int dW} \qquad \textbf{...(4-1)}$$

Here:

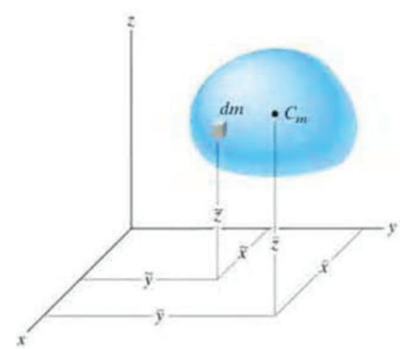
$\bar{x}, \bar{y}, \bar{z}$ are the coordinate of center of gravity *G,* (Figure (b) above).

$x`, y`, z`$ are the coordinate of each particle, (Figure (a) above).

2. Center of Mass of a Body:

The ***center of mass*** is a property of the distribution of mass within the body.

This location can be determined by substituting [$dW = g\ dm$] into Eqs. 4-1. Since *g* is constant, it cancels out, and so,

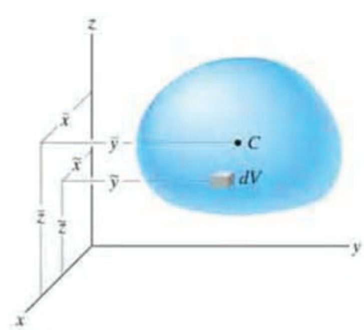


$$\bar{x} = \frac{\int x`dm}{\int dm}, \qquad \bar{y} = \frac{\int y`dm}{\int dm}, \qquad \bar{z} = \frac{\int z`dm}{\int dm} \qquad \textbf{...(4-2)}$$

3. Centroid:

The ***centroid*** is referred to as the geometric center of the region.

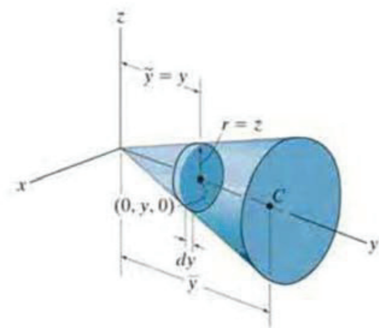
a. Centroid of a Volume:



If the volume in the figure is made from a homogeneous material, then its density ρ will be constant. Therefore, a differential element of volume dV has a mass [$dm = \rho.dV$]. Substituting this into Eqs. 4-2 and canceling out the constant ρ, we obtain formulas that locate the ***centroid C*** or geometric center of the body; namely,

$$\bar{x} = \frac{\int_V x^{`}.dV}{\int_V dV}, \qquad \bar{y} = \frac{\int_V y^{`}.dV}{\int_V dV}, \qquad \bar{z} = \frac{\int_V z^{`}.dV}{\int_V dV}$$

…(4-3)

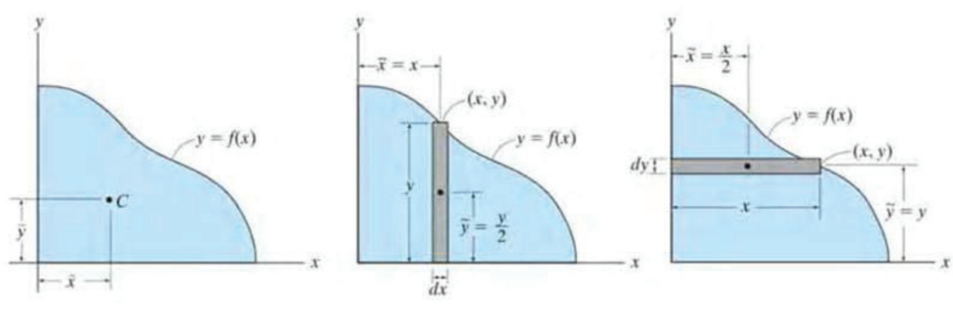


If the volume possesses two planes of symmetry, then its centroid must lie along the line of intersections of these planes. For example for the cone shown in Figure the centroid C must be located along y-axis, i.e. [$x = 0$ and $z = 0$]

b. Centroid of Area:

If an area lies in the x-y plane and is bounded by the curve $y = f(x)$, as shown in Figure (a) below, then its centroid will be in this plane and can be determined from integral similar to Eqs 4-3 namely,

$$\bar{x} = \frac{\int_A x^{`}.dA}{\int_A dA}, \qquad \bar{y} = \frac{\int_A y^{`}.dA}{\int_A dA} \qquad \text{…(4-4)}$$

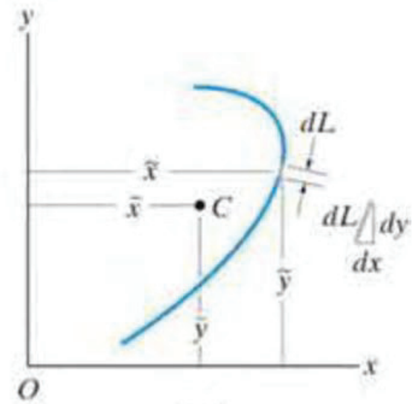


(a) (b) (c)

The terms $\int_A y^{`}.dA$ and $\int_A x^{`}.dA$ are the *first moments of the area* about the x- and y-axes respectively.

c. Centroid of a Line

If a line segment (or rod) lies within the *x-y* plane and it can be described by a thin curve $y = f(x)$, then its centroid is determined from,



(a)

$$\bar{x} = \frac{\int_L x`.dL}{\int_L dL}, \qquad \bar{y} = \frac{\int_L y`.dL}{\int_L dL} \qquad \textbf{...(4-5)}$$

Here, the length of the differential element is given by Pythagorean theorem,

$$dL = \sqrt{(dx)^2 + (dy)^2} \qquad \textbf{...(4-6a)}$$

Which also can be written in the form,

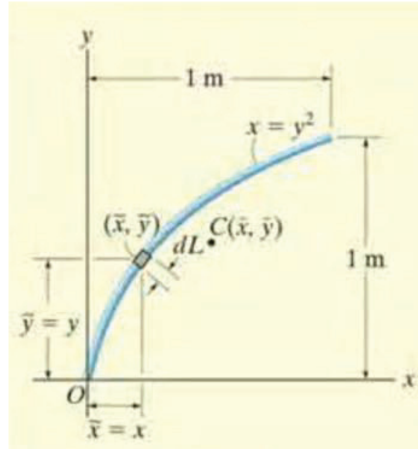
$$dL = \left(\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\right).dx \qquad \textbf{...(4-6b)}$$

or $$dL = \left(\sqrt{\left(\frac{dx}{dy}\right)^2 + 1}\right).dy \qquad \textbf{...(4-6c)}$$

***Note*:** Centroids and mass centers coincide only if the distribution of mass is uniform (ρ is constant), that is, if the body is homogeneous.

However, because weight and mass differ only by a constant factor (provided that the gravitational field is uniform), we find that the centers of mass and gravity coincide in most applications.

Example 1: Locate the centroid of the rod bent into the shape of a parabolic arc



Solution: Area and Moment Arms:

$$dL=\sqrt{(dx)^2+(dy)^2}=\left(\sqrt{\left(\frac{dx}{dy}\right)^2+1}\right).dy$$

Since $x = y^2$, then $\frac{dx}{dy}=2y$.

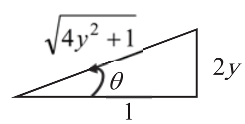
$$dL=\left(\sqrt{(2y)^2+1}\right).dy$$

The centroid of the element $x' = x,\ y' = y,$

$$\bar{x}=\frac{\int_L x'dL}{\int_L dL}=\frac{\int_0^1 x\sqrt{4y^2+1}dy}{\int_0^1\sqrt{4y^2+1}dy}=\frac{\int_0^1 y^2\sqrt{4y^2+1}dy}{\int_0^1\sqrt{4y^2+1}dy}=\frac{0.6063}{1.479}=0.410\,\text{m}$$

$$\bar{y}=\frac{\int_L y'dL}{\int_L dL}=\frac{\int_0^1 y\sqrt{4y^2+1}dy}{\int_0^1\sqrt{4y^2+1}dy}=\frac{0.8484}{1.479}=0.574\,\text{m}$$

To find the integral $\int\sqrt{4y^2+1}.dy$:



Let $2y=\tan\theta$ $\qquad 2dy=\sec^2\theta.d\theta$

$$\int\sqrt{4y^2+1}.dy=\int\sqrt{\tan^2\theta+1}\,\frac{\sec^2\theta}{2}d\theta=\frac{1}{2}\int\sqrt{\sec^2\theta}\sec^2\theta.d\theta$$

$$=\frac{1}{2}\int\sec\theta\sec^2\theta.d\theta$$

$u=\sec\theta$ $\qquad dv=\sec^2\theta.d\theta$

$du=\sec\theta\tan\theta.d\theta$ $\qquad v=\tan\theta$

$$\frac{1}{2}\int\sec\theta\sec^2\theta.d\theta=\frac{1}{2}\left[\sec\theta\tan\theta-\int\sec\theta\tan^2\theta.d\theta\right]$$

$$=\frac{1}{2}\left[\sec\theta\tan\theta-\int\sec\theta[\sec^2\theta-1].d\theta\right]$$

$$=\frac{1}{2}\left[\sec\theta\tan\theta-\int[\sec^3\theta-\sec\theta].d\theta\right]$$

$$=\frac{1}{2}[\sec\theta\tan\theta]-\frac{1}{2}\int[\sec^3\theta].d\theta+\frac{1}{2}\int\sec\theta.d\theta$$

$$\frac{1}{2}\int[\sec^3\theta].d\theta+\frac{1}{2}\int[\sec^3\theta].d\theta=\frac{1}{2}[\sec\theta\tan\theta]+\frac{1}{2}\int\sec\theta.d\theta$$

$$\int[\sec^3\theta].d\theta=\frac{1}{2}[\sec\theta\tan\theta]+\frac{1}{2}\ln|\sec\theta+\tan\theta|+C$$

$$\frac{1}{2}\int[\sec^3\theta].d\theta=\frac{1}{4}[\sec\theta\tan\theta]+\frac{1}{4}\ln|\sec\theta+\tan\theta|+C$$

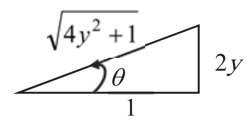
$$\frac{1}{2}\int\sec\theta\sec^2\theta.d\theta=\frac{1}{4}\left[2y\sqrt{4y^2+1}+\ln\left|\sqrt{4y^2+1}+2y\right|\right]+C$$

$$\int_0^1\sqrt{4y^2+1}.dy=\frac{1}{4}\left[\left(2\sqrt{4+1}+\ln\left|\sqrt{4+1}+2\right|\right)-\left(0+\ln\left|\sqrt{0+1}+0\right|\right)\right]=1.479$$

To find the integral $\int_0^1 y\sqrt{4y^2+1}.dy=\frac{1}{8}\int_0^1\left(4y^2+1\right)^{1/2}(8y.dy)$:

$$=\left[\frac{1}{8}\frac{\left(4y^2+1\right)^{3/2}}{3/2}\right]_0^1=\frac{1}{12}\left[(4+1)^{3/2}-(0+1)^{3/2}\right]=0.8484$$

To find the integral $\int y^2\sqrt{4y^2+1}.dy$:

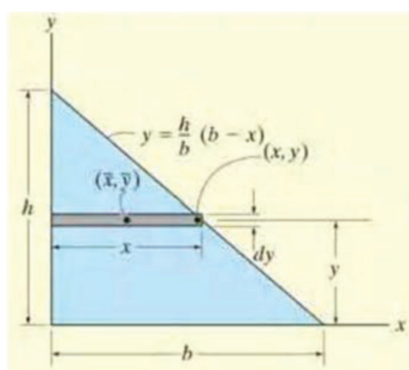


$$\int y^2\sqrt{4y^2+1}.dy=\int\left(\frac{\tan\theta}{2}\right)^2\sqrt{\sec^2\theta}\sec^2\theta.d\theta$$

$$=\frac{1}{2}\int\tan^2\theta\sec^3\theta.d\theta=\frac{1}{2}\int(\sec^2\theta-1)\sec^3\theta.d\theta$$

$$=\frac{1}{2}\int\sec^5\theta.d\theta-\frac{1}{2}\int\sec^3\theta.d\theta$$ and go on as the previous integrand.

Example 2: Determine the distance $\bar{y}$ measured from x-axis to the centroid of the area of the triangle shown in Figure below:



Solution: Use horizontal differential element

The area of the element is:

$$dA=xdy=\frac{b}{h}(h-y)dy,$$

and its centroid is located a distance $y'=y$ from the x-axis.

$$\bar{y}=\frac{\int_A y'dA}{\int_A dA}=\frac{\int_0^h y\left[\frac{b}{h}(h-y)\right].dy}{\int_0^h\frac{b}{h}(h-y).dy}=\frac{\frac{b}{h}\int_0^h(hy-y^2).dy}{\frac{b}{h}\int_0^h(h-y).dy}$$

$$= \frac{\frac{b}{h}\left[h\frac{y^2}{2}-\frac{y^3}{3}\right]_0^h}{\frac{b}{h}\left[hy-\frac{y^2}{2}\right]_0^h} = \frac{\frac{b}{h}\left[\frac{h^3}{2}-\frac{h^3}{3}\right]}{\frac{b}{h}\left[h^2-\frac{h^2}{2}\right]} = \frac{\frac{b}{h}\left[\frac{h^3}{6}\right]}{\frac{b}{h}\left[\frac{h^2}{2}\right]} = \frac{\frac{bh^2}{6}}{\frac{bh}{2}}$$

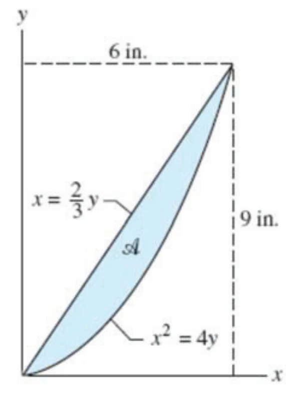
$$\bar{y} = \frac{h}{3}$$

Note**:** This result is valid for any shape of triangle. It states that the centroid is located at one-third the height, measured from the base of the triangle.

Example 3: Determine the coordinates of the centroid of the area that lies between the straight line $x = 2y/3$ and the parabola $x^2 = 4y$, where x and y are measured in inches [see Fig. (a)]. Use the following methods:

(1) Using a horizontal differential area element; and

(2) Using a vertical differential area element.



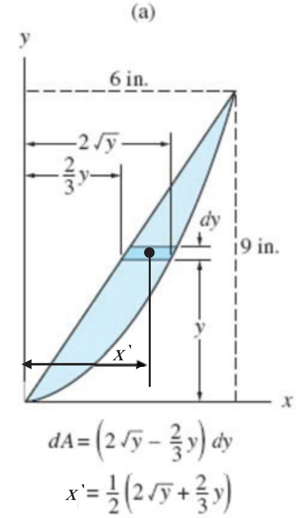
(a)

Solution: Part 1: Horizontal Differential Area Element

For the area we have

$$A = \int_A dA = \int_y (x_R - x_L)dy$$

$$= \int_0^9 \left(2\sqrt{y} - \frac{2y}{3}\right)dy = \left[\frac{4y^{3/2}}{3} - \frac{y^2}{3}\right]_0^9 = 9\,in^2$$



(b)

The first moment of the area about the y-axis becomes

$$M_y = \int_A x`dA = \int_y \frac{1}{2}\left(2\sqrt{y} + \frac{2y}{3}\right)(x_R - x_L).dy$$

$$= \frac{1}{2}\int_0^9 \left(2\sqrt{y} + \frac{2y}{3}\right)\left(2\sqrt{y} - \frac{2y}{3}\right)dy = \int_0^9 \left(2y - \frac{2y^2}{9}\right)dy$$

$$= \left[y^2 - \frac{2y^3}{27}\right]_0^9 = 27\,in^3$$

And the first moment about the x-axis is

$$M_x = \int_A y`dA = \int_y y(x_R - x_L).dy$$

$$= \int_0^9 y\left(2\sqrt{y} - \frac{2y}{3}\right)dy = \int_0^9 \left(2y^{3/2} - \frac{2y^2}{3}\right)dy$$

$$= \left[\frac{4y^{5/2}}{5} - \frac{2y^3}{9}\right]_0^9 = 32.4\,in^3$$

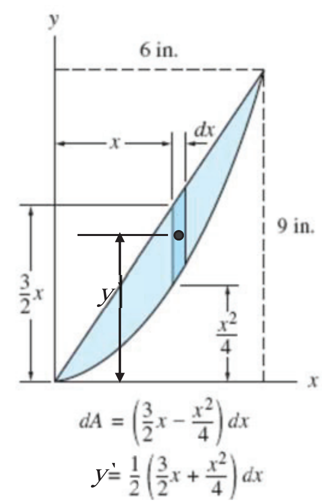
Therefore, the coordinates of the centroid of the area are

$$\bar{x} = \frac{M_y}{A} = \frac{27}{9} = 3\,in$$
$$\bar{y} = \frac{M_x}{A} = \frac{32.4}{9} = 3.6\,in$$

Part 2: Vertical Differential Area Element

The area of the region is

$$A = \int_A dA = \int_x (y_T - y_B)dx$$
$$= \int_0^6 \left(\frac{3x}{2} - \frac{x^2}{4}\right)dx = \left[\frac{3x^2}{4} - \frac{x^3}{12}\right]_0^6 = 9\,in^2$$



(c)

The first moment of the area about the y-axis becomes

$$M_y = \int_A x`dA = \int_x x`(y_T - y_B)dx$$
$$= \int_0^6 x\left(\frac{3x}{2} - \frac{x^2}{4}\right).dx$$
$$= \int_0^6 \left(\frac{3x^2}{2} - \frac{x^3}{4}\right).dx = \left[\frac{x^3}{2} - \frac{x^4}{16}\right]_0^8 = 27\,in^3$$

And the first moment about the x-axis is

$$M_x = \int_A y`dA = \int_x y`(y_T - x_B).dx$$

$$= \int_0^6 \frac{1}{2}\left(\frac{3x}{2} + \frac{x^2}{4}\right)\left(\frac{3x}{2} - \frac{x^2}{4}\right)dx$$

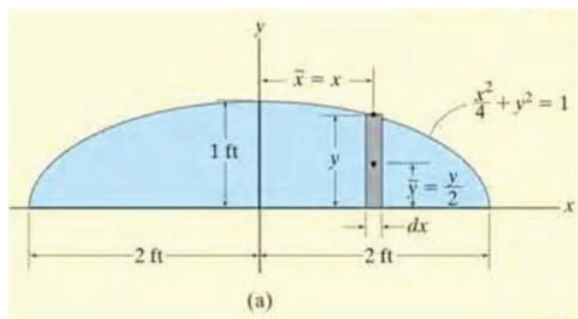
$$= \int_0^6 \frac{1}{2}\left(\frac{9x^2}{4} - \frac{x^4}{16}\right)dx = \frac{1}{2}\left[\frac{9x^3}{12} - \frac{x^5}{80}\right]_0^6 = 32.4\,in^3$$

Therefore, the coordinates of the centroid of the area are

$$\bar{x} = \frac{M_y}{A} = \frac{27}{9} = 3\,in$$

$$\bar{y} = \frac{M_x}{A} = \frac{32.4}{9} = 3.6\,in$$

Example 4: Locate the centroid of the semi-elliptical area shown below.



Solution I: The rectangular differential element parallel to the y-axis shown shaded in Figure (a) will be considered. This element has a thickness of $[t = dx]$ and a height of $[h = y]$.

Thus, the area is

$$dA = y.dx$$

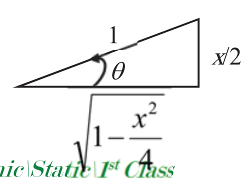
and its centroid is located at $[x' = x]$ and $[y' = y/2]$.

Since the area is symmetrical about y-axis, $\bar{x} = 0$

Integration: Apply $y = \sqrt{1 - \frac{x^2}{4}}$, we have

$$A = \int_A dA = \int_{-2}^{2} y.dx = \int_{-2}^{2} \sqrt{1 - \frac{x^2}{4}}.dx$$

Using trigonometric substitution:



Let $\frac{x}{2}=\sin\theta \qquad \frac{dx}{2}=\cos\theta.d\theta$

$$\therefore \int\sqrt{1-\frac{x^2}{4}}dx=\int\sqrt{1-\sin^2\theta}(2)\cos\theta.d\theta$$

$$=2\int\cos^2\theta.d\theta=2\int\frac{1+\cos 2\theta}{2}d\theta=\int d\theta+\int\cos 2\theta.d\theta$$

$$=\theta+\frac{\sin 2\theta}{2}+C=\theta+\frac{2\sin\theta\cos\theta}{2}+C=\theta+\sin\theta\cos\theta+C$$

$$=\sin^{-1}\left(\frac{x}{2}\right)+\frac{x}{2}\sqrt{1-\frac{x^2}{4}}+C$$

$$A=\int_A dA=\int_{-2}^{2}y.dx=\int_{-2}^{2}\sqrt{1-\frac{x^2}{4}}.dx=\left[\sin^{-1}\left(\frac{x}{2}\right)+\frac{x}{2}\sqrt{1-\frac{x^2}{4}}\right]_{-2}^{2}$$

$$=\left[\left(\sin^{-1}\left(\frac{2}{2}\right)+\frac{(2)}{2}\sqrt{1-\frac{2^2}{4}}\right)-\left(\sin^{-1}\left(\frac{(-2)}{2}\right)+\frac{(-2)}{2}\sqrt{1-\frac{(-2)^2}{4}}\right)\right]$$

$$=\frac{\pi}{2}-\left(-\frac{\pi}{2}\right)=\pi\ \text{ft}^2$$

$$M_x=\int_A y'dA=\int_{-2}^{2}\left(\frac{y}{2}\right)y.dx=\frac{1}{2}\int_{-2}^{2}\left(1-\frac{x^2}{4}\right)dx=\frac{1}{2}\left[x-\frac{x^3}{3*4}\right]_{-2}^{2}$$

$$=\frac{1}{2}\left[\left(2-\frac{2^3}{12}\right)-\left((-2)-\frac{(-2)^3}{12}\right)\right]=\frac{4}{3}\ \text{ft}^3$$

$$\bar{y}=\frac{\int_A y'dA}{\int_A dA}=\frac{M_x}{A}=\frac{4/3}{\pi}=0.424\ \text{ft}$$

Composite Bodies

A ***composite body*** consists of a series of connected "simpler" shaped bodies, which may be rectangular, triangular, semicircular, etc. Such body can often be sectioned or divided into its composite parts and, provided the *weight* and location of the center of gravity of each of these parts are known, we can then eliminate the need for integration to determine the center of gravity for the entire body. Therefore,

$$\bar{x} = \frac{\Sigma x`W}{\Sigma W}, \quad \bar{y} = \frac{\Sigma y`W}{\Sigma W}, \quad \bar{z} = \frac{\Sigma z`W}{\Sigma W} \qquad \text{...(4-6)}$$

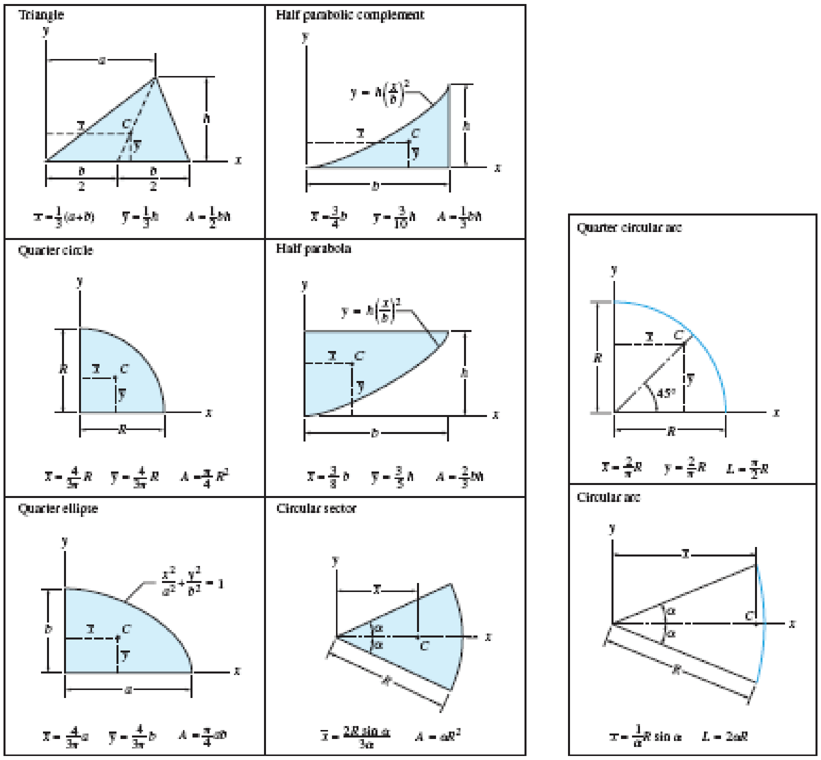
Here

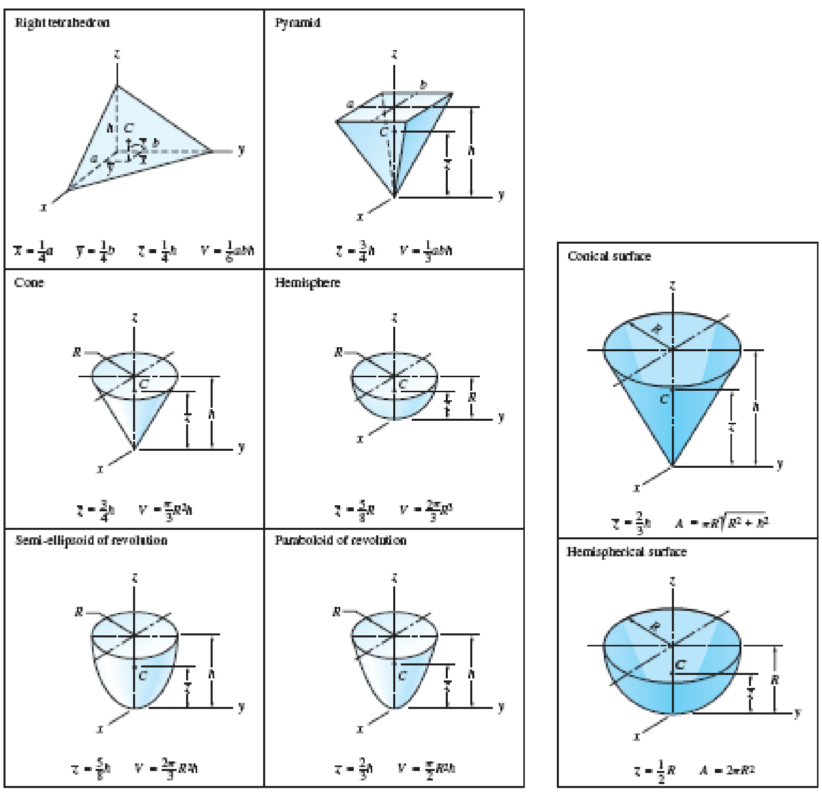
$\bar{x}, \bar{y}, \bar{z}$ represent the coordinate of center of gravity G of the composite body.

$x`, y`, z`$ represent the coordinate of center of gravity of each composite part of the body.

ΣW is the sum of the weights of all composite parts of the body or simply the total weight of the body.

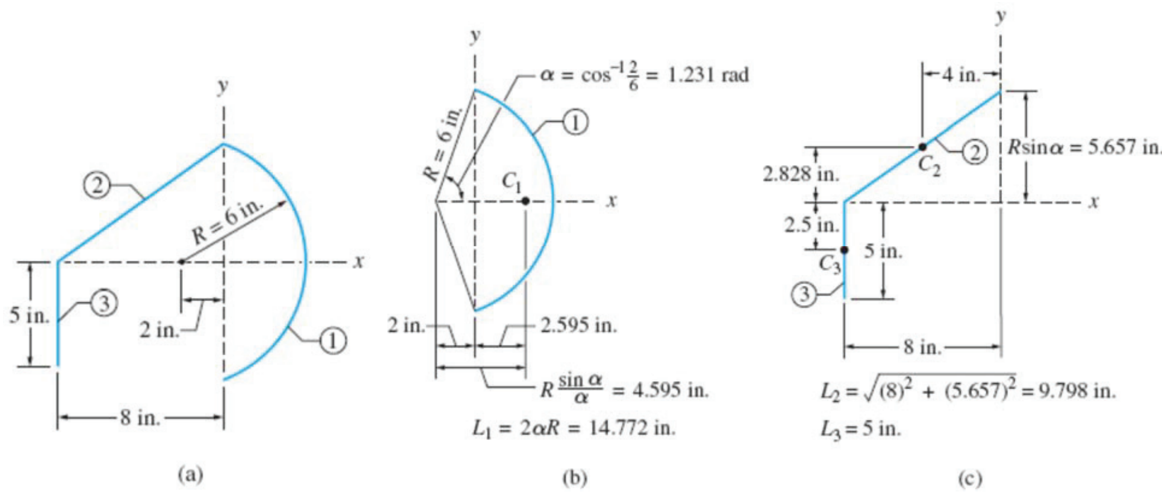
When the body has a *constant density,* or *specific weight*, the center of gravity coincides with the centroid of the body. The centroid for composite lines, area, and volumes can be found using relations analogous to Eqs. 4-6; however the W's are replaced L's, A's, and V's respectively.


Triangle
$\bar{x} = \frac{1}{3}(a+b)$ $\bar{y} = \frac{1}{3}h$ $A = \frac{1}{2}bh$
Half parabolic complement
$y = h\left(\frac{x}{b}\right)^2$
$\bar{x} = \frac{3}{4}b$ $\bar{y} = \frac{3}{10}h$ $A = \frac{1}{3}bh$
Quarter circle
$\bar{x} = \frac{4}{3\pi}R$ $\bar{y} = \frac{4}{3\pi}R$ $A = \frac{\pi}{4}R^2$
Half parabola
$y = h\left(\frac{x}{b}\right)^2$
$\bar{x} = \frac{3}{8}b$ $\bar{y} = \frac{3}{5}h$ $A = \frac{2}{3}bh$
Quarter ellipse
$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$
$\bar{x} = \frac{4}{3\pi}a$ $\bar{y} = \frac{4}{3\pi}b$ $A = \frac{\pi}{4}ab$
Circular sector
$\bar{x} = \frac{2R\sin\alpha}{3\alpha}$ $A = \alpha R^2$
Quarter circular arc
$\bar{x} = \frac{2}{\pi}R$ $\bar{y} = \frac{2}{\pi}R$ $L = \frac{\pi}{2}R$
Circular arc
$\bar{x} = \frac{1}{\alpha}R\sin\alpha$ $L = 2\alpha R$


Right tetrahedron
$\bar{x} = \frac{1}{4}a$ $\bar{y} = \frac{1}{4}b$ $\bar{z} = \frac{1}{4}h$ $V = \frac{1}{6}abh$
Pyramid
$\bar{z} = \frac{3}{4}h$ $V = \frac{1}{3}abh$
Cone
$\bar{z} = \frac{3}{4}h$ $V = \frac{\pi}{3}R^2h$
Hemisphere
$\bar{z} = \frac{5}{8}R$ $V = \frac{2\pi}{3}R^3$
Semi-ellipsoid of revolution
$\bar{z} = \frac{5}{8}h$ $V = \frac{2\pi}{3}R^2h$
Paraboloid of revolution
$\bar{z} = \frac{2}{3}h$ $V = \frac{\pi}{2}R^2h$
Conical surface
$\bar{z} = \frac{2}{3}h$ $A = \pi R\sqrt{R^2 + h^2}$
Hemispherical surface
$\bar{z} = \frac{1}{2}R$ $A = 2\pi R^2$

Example 7: Using the method of composite curves, determine the centroidal coordinates of the line in Fig. (a) that consists of the circular arc 1, and the straight lines 2 and 3.

Solution:



It is convenient to organize the analysis in tabular form, as follows:

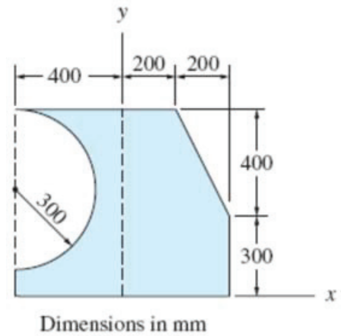
Segment	Length (in)	$x`$ (in)	$Lx`$ (in^2)	$y`$ (in)	**$Ly`$ (in^2)**
1	14.772	+2.595	+38.33	0	**0**
2	9.798	-4.0	-39.19	+2.828	**+27.71**
3	5	-8.0	-40.0	-2.5	**-12.50**
Σ	**29.570**	**…**	**-40.86**	**…**	**+15.21**

Therefore, the coordinates of the centroid of the composite curve are:

$$\bar{x} = \frac{\sum Lx`}{\sum L} = \frac{-40.86}{29.570} = -1.383\,in$$

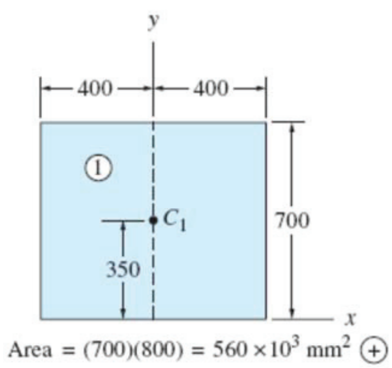
$$\bar{y} = \frac{\sum Ly`}{\sum L} = \frac{+15.21}{29.570} = +0.514\,in$$

Example 8: Using the method of composite areas, determine the location of the centroid of the shaded area shown in Fig. (a).

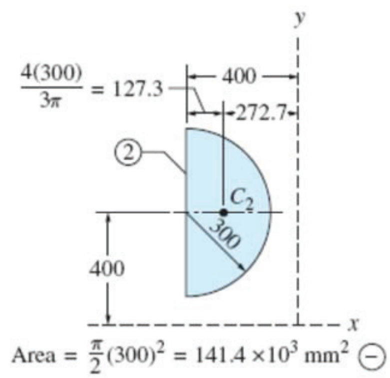


(a)

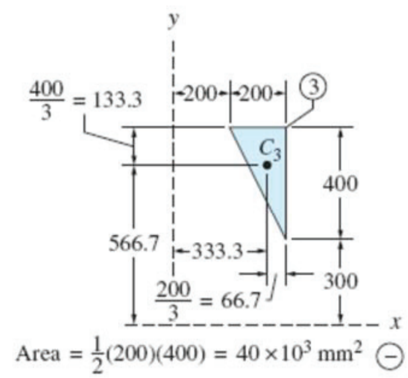
Solution:



(b)



(c)



(d)

Shape	Area (mm²)	x\` (mm)	Ax\` (mm³)	y\` (mm)	**Ay\` (mm³)**
1 (Rectangle)	+560.0 × 10³	0	0	+350.0	**196.0 × 10⁶**
2 (Semicircle)	-141.4 × 10³	-272.7	+38.56 × 10⁶	+400.0	**-56.56 × 10⁶**
3 (Triangle)	-40.0 × 10³	+333.3	-13.33 × 10⁶	+566.6	**-22.67 × 10⁶**
Σ	**+378.6 × 10³**	…	**+35.23 × 10⁶**	…	**+116.77 × 10⁶**

The centroid of the composite area are:

$$\bar{x} = \frac{\sum Ax`}{\sum A} = \frac{+35.23 \times 10^6}{+378.6 \times 10^3} = 66.6\,mm$$

$$\bar{y} = \frac{\sum Ay`}{\sum A} = \frac{+116.77 \times 10^6}{+378.6 \times 10^3} = 308\,mm$$