I-b- Solution of a set of 1st order ODEs

To solve a set of ordinary differential equations we can use the previous methods (either Euler's or Runge-Kutta method).

Example : For the following set of ordinary differential equations, if at $x=0$, $y=4$ and $z=6$, then by one step of the 2nd order Runge-Kutta method, find y and z at $x=0.5$.

$$\frac{dy}{dx}=x-0.5y+z\,, \qquad \frac{dz}{dx}=x-y+2z\,.$$

Solution:

Let $f_1(x,y,z)=y'=x-0.5y+z$ (which is used to find y),

and $f_2(x,y,z)=z'=x-y+2z$ (which is used to find z).

From the start point $x=0$ to the end point $x=0.5$, by one step, we need a step size of $h=0.5$.

By using the 2nd order Runge-Kutta method,

$$y_{j+1} = y_j + h.(k_2)_1 \quad \text{and} \quad z_{j+1} = z_j + h.(k_2)_2 \quad \text{where,}$$

$$(k_1)_1 = f_1(x_j, y_j, z_j) \quad \text{and} \quad (k_2)_1 = f_1(x_j + \frac{h}{2}, y_j + \frac{h}{2}(k_1)_1, z_j + \frac{h}{2}(k_1)_2).$$

$$(k_1)_2 = f_2(x_j, y_j, z_j) \quad \text{and} \quad (k_2)_2 = f_2(x_j + \frac{h}{2}, y_j + \frac{h}{2}(k_1)_1, z_j + \frac{h}{2}(k_1)_2).$$

$x_j = 0$, $y_j = y(x_j) = y(0) = 4$, and $z_j = z(x_j) = z(0) = 6$.

$$(k_1)_1 = f_1(0,4,6) = 0 - 0.5(4) + 6 = 4,$$

$$(k_1)_2 = f_2(0,4,6) = 0 - 4 + 2(6) = 8,$$

$$(k_2)_1 = f_1((0 + \frac{0.5}{2}),(4 + \frac{0.5}{2} \times 4),(6 + \frac{0.5}{2} \times 8)) = f_1(0.25,5,8) = 0.25 - 0.5(5) + 8 = 5.75,$$

$$(k_2)_2 = f_2((0 + \frac{0.5}{2}),(4 + \frac{0.5}{2} \times 4),(6 + \frac{0.5}{2} \times 8)) = f_2(0.25,5,8) = 0.25 - 5 + 2(8) = 11.25,$$

$$\therefore y_{0.5} = 4 + (0.5)(5.75) = 6.875, \text{ and}$$

$$z_{0.5} = 6 + (0.5)(11.25) = 11.625.$$

I-c- Solution of second order ODEs

To solve a 2nd order ordinary differential equations we can use ***either*** the previous methods (but first we must transform the problem into a set of two 1st order ODEs.) ***or*** we use suitable finite differences approximations.

Example 1: Using $h = 0.1$, find $y(0.1)$ to $O(h)^2$ if

$$\frac{d^2y}{dt^2} = y + e^t, \qquad y(0) = 1, \ \frac{dy}{dt}(0) = 0.$$

Solution I: By using the 2nd order Runge-Kutta method which is of $O(h)^2$.

We must first transform the problem into a set of two 1st order ODEs.

Let $\frac{dy}{dt} = z \quad \Rightarrow \quad \frac{dz}{dt} = y + e^t$.

Put $f_1(z) = y' = z$ (which is used to find *y*),

and $f_2(t, y) = z' = y + e^t$ (which is used to find *z*).

$y_{j+1} = y_j + h.(k_2)_1$ and $z_{j+1} = z_j + h.(k_2)_2$ where,

$(k_1)_1 = f_1(t_j, y_j, z_j)$ and $(k_2)_1 = f_1(t_j + \frac{h}{2}, y_j + \frac{h}{2}(k_1)_1, z_j + \frac{h}{2}(k_1)_2)$.

$(k_1)_2 = f_2(t_j, y_j, z_j)$ and $(k_2)_2 = f_2(t_j + \frac{h}{2}, y_j + \frac{h}{2}(k_1)_1, z_j + \frac{h}{2}(k_1)_2)$.

Since $h = 0.1$, then we need one step to move from the start point $t = 0$ to the end point $t = 0.1$.

$$t_j = 0, \quad y_j = y(t_j) = y(0) = 1, \text{ and } \quad z_j = \frac{dy}{dt}(t_j) = 0.$$

$(k_1)_1 = f_1(0,1,0) = 0$,

$(k_1)_2 = f_2(0,1,0) = 1 + e^0 = 2$,

$(k_2)_1 = f_1((0 + \frac{0.1}{2}),(1 + \frac{0.1}{2} \times 0),(0 + \frac{0.1}{2} \times 2)) = f_1(0.05,1,0.1) = 0.1$,

$(k_2)_2 = f_2((0 + \frac{0.1}{2}),(1 + \frac{0.1}{2} \times 0),(0 + \frac{0.1}{2} \times 2)) = f_2(0.05,1,0.1) = 1 + e^{0.05} = 2.051271$,

$\therefore y_{0.1} = 1 + (0.1)(0.1) = 1.01$, and

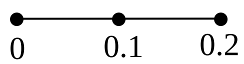
$z_{0.1} = 0 + (0.1)(2.051271) = 0.205127$. (Not required, representing the slope)

Solution II: By using the finite differences approximations:

For the given ODE, using central finite differences approximations of $O(h)^2$ we get,

$f''_j = \frac{f_{j-1} - 2f_j + f_{j+1}}{h^2}$, substituting this derivative into the given ODE yields,

$\frac{y_{j-1} - 2y_j + y_{j+1}}{h^2} = y_j + e^{t_j}$,



$\therefore y_{j-1} - (2 + h^2)y_j + y_{j+1} = h^2.e^{t_j}$.

At $t_j = 0.1$, (*Note*: from the first condition $y_0 = y(0) = 1$)

$y_0 - (2 + 0.1^2)y_{0.1} + y_{0.2} = (0.1)^2.e^{0.1} \Rightarrow -2.01y_{0.1} + y_{0.2} = -0.988948$ ……(1)

For the second condition $\frac{dy}{dt}(0) = 0$, using forward differences of $O(h)^2$, we get

$f'_j = \dfrac{-3f_j + 4f_{j+1} - f_{j+2}}{2h}$, substituting into the 2nd condition yields:

$$\frac{-3y_0 + 4y_{0.1} - y_{0.2}}{2h} = 0 \qquad \Rightarrow \qquad 4y_{0.1} - y_{0.2} = 3 \qquad \ldots\ldots(2)$$

Adding Eqs. (1) and (2) gives: $1.99y_{0.1} = 2.011052 \quad \Rightarrow \quad y_{0.1} = 1.010579$.