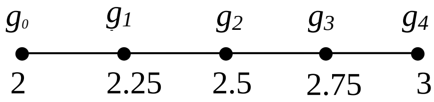
Example 5: Evaluate $I = \int_{2}^{3} \int_{x}^{2x^3} (x^2 + y)dydx$. (Use 4 segments in each direction)

Solution:

Let $f(x,y) = x^2 + y \quad \Rightarrow \quad g(x) = \int_{x}^{2x^3} f(x,y)dy$ (the inner integral)

$\therefore I = \int_{2}^{3} g(x)dx$. $\qquad h_x = \dfrac{3-2}{4} = 0.25$

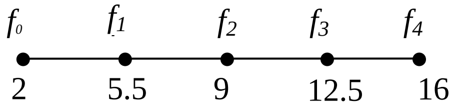


By using the Simpson's 1/3 rule, $I = \dfrac{h}{3}.(f_0 + 4\sum_{i=1,3,5,..}^{n-1} f_i + 2\sum_{i=2,4,6,..}^{n-2} f_i + f_n)$,

$$I = \frac{h_x}{3}.\{g_0 + 4(g_1 + g_3) + 2g_2 + g_4\},$$

$$= \frac{0.25}{3}.[g(2) + 4\{g(2.25) + g(2.75)\} + 2g(2.5) + g(3)].$$

To find $g(2)$: $\quad g(2) = \int_{2}^{2(2)^3} f(2,y)dy = \int_{2}^{16} f(2,y)dy$.



$h_y = \dfrac{16-2}{4} = 3.5$

$$\therefore g(2) = \frac{h_y}{3}.[f_0 + 4(f_1 + f_3) + 2f_2 + f_4]$$

$$= \frac{3.5}{3}.[f(2,2) + 4\{f(2,5.5) + f(2,12.5)\} + 2f(2,9) + f(2,16)]$$

$f(2,2) = 2^2 + 2 = 6$, $\quad f(2,5.5) = 2^2 + 5.5 = 9.5$, $\quad f(2,9) = 2^2 + 9 = 13$,
$f(2,12.5) = 2^2 + 12.5 = 16.5$, and $f(2,16) = 2^2 + 16 = 20$,

$$\therefore g(2) = \frac{3.5}{3}.[6 + 4(9.5 + 16.5) + 2(13) + 20] = 182.$$

Similarly,
$g(2.25) = 360.9009$, $\quad g(2.5) = 664.8438$, $\quad g(2.75) = 1154.995$, and $\quad g(3) = 1912.5$,

(Note: h_y is different for each of these inner integrals)

$$\therefore I \approx \frac{0.25}{3}.[182 + 4(360.9009 + 1154.995) + 2(664.8438) + 1912.5] \approx 790.6478.$$

The exact answer is: $I = \int_{2}^{3} \int_{x}^{2x^3} (x^2 + y)dydx = \int_{2}^{3} \left[x^2 y + \dfrac{y^2}{2} \right]_{x}^{2x^3} dx$,

$$= \int_{2}^{3} \left(2x^5 + 2x^6 - x^3 - \frac{x^2}{2} \right) dx = \left[\frac{2x^6}{6} + \frac{2x^7}{7} - \frac{x^4}{4} - \frac{x^3}{2(3)} \right]_{2}^{3} = 790.5357.$$

Romberg integration

This powerful and efficient numerical integration technique is based on the use of the trapezoidal rule combined with Richardson extrapolation. Richardson extrapolation is carried out according to:

$$I_k = \frac{1}{4^{k-1} - 1}.\left(4^{k-1} I_m - I_l\right),$$

where I_m and I_l are the more and less accurate integrals, respectively.

If $k = 2$, then $I_2 = \frac{1}{3}.\left(4I_m - I_l\right)$ which gives approximations with $O(h)^4$.

If $k = 3$, then $I_3 = \frac{1}{15}.\left(16I_m - I_l\right)$ which gives approximations with $O(h)^6$.

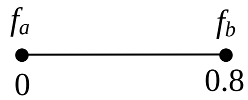
If $k = 4$, then $I_4 = \frac{1}{63}.\left(64I_m - I_l\right)$ which gives approximations with $O(h)^8$.

If $k = 5$, then $I_5 = \frac{1}{255}.\left(256I_m - I_l\right)$ which gives approximations with $O(h)^{10}$.

Example 1: Evaluate $\int_0^{0.8} e^{-x^2} dx$ using Romberg integration with an absolute convergence criterion of $\varepsilon = 10^{-6}$.

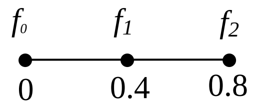
Solution:

1st iteration: Take $n = 1$ $\Rightarrow$ $h = \frac{b-a}{n} = \frac{0.8-0}{1} = 0.8$,



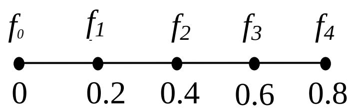
$$I = \frac{h}{2}.(f_a + f_b) = \frac{h}{2}.[f(0) + f(0.8)] = \frac{0.8}{2}.[e^{-(0)^2} + e^{-(0.8)^2}] = 0.610917.$$

2nd iteration: Take $n = 2$ $\Rightarrow$ $h = \frac{b-a}{n} = \frac{0.8-0}{2} = 0.4$,



$$I = \frac{h}{2}.(f_0 + 2f_1 + f_2) = \frac{h}{2}.[f(0) + 2f(0.4) + f(0.8)] = \frac{0.4}{2}.[e^{-(0)^2} + e^{-(0.4)^2} + e^{-(0.8)^2}] = 0.646316$$

3rd iteration: Take $n = 4$ $\Rightarrow$ $h = \frac{b-a}{n} = \frac{0.8-0}{4} = 0.2$,



$$I = \frac{h}{2}.(f_0 + \sum f_i + f_n) = \frac{h}{2}.[f(0) + 2\{f(0.2) + f(0.4) + f(0.6)\} + f(0.8)],$$

$$= \frac{0.2}{2}.[e^{-(0)^2} + 2\{e^{-(0.2)^2} + e^{-(0.4)^2} + e^{-(0.6)^2}\} + e^{-(0.8)^2}] = 0.654851.$$

The calculations must be continued until $\Delta \le \varepsilon$.

i	n	I_1	$I_2 = \frac{1}{3}.\left(4I_m - I_l\right)$	$I_3 = \frac{1}{15}.\left(16I_m - I_l\right)$	$I_4 = \frac{1}{63}.\left(64I_m - I_l\right)$
1	1	0.610917			
2	2	0.646316	0.658116		
3	4	0.654851	0.657696	0.657668	
4	8	0.656966	0.657671	0.657669	0.657669
$\lvert\Delta\rvert$			0.04….	4.4×10^{-4}	$1\times10^{-6}\le\varepsilon$

$$\therefore \int_0^{0.8} e^{-x^2}\,dx \approx 0.657669.$$

<u>Example 2</u>: (Final 2014) A rod is subjected to an axial tensile load and the stress-strain data, up to the point of rupture, is tabulated below. The area under the stress-strain curve, up to the point of rupture, is called the modulus of toughness. Compute this modulus to $O(h)^8$.

Strain, ε $(\times10^{-3})$	0	5	10	15	20	25	30	35	40
Stress, σ, (N/mm^2)	0	5	10	16	21	25	28	30	31

Solution:

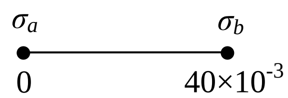
Since the modulus of toughness represents the area under the stress-strain curve,

$$\therefore \text{ the modulus of toughness } = \int_0^{40\times10^{-3}} \sigma.d\varepsilon$$

Since the answer is required to $O(h)^8$, then we must use Romberg integration .

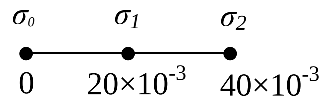
<u>1st iteration</u>: For $n = 1$ $\Rightarrow$ $h = \frac{b-a}{n} = \frac{40\times10^{-3}-0}{1} = 40\times10^{-3}$,

$$I = \frac{h}{2}.(\sigma_a + \sigma_b) = \frac{40\times10^{-3}}{2}.[0+31] = 0.62.$$



<u>2nd iteration:</u> Take $n = 2$ $\Rightarrow$ $h = \frac{b-a}{n} = \frac{40\times10^{-3}-0}{2} = 20\times10^{-3}$,

$$I = \frac{h}{2}.(\sigma_0 + 2\sigma_1 + \sigma_2) = \frac{20\times10^{-3}}{2}.[0+2(21)+31] = 0.73.$$



The calculations must be continued until the required order of error is achieved.

i	n	I_1	$I_2 = \frac{1}{3}.(4I_m - I_l)$	$I_3 = \frac{1}{15}.(16I_m - I_l)$	$I_4 = \frac{1}{63}.(64I_m - I_l)$
1	1	0.62			
2	2	0.73	0.766667		
3	4	0.745	0.75	0.748889	
4	8	0.7525	0.755	0.755333	0.755435
Order of error			$O(h)^4$	$O(h)^6$	$O(h)^8$

$\therefore$ The modulus of toughness ≈ 0.755435 N/mm^2.