**Example 3**: If the curve $y = a + bx + \frac{c}{x}$ is to be used to represent the points $(1,4.5)$, $(2,4.75)$, and $(4,7.125)$, find the values of *a*, *b*, and *c* by using linear least squares regression.

**Solution:**

Since the given curve is not a polynomial, we cannot use the general least squares matrix, and we must use the general least squares derivation.

Let the approximating equation (model) $g(x) = a + bx + \frac{c}{x}$.

The local error is $e_i = g_i - y_i$ and the total error is $E = \sum_{i=1}^{n} e_i^2$ which must be minimized by letting $\frac{\partial E}{\partial a} = 0$, $\frac{\partial E}{\partial b} = 0$, and $\frac{\partial E}{\partial c} = 0$.

$$E = \sum_{i=1}^{n} (g_i - y_i)^2 \quad \Rightarrow \quad E = \sum_{i=1}^{n} (a + bx_i + \frac{c}{x_i} - y_i)^2 ,$$

$$\frac{\partial E}{\partial a} = 2\sum_{i=1}^{n} (a + bx_i + \frac{c}{x_i} - y_i), \quad \frac{\partial E}{\partial a} = 0 \quad \Rightarrow \quad 2\sum_{i=1}^{n} (a + bx_i + \frac{c}{x_i} - y_i) = 0,$$

$$\sum_{i=1}^{n} a + \sum_{i=1}^{n} bx_i + \sum_{i=1}^{n} \frac{c}{x_i} - \sum_{i=1}^{n} y_i = 0, \qquad \text{but} \quad \sum_{i=1}^{n} a = n.a ,$$

$$\therefore \quad n.a + \sum_{i=1}^{n} bx_i + \sum_{i=1}^{n} \frac{c}{x_i} = \sum_{i=1}^{n} y_i . \qquad \ldots\ldots\ldots\ldots\ldots (1)$$

$$\frac{\partial E}{\partial b} = 2\sum_{i=1}^{n} (a + bx_i + \frac{c}{x_i} - y_i)x_i , \quad \frac{\partial E}{\partial b} = 0 \quad \Rightarrow \quad 2\sum_{i=1}^{n} (ax_i + bx_i^2 + c - x_i y_i) = 0,$$

$$\sum_{i=1}^{n} x_i a + \sum_{i=1}^{n} x_i^2 b + \sum_{i=1}^{n} c - \sum_{i=1}^{n} x_i y_i = 0, \qquad \text{but} \quad \sum_{i=1}^{n} c = n.c ,$$

$$\therefore \quad \sum_{i=1}^{n} x_i a + \sum_{i=1}^{n} x_i^2 b + n.c = \sum_{i=1}^{n} x_i y_i \qquad \ldots\ldots\ldots\ldots\ldots (2)$$

$$\frac{\partial E}{\partial c} = 2\sum_{i=1}^{n} (a + bx_i + \frac{c}{x_i} - y_i)\frac{1}{x_i}, \quad \frac{\partial E}{\partial c} = 0 \quad \Rightarrow \quad 2\sum_{i=1}^{n} (\frac{1}{x_i} a + b + \frac{1}{x_i^2} c - \frac{1}{x_i} y_i) = 0,$$

$$\sum_{i=1}^{n} \frac{1}{x_i} a + \sum_{i=1}^{n} b + \sum_{i=1}^{n} \frac{1}{x_i^2} c - \sum_{i=1}^{n} \frac{y_i}{x_i} = 0, \qquad \text{but} \quad \sum_{i=1}^{n} b = n.b ,$$

$$\therefore \quad \sum_{i=1}^{n} \frac{1}{x_i} a + n.b + \sum_{i=1}^{n} \frac{1}{x_i^2} c = \sum_{i=1}^{n} \frac{y_i}{x_i} . \qquad \ldots\ldots\ldots\ldots\ldots (3)$$

In matrix form:

$$\begin{bmatrix} n & \sum_{i=1}^{n} x_i & \sum_{i=1}^{n} \frac{1}{x_i} \\ \sum_{i=1}^{n} x_i & \sum_{i=1}^{n} x_i^2 & n \\ \sum_{i=1}^{n} \frac{1}{x_i} & n & \sum_{i=1}^{n} \frac{1}{x_i^2} \end{bmatrix} \begin{Bmatrix} a \\ b \\ c \end{Bmatrix} = \begin{bmatrix} \sum_{i=1}^{n} y_i \\ \sum_{i=1}^{n} x_i y_i \\ \sum_{i=1}^{n} \frac{y_i}{x_i} \end{bmatrix},$$

$n = 3$, $\sum_{i=1}^{n} x_i = 1 + 2 + 4 = 7$, $\sum_{i=1}^{n} x_i^2 = 1^2 + 2^2 + 4^2 = 21$,

$\sum_{i=1}^{n} \frac{1}{x_i} = \frac{1}{1} + \frac{1}{2} + \frac{1}{4} = 1.75$, $\sum_{i=1}^{n} \frac{1}{x_i^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{4^2} = 1.3125$,

$\sum_{i=1}^{n} y_i = 4.5 + 4.75 + 7.125 = 16.375$, $\sum_{i=1}^{n} x_i y_i = 1(4.5) + 2(4.75) + 4(7.125) = 42.5$,

$\sum_{i=1}^{n} \frac{y_i}{x_i} = \frac{4.5}{1} + \frac{4.75}{2} + \frac{7.125}{4} = 8.65625$.

$$\therefore \begin{bmatrix} 3 & 7 & 1.75 \\ 7 & 21 & 3 \\ 1.75 & 3 & 1.3125 \end{bmatrix} \begin{Bmatrix} a \\ b \\ c \end{Bmatrix} = \begin{bmatrix} 16.375 \\ 42.5 \\ 8.65625 \end{bmatrix}.$$

Solving the above matrix, we get: $a = 0.5$, $b = 1.5$, and $c = 2.5$.

## **Non-polynomial models**

Linear least-squares regression may be used to fit a non-polynomial model by transforming it to a polynomial model, such as

* $y = \alpha e^{\beta x} \Rightarrow \ln y = \ln \alpha + \beta . x \Rightarrow y^* = a + \beta . x$ (polynomial),

where $y^* = \ln y$ and $a = \ln \alpha$.

* $y = \alpha x^{\beta} \Rightarrow \log y = \log \alpha + \beta \log x \Rightarrow y^* = a + \beta x^*$ (polynomial),

where $x^* = \log x$, $y^* = \log y$, and $a = \log \alpha$.

* $y = \frac{\alpha . x}{\beta + x} \Rightarrow \frac{1}{y} = \frac{\beta + x}{\alpha . x} \Rightarrow \frac{1}{y} = \frac{1}{\alpha} + \frac{\beta}{\alpha} . \frac{1}{x} \Rightarrow y^* = a + bx^*$ (polynomial),

where $x^* = \frac{1}{x}$, $y^* = \frac{1}{y}$, $a = \frac{1}{\alpha}$, and $b = \frac{\beta}{\alpha}$.

**Example**: The stress-strain data obtained from a compression test of a concrete cylinder is listed below. Perform a least-squares fit using the equation $\sigma = Ae^{B\varepsilon}$, where *A* and *B* are constants.

| Strain $\varepsilon$ $(\times 10^{-6})$ | 500 | 1000 | 1500 | 2000 | 2375 |
|---|---|---|---|---|---|
| Stress $\sigma$ (MPa) | 15.5 | 24.6 | 29.3 | 30.3 | 30.6 |

**Solution:**

Since the given model $\sigma = Ae^{B\varepsilon}$ is a non-polynomial, thus we must first transform it to a polynomial form.

$$\sigma = Ae^{B\varepsilon} \quad \Rightarrow \quad \ln\sigma = \ln A + B\varepsilon \quad \Rightarrow \quad y = a + Bx \quad \text{(polynomial)},$$

where $x = \varepsilon$, $y = \ln\sigma$ and $a = \ln A$.

| $x$, $(=\varepsilon)$, $(\times 10^{-6})$ | 500 | 1000 | 1500 | 2000 | 2375 |
|---|---|---|---|---|---|
| $y$, $(=\ln\sigma)$ | 2.74084 | 3.202746 | 3.377588 | 3.411148 | 3.421 |

Now, use the least squares criterion,

$$\begin{bmatrix} n & \sum_{i=1}^{n} x_i \\ \sum_{i=1}^{n} x_i & \sum_{i=1}^{n} x_i^2 \end{bmatrix} \begin{Bmatrix} a \\ B \end{Bmatrix} = \begin{bmatrix} \sum_{i=1}^{n} y_i \\ \sum_{i=1}^{n} x_i y_i \end{bmatrix},$$

$n = 5$, $\quad \sum_{i=1}^{n} x_i = (500 + 1000 + 1500 + 2000 + 2375) \times 10^{-6} = 7375 \times 10^{-6}$,

$\sum_{i=1}^{n} x_i^2 = 1.314 \times 10^{-5}$, $\sum_{i=1}^{n} y_i = 16.153322$, and $\sum_{i=1}^{n} x_i y_i = 0.024587$.

$$\therefore \begin{bmatrix} 5 & 7375 \times 10^{-6} \\ 7375 \times 10^{-6} & 1.314 \times 10^{-5} \end{bmatrix} \begin{Bmatrix} a \\ B \end{Bmatrix} = \begin{bmatrix} 16.153322 \\ 0.024587 \end{bmatrix}.$$

Solving the above matrix, we get: $a = 2.734504$ and $B = 336.380242$.

But $a = \ln A \quad \Rightarrow \quad A = e^a \quad \Rightarrow \quad A = e^{2.734504} = 15.402102$.

$\therefore$ The required equation is $\sigma \approx 15.4 e^{336.38\varepsilon}$.