7- Numerical Solution of Ordinary Differential Equations

Introduction

An n^{th} order differential equation requires n conditions to obtain a unique solution. If all conditions are specified at the same value of the independent variable, then the problem is called an *initial value problem*, such as

$$y'' + 2y = \ln x, \qquad y(0) = 1 \text{ and } y'(0) = 0.$$

If the conditions are specified at different values of the independent variable, then it is a *boundary value problem*, such as

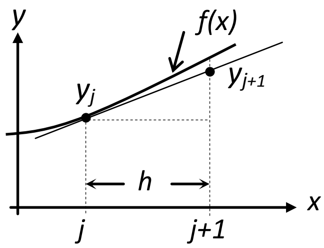
$$EIy'' = -M, \qquad y(0) = 0 \text{ and } y(L) = 0.$$

I- Solution of initial value problems

I-a- Solution of 1st order ODEs

Different numerical methods are used to solve 1st ordinary differential equations. Consider the following 1st order ordinary differential equation $y' = f(x, y)$:

1- Euler's method



From the figure $y'_j = \dfrac{y_{j+1} - y_j}{h}$,

$$\therefore \frac{y_{j+1} - y_j}{h} = f(x_j, y_j),$$

or $\quad y_{j+1} = y_j + h.f(x_j, y_j)$. (New value = old value + step size × slope)

<u>*Note*</u>: Euler's method gives approximations with an error of 1st order $O(h)$.

2- Second order Runge-Kutta method

$$y_{j+1} = y_j + h.k_2,$$

where $\quad k_1 = f(x_j, y_j) \quad$ and $\quad k_2 = f(x_j + \frac{h}{2}, y_j + \frac{h}{2}k_1)$.

<u>*Note*</u>: The 2nd order Runge-Kutta method gives approximations with an error of 2nd order $O(h)^2$.

3- Fourth order Runge-Kutta method

$$y_{j+1} = y_j + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4),$$

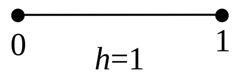
where $k_1 = f(x_j, y_j)$, $\qquad k_2 = f(x_j + \frac{h}{2}, y_j + \frac{h}{2}k_1)$,

$k_3 = f(x_j + \frac{h}{2}, y_j + \frac{h}{2}k_2)$, and $\quad k_4 = f(x_j + h, y_j + hk_3)$.

Note: The 4th order Runge-Kutta method gives approximations with an error of 4th order $O(h)^4$.

Example 1: Find $y(1)$ if $\frac{dy}{dx} = \frac{1}{2}(x - y)$, $\quad y(0) = 1$. $\quad$ (Use h = 1)

Solution:



The slope $f(x, y) = y' = \frac{1}{2}(x - y)$

With the given step size $h = 1$, we need one step to move from the start point $x = 0$ (where condition is given) to the end point $x = 1$ (where y is required).

Solution I: By Euler's method $\Rightarrow$ $y_{j+1} = y_j + h.f(x_j, y_j)$.

$x_j = 0$ and $\quad y_j = y(x_j) = y(0) = 1$,

$\therefore \quad y_1 = 1 + h.f(0,1) = 1 + (1)[\frac{1}{2}(0 - 1)] = 0.5$.

* From the analytical solution:

$y = x - 2 + 3e^{-x/2} \quad \Rightarrow \quad y(1) = 1 - 2 + 3e^{-1/2} = 0.819592$ [the (exact) answer].

* Percent relative error P= $\left|\frac{exact - approx.}{exact}\right| \times 100 = \left|\frac{0.819592 - 0.5}{0.819592}\right| \times 100 \approx 39\%$.

Solution II: By the 2nd order Runge-Kutta method $\Rightarrow$ $y_{j+1} = y_j + h.k_2$,

where $k_1 = f(x_j, y_j)$ and $\quad k_2 = f(x_j + \frac{h}{2}, y_j + \frac{h}{2}k_1)$.

$x_j = 0$ and $\quad y_j = y(x_j) = y(0) = 1$,

$k_1 = f(0,1) = \frac{1}{2}(0 - 1) = -0.5$,

$$k_2 = f((0+\frac{1}{2}),(1+\frac{1}{2}\times(-0.5))) = f(0.5,0.75) = \frac{1}{2}(0.5-0.75) = -0.125,$$

$$\therefore y_1 = 1 + (1)(-0.125) = 0.875.$$

* Percent relative error $P = \left|\frac{0.819592 - 0.875}{0.819592}\right| \times 100 \approx 6.8\%$.

Solution III: By the 4th order Runge-Kutta method,

$$y_{j+1} = y_j + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4),$$

where $k_1 = f(x_j, y_j)$, $\qquad k_2 = f(x_j + \frac{h}{2}, y_j + \frac{h}{2}k_1)$,

$k_3 = f(x_j + \frac{h}{2}, y_j + \frac{h}{2}k_2)$, and $\qquad k_4 = f(x_j + h, y_j + hk_3)$.

$x_j = 0$ and $\quad y_j = 1$,

$$k_1 = f(0,1) = \frac{1}{2}(0-1) = -0.5,$$

$$k_2 = f((0+\frac{1}{2}),(1+\frac{1}{2}\times(-0.5))) = f(0.5,0.75) = \frac{1}{2}(0.5-0.75) = -0.125,$$

$$k_3 = f((0+\frac{1}{2}),(1+\frac{1}{2}\times(-0.125))) = f(0.5,0.9375) = \frac{1}{2}(0.5-0.9375) = -0.21875,$$

$$k_4 = f((0+1),(1+1\times(-0.21875))) = f(1,0.78125) = \frac{1}{2}(1-0.78125) = 0.109375,$$

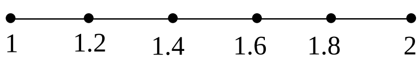
$$\therefore y_1 = 1 + \frac{1}{6}(-0.5 + 2(-0.125) + 2(-0.21875) + 0.109375) = 0.820313.$$

* Percent relative error $P = \left|\frac{0.819592 - 0.820313}{0.819592}\right| \times 100 \approx 0.09\%$.

Example 2: Use Euler's method to find y at $x = 2$, given that

$$dy = e^{x + 0.1y}dx, \qquad y(1) = 0. \qquad \text{(Use } h = 0.2\text{)}$$

Solution:



$$\frac{dy}{dx} = e^{x + 0.1y} \qquad \Rightarrow \qquad \text{The slope is } f(x,y) = e^{x + 0.1y}.$$

With the given step size $h = 0.2$, we need 5 steps to move from the start point $x = 1$ (where condition is given) to the end point $x = 2$ (where y is required).

Using Euler's method $\Rightarrow$ $y_{j+1} = y_j + h.f(x_j, y_j)$.

Step 1: $x_j = 1$ and $y_j = y(x_j) = y(1) = 0$,

$$y_{1.2} = y_1 + h.f(1,0) = 0 + 0.2 \times e^{1+0.1(0)} = 0.543656.$$

Step 2: $x_j = 1.2$ and $y_j = 0.543656$,

$$y_{1.4} = y_{1.2} + h.f(1.2, 0.543656) = 0.543656 + 0.2 \times e^{1.2+0.1(0.543656)} = 1.244779.$$

The calculations must be continued for 5 steps.

No. of Step (*j*)	x_j	y_j	$f(x_j, y_j) = e^{x_j + 0.1 y_j}$	$y_{j+1} = y_j + h.f(x_j, y_j)$
1	1	0	2.718282	0.543656
2	1.2	0.543656	3.505614	1.244779
3	1.4	1.244779	4.592745	2.163328
4	1.6	2.163328	6.149266	3.393181
5	1.8	3.393181	8.493644	5.091910

$\therefore \; y(2) \approx 5.091910$.