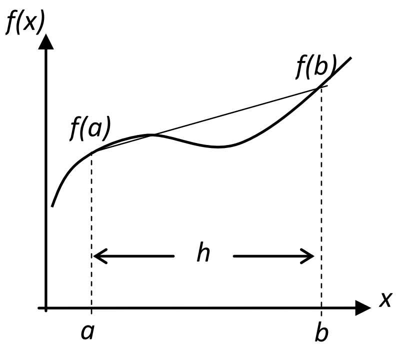
6- Numerical Integration

Introduction

The primary purpose of numerical integration (also called quadrature) is the evaluation of integrals which are either impossible or else very difficult to evaluate analytically. Numerical integration is also essential in the evaluation of integrals of functions available only at discrete points. Such functions often result from the numerical solution of differential equations or from experimental data taken at discrete intervals.

An integral of a given function represents the area enclosed by this function and the *x*-axis. So, evaluating this area is equivalent to evaluate the integral of this function. In the following, some of numerical techniques, which are used to evaluate an integral, are presented.



1- Trapezoidal rule

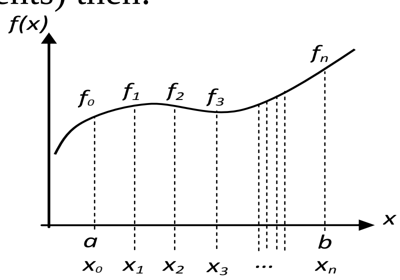
Consider the integral:

$$I = \int_a^b f(x)dx,$$

If $f(x)$ is replaced by a straight line (1st order polynomial) connecting two points, then the area under this function can be computed from:

$$I = \frac{h}{2}.(f_a + f_b).$$ [trapezoidal rule for one segment (panel)]

If we divide the interval [a,b] into n equal subintervals (segments) then:



$$h = \Delta x = \frac{b-a}{n},$$

$$I = \frac{h}{2}.(f_0 + f_1) + \frac{h}{2}.(f_1 + f_2) + ...+ \frac{h}{2}.(f_{n-1} + f_n),$$

or $$I = \frac{h}{2}.(f_0 + 2\sum_{i=1}^{n-1} f_i + f_n),$$ (trapezoidal rule for *n* segments)

where, $f_0 = f_a = f(a)$ and $f_n = f_b = f(b)$.

Notes:

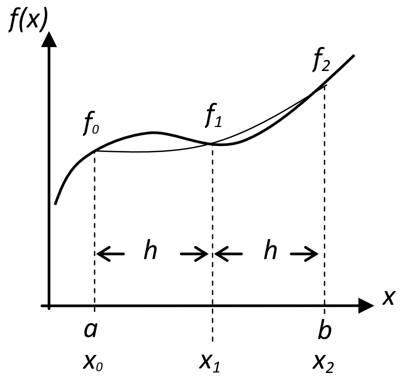
1- The trapezoidal rule gives an answer with an error of order $O(h)^2$.

2- The trapezoidal rule gives an answer which is exact for 1st degree polynomial and approximate for other polynomials of higher degree.

3- Reducing h will, in general, provide more accurate answers.

2- Simpson's rule

2.1- Simpson's 1/3 rule



If $f(x)$ is replaced by a 2nd order polynomial (parabola) connecting three points, then the area under this function can be computed from:

$I = \frac{h}{3}.(f_0 + 4f_1 + f_2)$.(Simpson's 1/3 rule for two segments)

If we divide the interval [a,b] into n equal subintervals (n is even) then:

$$I = \frac{h}{3}.(f_0 + 4f_1 + f_2) + \frac{h}{3}.(f_2 + 4f_3 + f_4) + ...+ \frac{h}{3}.(f_{n-2} + 4f_{n-1} + f_n),$$

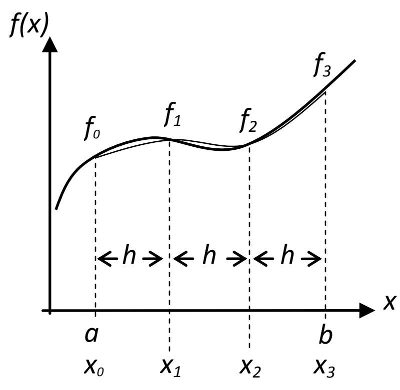
or $I = \frac{h}{3}.(f_0 + 4\sum_{i=1,3,5,..}^{n-1} f_i + 2\sum_{i=2,4,6,..}^{n-2} f_i + f_n)$. [Simpson's 1/3 rule for n (even) segments]

Notes:

1- Simpson's 1/3 rule gives answers with an error of order $O(h)^4$.

2- Simpson's 1/3 gives answers which are exact for polynomials of 2nd degree or lower and approximate for other polynomials of higher degree.

2.2- Simpson's 3/8 rule



If $f(x)$ is replaced by a 3rd order polynomial (cubic equation) connecting four points, then the area under this function can be computed from:

$$I = \frac{3h}{8}.(f_0 + 3f_1 + 3f_2 + f_3).$$ (Simpson's 3/8 rule for three segments)

If we divide the interval [a,b] into n equal subintervals (segments) then:

$$I = \frac{3h}{8}.(f_0 + 3f_1 + 3f_2 + f_3) + \frac{3h}{8}.(f_3 + 3f_4 + 3f_5 + f_6) + .. + \frac{3h}{8}.(f_{n-3} + 3f_{n-2} + 3f_{n-1} + f_n)$$

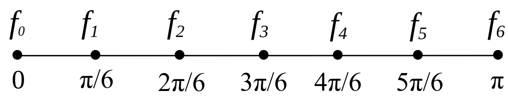
or $I = \frac{3h}{8}.[f_0 + 3(f_1 + f_2 + f_4 + f_5 + \ldots) + 2\sum_{i=3,6,9,..}^{n-4} f_i + f_n].$ (3/8 rule for n segments)

<u>*Notes*</u>:

1- Simpson's 3/8 rule gives answers with an error of order $O(h)^4$.

2- Simpson's 3/8 gives answers which are exact for polynomials of 3rd degree or lower and approximate for other polynomials of higher degree.

<u>Example 1</u>: Evaluate $I = \int_0^{\pi} \sin x dx$ using six segments. Compare with the exact answer.



Solution:

Since $n = 6$ $\Rightarrow$ $h = \Delta x = \frac{b-a}{n} = \frac{\pi - 0}{6} = \frac{\pi}{6}$.

By using the trapezoidal rule (which is of error of $O(h)^2$) $\Rightarrow I = \frac{h}{2}.(f_0 + 2\sum_{i=1}^{n-1} f_i + f_n)$,

$$I = \frac{\pi/6}{2}.\{f_0 + 2(f_1 + f_2 + f_3 + f_4 + f_5) + f_6\},$$

$$= \frac{\pi}{12}[\sin 0 + 2\{\sin(\pi/6) + \sin(2\pi/6) + \sin(3\pi/6) + \sin(4\pi/6) + \sin(5\pi/6)\} + \sin(\pi)]$$

$$\approx 1.954097.$$

The exact value is $I = [-\cos x]_0^{\pi} = (-\cos\pi) - (-\cos 0) = \{-(-1)\} - \{-(1)\} = 2$.

Percent relative error $P = \left|\frac{exact - approx.}{exact}\right| \times 100 = \left|\frac{2 - 1.954097}{2}\right| \times 100 = 2.3\%$.

<u>*Notes*</u>:

* If we use the Simpson's 1/3 rule (which is of error of $O(h)^4$) then,

$$I = \frac{h}{3}.(f_0 + 4\sum_{i=1,3,5,..}^{n-1} f_i + 2\sum_{i=2,4,..}^{n-2} f_i + f_n),$$

$$I = \frac{\pi/6}{3}.\{f_0 + 4(f_1 + f_3 + f_5) + 2(f_2 + f_4) + f_6\},$$

$$=\frac{\pi}{18}[\sin 0+4\{\sin(\pi/6)+\sin(3\pi/6)+\sin(5\pi/6)\}+2\{\sin(2\pi/6)+\sin(4\pi/6)\}+\sin(\pi)]$$

$$\approx 2.000863.$$

Percent relative error $P = \left|\frac{exact - approx.}{exact}\right|\times 100 = \left|\frac{2-2.000863}{2}\right|\times 100 = 0.04\%.$

* If we use the Simpson's 3/8 rule (which is of error of $O(h)^4$) then,

$$I=\frac{3h}{8}.[f_0+3(f_1+f_2+f_4+f_5+\ldots)+2\sum_{i=3,6,9,..}^{n-4} f_i+f_n],$$

$$I=\frac{3(\pi/6)}{8}.\{f_0+3(f_1+f_2+f_4+f_5)+2(f_3)+f_6\},$$

$$=\frac{\pi}{16}[\sin 0+3\{\sin(\pi/6)+\sin(2\pi/6)+\sin(4\pi/6)+\sin(5\pi/6)\}+2\sin(3\pi/6)+\sin(\pi)]$$

$$\approx 2.000005.$$

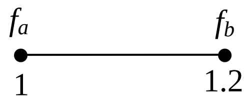
Percent relative error $P = \left|\frac{exact - approx.}{exact}\right|\times 100 = \left|\frac{2-2.000005}{2}\right|\times 100 = 0.0003\%.$

<u>Example 2</u>: Given the function $f(x)=(x+1)^x$, find $\int_1^{1.2} f(x)dx$ correct to three decimals.

Solution:

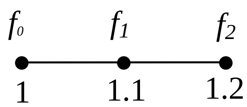
By using the trapezoidal rule,

<u>1^{st} iteration</u>: Take n = 1 $\Rightarrow$ $h=\frac{b-a}{n}=\frac{1.2-1}{1}=0.2,$



$$I=\frac{h}{2}.(f_a+f_b)=\frac{h}{2}.[f(1)+f(1.2)]=\frac{0.2}{2}.[(1+1)^1+(1.2+1)^{1.2}]=0.457577.$$

<u>2^{nd} iteration:</u> Take n = 2 $\Rightarrow$ $h=\frac{b-a}{n}=\frac{1.2-1}{2}=0.1,$



$$I=\frac{h}{2}.(f_0+2f_1+f_2)=\frac{h}{2}.[f(1)+2f(1.1)+f(1.2)]=\frac{0.1}{2}.[(1+1)^1+2(1.1+1)^{1.1}+(1.2+1)^{1.2}]=0.454962$$

The calculations must be continued until $\Delta \le \varepsilon$.

No. of Iteration (i)	h_i	I_i	$\Delta_i = \lvert I_i - I_{i-1} \rvert$
1	0.2	0.457577	----
2	0.1	0.454962	2.6×10^{-3}
3	0.05	0.454306	$6.5 \times 10^{-4} < \varepsilon$

$$\therefore \int_1^{1.2} f(x)dx \approx 0.454306.$$

<u>Example 3</u>: Evaluate $\int_0^{9.9} f(x)dx$ using the following data:

x	0	1.1	2.2	3.3	4.4	5.5	6.6	7.7	8.8	9.9
$f(x)$	0	0.6	0.8	0.6	0.1	- 0.2	- 0.1	0.1	0.3	0.4

Solution:

Here we have $n = 9$ and $h = 1.1$.

f_0	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8	f_9
0	1.1	2.2	3.3	4.4	5.5	6.6	7.7	8.8	9.9

<u>Solution I</u>: By using the trapezoidal rule $\Rightarrow$ $I = \frac{h}{2}.(f_0 + 2\sum_{i=1}^{n-1} f_i + f_n)$,

$$I = \frac{h}{2}.\{f_0 + 2(f_1 + f_2 + f_3 + f_4 + f_5 + f_6 + f_7 + f_8) + f_9\},$$

$$= \frac{1.1}{2}[0 + 2\{0.6 + 0.8 + 0.6 + 0.1 + (-0.2) + (-0.1) + 0.1 + 0.3\} + 0.4] = 2.64.$$

<u>Solution II</u>: Since n = 9 (odd), so we can not use the Simpson's 1/3 rule directly. Instead, we can apply it for the first 8 segments and the trapezoidal rule for the last segment:

$$I = \frac{h}{3}.\{f_0 + 4(f_1 + f_3 + f_5 + f_7) + 2(f_2 + f_4 + f_6) + f_8\} + \frac{h}{2}.(f_8 + f_9),$$

$$= \frac{1.1}{3}[0 + 4\{0.6 + 0.6 + (-0.2) + 0.1\} + 2\{0.8 + 0.1 + (-0.1)\} + 0.3] + \frac{1.1}{2}[0.3 + 0.4] = 2.695 \cdot$$

<u>Solution III</u>: We can apply the Simpson's 1/3 rule for the first 6 segments and the 3/8 rule for the last 3 segments, then:

$$I = \frac{h}{3}.\{f_0 + 4(f_1 + f_3 + f_5) + 2(f_2 + f_4) + f_6\} + \frac{3h}{8}.\{f_6 + 3(f_7 + f_8) + f_9\},$$

$$= \frac{1.1}{3}[0 + 4\{0.6 + 0.6 + (-0.2)\} + 2(0.8 + 0.1) + (-0.1)] + \frac{3(1.1)}{8}[-0.1 + 3(0.1 + 0.3) + 0.4] = 2.70875 \cdot$$

Solution IV: Since n = 9 = (3×3), so we can use the Simpson's 3/8 rule directly:

$$I=\frac{3h}{8}.\{f_0+3(f_1+f_2)+f_3\}+\frac{3h}{8}.\{f_3+3(f_4+f_5)+f_6\}+\frac{3h}{8}.\{f_6+3(f_7+f_8)+f_9\},$$

or $$I=\frac{3h}{8}.\{f_0+3(f_1+f_2+f_4+f_5+f_7+f_8)+2(f_3+f_6)+f_9\},$$

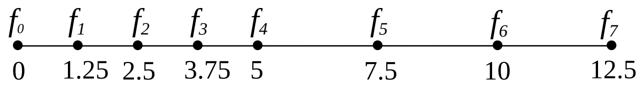
$$=\frac{3(1.1)}{8}[0+3\{0.6+0.8+0.1+(-0.2)+0.1+0.3\}+2\{0.6+(-0.1)\}+0.4]=2.68125.$$

Example 4: A rectangular swimming pool is (7.5 m) wide and (12.5 m) long. The depth of water (h) of distance (x) from one end of the pool is measured and found to be as follows:

Distance, x, (m)	0	1.25	2.5	3.75	5	7.5	10	12.5
Depth, h, (m)	1.5	2.05	2.275	2.475	2.625	2.875	3.075	3.25

Determine, as accurate as possible, the volume of water in the pool.

Solution:



$$\text{Volume of water} = \text{Lateral area of water} \times \text{wide} = \left(\int_0^{12.5} h.dx\right)\times 7.5\ .$$

Here we have 4 segments of h_1 = 1.25 m and 3 segments of h_2 = 2.5 m.

By using the Simpson's 1/3 rule for the first 4 segments and the 3/8 rule for the last 3 segments we get:

$$I=\frac{h_1}{3}.(f_0+4f_1+f_2)+\frac{h_1}{3}.(f_2+4f_3+f_4)+\frac{3h_2}{8}.\{f_4+3(f_5+f_6)+f_7\},$$

$$=\frac{1.25}{3}.\{1.5+4(2.05)+2.275\}+\frac{1.25}{3}.\{2.275+4(2.475)+2.625)+\frac{3(2.5)}{8}.\{2.625+3(2.875+3.075)+3.25\}$$

≈ 35.63 m^2.

$\therefore$ Volume of water $\approx$ 35.63 $\times$ 7.5 $\approx$ 267.225 m^3.