9- Interpolation and Extrapolation

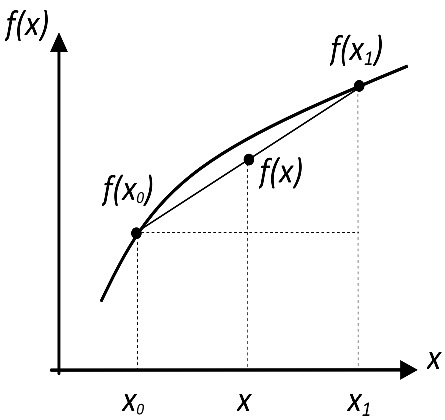
Introduction

By interpolation a functional value is approximated between the data points. While, by extrapolation a functional value is approximated beyond the data points.

The simplest form of interpolation is to connect two data points with a straight line then using similar triangles,

$$\frac{f(x)-f(x_o)}{x-x_o}=\frac{f(x_1)-f(x_o)}{x_1-x_o},$$

$$f(x)=f(x_o)+(x-x_o)\frac{f(x_1)-f(x_o)}{x_1-x_o}.$$



If $x_o=0$, then

$$f(x)=f(0)+x\frac{f(x_1)-f(x_o)}{h},$$

Or $$f(x)=f(0)+\frac{x}{h}\Delta f_o.$$

Interpolation with equally spaced data

1- Gregory-Newton forward interpolation formula

From Taylor series

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+\ldots\ldots.$$

Since $f'(0)=\frac{\Delta f_o}{h}-\frac{h}{2!}f''(0)-\frac{h^2}{3!}f'''(0)-\ldots\ldots,$

and $f''(0)=\frac{\Delta^2 f_o}{h^2}-hf'''(0)-\ldots\ldots,$

$$f(x)=f(0)+\frac{x}{h}\Delta f_o+\frac{x(x-h)}{2!h^2}\Delta^2 f_o+\frac{x(x-h)(x-2h)}{3!h^3}\Delta^3 f_o+\ldots\ldots \text{ (General formula)}$$

If $h=1$,

$$f(x)=f(0)+x\Delta f_o+\frac{x(x-1)}{2!}\Delta^2 f_o+\frac{x(x-1)(x-2)}{3!}\Delta^3 f_o+\ldots\ldots \text{ (Particular formula)}$$

2- Lagrange interpolation polynomial

The Lagrange interpolation polynomial is simply a reformulation of the Gregory-Newton polynomial that avoids the computation of divided differences. It can be represented as

$$f_n(x) = \sum_{i=0}^{n} L_i(x) . f(x_i) ,$$

where $L_i(x) = \prod_{\substack{i=0 \\ j \neq i}}^{n} \frac{x - x_j}{x - x_i}$. ($\prod$ designates the " *product of* ")

Or

$$f(x) = \frac{(x - x_1)(x - x_2)\ldots(x - x_n)}{(x_o - x_1)(x_o - x_2)\ldots(x_o - x_n)} f(x_o) + \frac{(x - x_o)(x - x_2)\ldots(x - x_n)}{(x_1 - x_o)(x_1 - x_2)\ldots(x_1 - x_n)} f(x_1) +$$

$$+ \ldots\ldots\ldots\ldots + \frac{(x - x_o)(x - x_1)\ldots(x - x_{n-1})}{(x_n - x_o)(x_n - x_1)\ldots(x_n - x_{n-1})} f(x_n) \; .$$

Example 1: Given the following data:

x	0	1	2	3
$f(x)$	- 7	- 3	6	25

Find $f(1.1)$ and $f(3.5)$.

Solution:

***Solution I*:** By Gregory-Newton interpolation formula,

Since $h = 1 \Rightarrow$ we can use the particular formula directly (rescaling is not required).

$x_o = 0 \Rightarrow$ Shifting is not required.

x	$f(x)$	Δf	$\Delta^2 f$	$\Delta^3 f$
0	- 7	4	5	5
1	- 3	9	10	
2	6	19		
3	25			

$$f(x) = f(0) + x\Delta f_o + \frac{x(x-1)}{2!} \Delta^2 f_o + \frac{x(x-1)(x-2)}{3!} \Delta^3 f_o + \ldots\ldots.$$

To get the most accurate interpolation we choose the first row, in the above forward differences table, as the base line (since it contains more entries).

$$f(x) = -7 + x(4) + \frac{x(x-1)}{2}(5) + \frac{x(x-1)(x-2)}{6}(5) .$$

$$\therefore \quad f(1.1) = -7 + 1.1(4) + \frac{1.1(1.1-1)}{2}(5) + \frac{1.1(1.1-1)(1.1-2)}{6}(5) = -2.4075.$$

$$f(3.5) = -7 + 3.5(4) + \frac{3.5(3.5-1)}{2}(5) + \frac{3.5(3.5-1)(3.5-2)}{6}(5) = 39.8125.$$

***Solution II*:** By Lagrange interpolation formula,

$$f(x) = \frac{(x-x_1)(x-x_2)\ldots(x-x_n)}{(x_o-x_1)(x_o-x_2)\ldots(x_o-x_n)} f(x_o) + \frac{(x-x_o)(x-x_2)\ldots(x-x_n)}{(x_1-x_o)(x_1-x_2)\ldots(x_1-x_n)} f(x_1) + \ldots$$

$$f(x) = \frac{(x-1)(x-2)(x-3)}{(0-1)(0-2)(0-3)}(-7) + \frac{(x-0)(x-2)(x-3)}{(1-0)(1-2)(1-3)}(-3) + \frac{(x-0)(x-1)(x-3)}{(2-0)(2-1)(2-3)}(6) + \frac{(x-0)(x-1)(x-2)}{(3-0)(3-1)(3-2)}(25).$$

$$\therefore \quad f(1.1) = \frac{(1.1-1)(1.1-2)(1.1-3)}{(0-1)(0-2)(0-3)}(-7) + \frac{(1.1-0)(1.1-2)(1.1-3)}{(1-0)(1-2)(1-3)}(-3) + \frac{(1.1-0)(1.1-1)(1.1-3)}{(2-0)(2-1)(2-3)}(6) + \frac{(1.1-0)(1.1-1)(1.1-2)}{(3-0)(3-1)(3-2)}(25) = -2.4075.$$

$$f(3.5) = \frac{(3.5-1)(3.5-2)(3.5-3)}{(0-1)(0-2)(0-3)}(-7) + \frac{(3.5-0)(3.5-2)(3.5-3)}{(1-0)(1-2)(1-3)}(-3) + \frac{(3.5-0)(3.5-1)(3.5-3)}{(2-0)(2-1)(2-3)}(6) + \frac{(3.5-0)(3.5-1)(3.5-2)}{(3-0)(3-1)(3-2)}(25) = 39.8125.$$

<u>Example 2</u>: Approximate the functional value at $x = 4.31$.

x	1	2	3	4	5
$f(x)$	6	10	46	138	430

Solution:

By Gregory-Newton interpolation formula,

Since $h=1 \Rightarrow$ Rescaling is not required. (we can use the particular formula directly)

$x_o \neq 0 \Rightarrow$ Shifting is required.

x	$x_{shifted}$	$f(x)$	Δf	$\Delta^2 f$	$\Delta^3 f$	$\Delta^4 f$
1	0	6	4	32	24	120
2	1	10	36	56	144	
3	2	46	92	200		
4	3	138	292			
5	4	430				

$$f(x) = f(0) + x\Delta f_o + \frac{x(x-1)}{2!}\Delta^2 f_o + \frac{x(x-1)(x-2)}{3!}\Delta^3 f_o + \ldots\ldots.$$

$$f(x) = 6 + x(4) + \frac{x(x-1)}{2}(32) + \frac{x(x-1)(x-2)}{6}(24) + \frac{x(x-1)(x-2)(x-3)}{24}(120).$$

At $x_{old} = 4.31 \quad \Rightarrow \quad x_{new} = 4.31 - 1 = 3.31,$

$$\therefore \quad f(x_{new}) = 6 + 3.31(4) + \frac{3.31(3.31-1)}{2}(32) + \frac{3.31(3.31-1)(3.31-2)}{6}(24) +$$

$$+ \frac{3.31(3.31-1)(3.31-2)(3.31-3)}{24}(120) = 197.16857.$$

$$\therefore \quad f(4.31) = 197.16857.$$