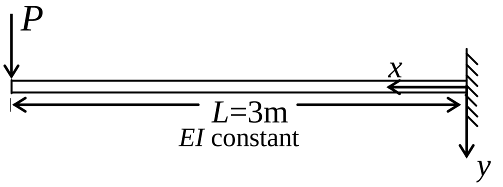
Example 2: For the shown cantilever, find numerically the deflection at the free end.

(Use $h = 1$ m)



Solution:

$$EIy'' = -M\ .$$

From left, $M = -P(L-x) \quad \Rightarrow \quad EIy'' = P(L-x)$,

Or $\quad y'' = \dfrac{P}{EI}(3-x)$, $\qquad y(0) = 0,\ \ y'(0) = 0$.

Solution I: By using the 2nd order Runge-Kutta method which is of $O(h)^2$.

We must first transform the problem into a set of two 1st order ODEs.

Let $\quad y' = z \qquad \Rightarrow \qquad z' = \dfrac{P}{EI}(3-x)$.

Put $f_1(z) = y' = z$ $\qquad$ (which is used to find y),

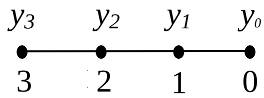
and $f_2(x) = z' = \dfrac{P}{EI}(3-x)$ (which is used to find z).

$$y_{j+1} = y_j + h.(k_2)_1 \quad \text{and} \quad z_{j+1} = z_j + h.(k_2)_2 \quad \text{where,}$$

$$(k_1)_1 = f_1(x_j, y_j, z_j) \quad \text{and} \quad (k_2)_1 = f_1(x_j + \frac{h}{2}, y_j + \frac{h}{2}(k_1)_1, z_j + \frac{h}{2}(k_1)_2).$$

$$(k_1)_2 = f_2(x_j, y_j, z_j) \quad \text{and} \quad (k_2)_2 = f_2(x_j + \frac{h}{2}, y_j + \frac{h}{2}(k_1)_1, z_j + \frac{h}{2}(k_1)_2).$$

Since $h = 1$ m, then we need three steps to move from the start point $x = 0$ to the end point $x = 3$ m.



Step 1: $\quad x_j = 0, \quad y_j = 0,$ and $\quad z_j = y'(x_j) = 0$.

$$(k_1)_1 = f_1(0,0,0) = 0,$$

$$(k_1)_2 = f_2(0,0,0) = \frac{P}{EI}(3-0) = \frac{3P}{EI},$$

$$(k_2)_1 = f_1((0+\frac{1}{2}),(0+\frac{1}{2}\times 0),(0+\frac{1}{2}\times\frac{3P}{EI})) = f_1(\frac{1}{2},0,\frac{3P}{2EI}) = \frac{3P}{2EI},$$

$$(k_2)_2 = f_2(\frac{1}{2},0,\frac{3P}{2EI}) = \frac{P}{EI}(3-\frac{1}{2}) = \frac{5P}{2EI},$$

$$\therefore y_1 = 0 + (1)(\frac{3P}{2EI}) = \frac{3P}{2EI}, \qquad \text{(deflection at } x = 1 \text{ m)}$$

$$\therefore z_1 = 0 + (1)(\frac{5P}{2EI}) = \frac{5P}{2EI}. \qquad \text{(slope at } x = 1 \text{ m)}$$

<u>Step 2</u>: $x_j = 1$, $y_j = \frac{3P}{2EI}$, and $z_j = \frac{5P}{2EI}$.

$$(k_1)_1 = f_1(1,\frac{3P}{2EI},\frac{5P}{2EI}) = \frac{5P}{2EI},$$

$$(k_1)_2 = f_2(1,\frac{3P}{2EI},\frac{5P}{2EI}) = \frac{P}{EI}(3-1) = \frac{2P}{EI},$$

$$(k_2)_1 = f_1((1+\frac{1}{2}),\ldots,(\frac{5P}{2EI}+\frac{1}{2}\times\frac{2P}{EI})) = f_1(\frac{3}{2},\ldots,\frac{7P}{2EI}) = \frac{7P}{2EI},$$

$$(k_2)_2 = f_2(\frac{3}{2},\ldots,\frac{7P}{2EI}) = \frac{P}{EI}(3-\frac{3}{2}) = \frac{3P}{2EI},$$

$$\therefore y_2 = \frac{3P}{2EI} + (1)(\frac{7P}{2EI}) = \frac{5P}{EI}, \qquad \text{(deflection at } x = 2 \text{ m)}$$

$$\therefore z_2 = \frac{5P}{2EI} + (1)(\frac{3P}{2EI}) = \frac{4P}{EI}. \qquad \text{(slope at } x = 2 \text{ m)}$$

<u>Step 3</u>: $x_j = 2$, $y_j = \frac{5P}{EI}$, and $z_j = \frac{4P}{EI}$.

$$(k_1)_1 = f_1(2,\frac{5P}{EI},\frac{4P}{EI}) = \frac{4P}{EI},$$

$$(k_1)_2 = f_2(2,\frac{5P}{EI},\frac{4P}{EI}) = \frac{P}{EI}(3-2) = \frac{P}{EI},$$

$$(k_2)_1 = f_1((2+\frac{1}{2}),\ldots,(\frac{4P}{EI}+\frac{1}{2}\times\frac{P}{EI})) = f_1(\frac{5}{2},\ldots,\frac{9P}{2EI}) = \frac{9P}{2EI},$$

$$(k_2)_2 = f_2(\frac{5}{2},\ldots,\frac{9P}{2EI}) = \frac{P}{EI}(3-\frac{5}{2}) = \frac{P}{2EI},$$

$$\therefore y_3 = \frac{5P}{EI} + (1)(\frac{9P}{2EI}) = \frac{19P}{2EI}, \qquad \text{(deflection at } x = 3 \text{ m)}$$

$$\therefore z_3 = \frac{4P}{EI} + (1)(\frac{P}{2EI}) = \frac{9P}{2EI}. \qquad \text{(slope at } x = 3 \text{ m)}$$

$\therefore$ The deflection at the free end is $y_3 \approx \frac{19P}{2EI}$.

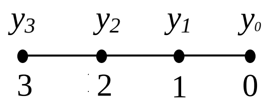
Solution II: By using the finite differences approximations:

For the obtained ODE, using central finite differences of $O(h)^2$ we get,

$f''_j = \dfrac{f_{j-1} - 2f_j + f_{j+1}}{h^2}$, substituting this derivative into the ODE yields,

$$\frac{y_{j-1} - 2y_j + y_{j+1}}{h^2} = \frac{P}{EI}(3 - x_j),$$

$$\therefore y_{j-1} - 2y_j + y_{j+1} = \frac{Ph^2}{EI}(3 - x_j)$$



(*Three unknowns: y_1, y_2, and y_3*)

At $x_j = 1$, (*Note*: from the first condition $y_0 = y(0) = 0$)

$$y_0 - 2y_1 + y_2 = \frac{P(1)^2}{EI}(3-1) \quad \Rightarrow \quad -2y_1 + y_2 = \frac{2P}{EI} \qquad \ldots\ldots(1)$$

At $x_j = 2$,

$$y_1 - 2y_2 + y_3 = \frac{P(1)^2}{EI}(3-2) \quad \Rightarrow \quad y_1 - 2y_2 + y_3 = \frac{P}{EI} \qquad \ldots\ldots(2)$$

For the second condition $y'(0) = 0$, using forward differences of $O(h)^2$, we get,

$f'_j = \dfrac{-3f_j + 4f_{j+1} - f_{j+2}}{2h}$, substituting into the 2nd condition yields:

$$\frac{-3y_0 + 4y_1 - y_2}{2h} = 0 \quad \Rightarrow \quad 4y_1 - y_2 = 0 \qquad \ldots\ldots(3)$$

In matrix form: $\begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 4 & -1 & 0 \end{bmatrix} \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \end{Bmatrix} = \begin{bmatrix} 2P/EI \\ P/EI \\ 0 \end{bmatrix}.$

Use Cramer's rule:

$$y_3 = \frac{\begin{vmatrix} -2 & 1 & 2P/EI \\ 1 & -2 & P/EI \\ 4 & -1 & 0 \end{vmatrix}}{\begin{vmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 4 & -1 & 0 \end{vmatrix}} = \frac{\frac{2P}{EI}\begin{vmatrix} 1 & -2 \\ 4 & -1 \end{vmatrix} + (-1)\frac{P}{EI}\begin{vmatrix} -2 & 1 \\ 4 & -1 \end{vmatrix} + 0}{0 + (-1)(1)\begin{vmatrix} -2 & 1 \\ 4 & -1 \end{vmatrix} + 0} = \frac{\frac{14P}{EI} + \frac{2P}{EI}}{2} = \frac{8P}{EI}.$$

$\therefore$ The deflection at the free end is $y_3 \approx \dfrac{8P}{EI}$.

II- Solution of boundary value problems

To solve this type of ordinary differential equations, finite differences approximations are used.

Example 1: Find $y(2)$ and $y(3)$: (Use $h=1$)

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}-y=0, \qquad y(1)=6,\ \ y(4)=9.$$

Solution:

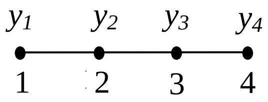
By using the finite differences approximations:

For the given ODE, using central finite differences approximations of $O(h)^2$ we get,

$$x_j^2\left(\frac{y_{j-1}-2y_j+y_{j+1}}{h^2}\right)+x_j\left(\frac{-y_{j-1}+y_{j+1}}{2h}\right)-y_j=0,$$

$$x_j^2(y_{j-1}-2y_j+y_{j+1})+\frac{hx_j}{2}(-y_{j-1}+y_{j+1})-h^2y_j=0,$$

$$(x_j^2-\frac{hx_j}{2})y_{j-1}-(2x_j^2+h^2)y_j+(x_j^2+\frac{hx_j}{2})y_{j+1}=0.$$



(*Two unknowns*)

At $x_j=2$, (*Note*: from the given conditions $y_1=6$ and $y_4=9$)

$$(2^2-\frac{1\times2}{2})y_1-(2\times2^2+1^2)y_2+(2^2+\frac{1\times2}{2})y_3=0,$$

$$3(6)-9y_2+5y_3=0 \quad\Rightarrow\quad -9y_2+5y_3=-18. \qquad \ldots\ldots(1)$$

At $x_j=3$,

$$(3^2-\frac{1\times3}{2})y_2-(2\times3^2+1^2)y_3+(3^2+\frac{1\times3}{2})y_4=0,$$

$$7.5y_2-19y_3+10.5(9)=0 \quad\Rightarrow\quad 7.5y_2-19y_3=-94.5. \qquad \ldots\ldots(2)$$

In matrix form: $\begin{bmatrix}-9 & 5\\ 7.5 & -19\end{bmatrix}\begin{Bmatrix}y_2\\ y_3\end{Bmatrix}=\begin{bmatrix}-18\\ -94.5\end{bmatrix}.$

Use Cramer's rule:

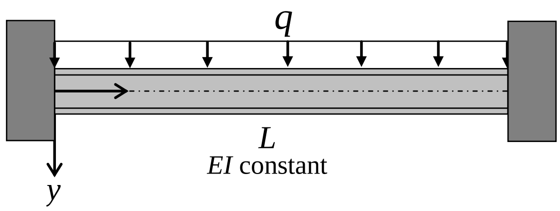
$$y_2=\frac{\begin{vmatrix}-18 & 5\\ -94.5 & -19\end{vmatrix}}{\begin{vmatrix}-9 & 5\\ 7.5 & -19\end{vmatrix}}=\frac{-18(-19)-5(-94.5)}{-9(-19)-5(7.5)}=\frac{814.5}{133.5}=6.101124.$$

$$y_3 = \frac{\begin{vmatrix} -9 & -18 \\ 7.5 & -94.5 \end{vmatrix}}{\begin{vmatrix} -9 & 5 \\ 7.5 & -19 \end{vmatrix}} = \frac{-9(-94.5)-(-18)(7.5)}{133.5} = \frac{985.5}{133.5} = 7.382023.$$

<u>*Note*</u>: The analytical solution is $y = \frac{4}{x} + 2x \;\Rightarrow\; y(2) = 6$ and $y(3) = 7.333333$.

<u>Example 2</u>: Estimate, numerically, the deflection at midspan. (Use $h = L/4$)

Solution:



$$EI\frac{d^4y}{dx^4} = w \quad\Rightarrow\quad EI\frac{d^4y}{dx^4} = q$$

or $\quad \frac{d^4y}{dx^4} = \frac{q}{EI}, \qquad y(0)=0,\; y'(0)=0,\; y(L)=0,$ and $y'(L)=0.$

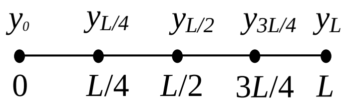
By using the finite differences approximations:

For the obtained ODE, using central finite differences of $O(h)^2$ we get,

$$\frac{f_{j-2} - 4f_{j-1} + 6f_j - 4f_{j+1} + f_{j+1}}{h^4}$$, substituting into the ODE yields:

$$\frac{y_{j-2} - 4y_{j-1} + 6y_j - 4y_{j+1} + y_{j+1}}{h^4} = \frac{q_j}{EI},$$

or $\quad y_{j-2} - 4y_{j-1} + 6y_j - 4y_{j+1} + y_{j+1} = \frac{q_j h^4}{EI}.$



At $x_j = L/2$, (*Note*: from the conditions $y_0 = 0$ and $y_L = 0$)

$$y_0 - 4y_{L/4} + 6y_{L/2} - 4y_{3L/4} + y_L = \frac{q(L/4)^4}{EI},$$

$$\therefore -4y_{L/4} + 6y_{L/2} - 4y_{3L/4} = \frac{qL^4}{256EI}. \qquad \text{.........(1)}$$

For the condition $y'(0) = 0$, using forward differences of $O(h)^2$, we get,

$$f'_j = \frac{-3f_j + 4f_{j+1} - f_{j+2}}{2h}$$, substituting into this condition yields:

$$\frac{-3y_0 + 4y_{L/4} - y_{L/2}}{2h} = 0 \quad \Rightarrow \quad 4y_{L/4} - y_{L/2} = 0. \qquad \ldots\ldots(2)$$

For the condition $y'(L) = 0$, using backward differences of $O(h)^2$, we get,

$f'_j = \dfrac{3f_j - 4f_{j-1} + f_{j-2}}{2h}$, substituting into this condition yields:

$$\therefore \frac{3y_L - 4y_{3L/4} + y_{L/2}}{2h} = 0 \quad \Rightarrow \quad y_{L/2} - 4y_{3L/4} = 0. \qquad \ldots\ldots(3)$$

In matrix form: $\begin{bmatrix} -4 & 6 & -4 \\ 4 & -1 & 0 \\ 0 & 1 & -4 \end{bmatrix} \begin{Bmatrix} y_{L/4} \\ y_{L/2} \\ y_{3L/4} \end{Bmatrix} = \begin{bmatrix} qL^4/256EI \\ 0 \\ 0 \end{bmatrix}$.

Use Cramer's rule:

$$y_{L/2} = \frac{\begin{vmatrix} -4 & \dfrac{qL^4}{256EI} & -4 \\ 4 & 0 & 0 \\ 0 & 0 & -4 \end{vmatrix}}{\begin{vmatrix} -4 & 6 & -4 \\ 4 & -1 & 0 \\ 0 & 1 & -4 \end{vmatrix}} = \frac{(-1)\dfrac{qL^4}{256EI}\begin{vmatrix} 4 & 0 \\ 0 & -4 \end{vmatrix} + 0}{-4\begin{vmatrix} -1 & 0 \\ 1 & -4 \end{vmatrix} + (-1)(4)\begin{vmatrix} 6 & -4 \\ 1 & -4 \end{vmatrix} + 0} = \frac{\dfrac{qL^4}{16EI}}{64} = \frac{qL^4}{1024EI}.$$

$\therefore$ The deflection at midspan is $y_{L/2} \approx \dfrac{qL^4}{1024EI}$.