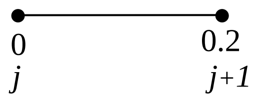
Example 3: Find $f'(0)$ for the function $f(x)=\sqrt{x}+7x$. $(\varepsilon = 1\times10^{-3})$

Solution:

Use forward difference approximations with $O(h)$,

$$f'_j = \frac{-f_j + f_{j+1}}{h} + O(h).$$



1st iteration: Take $h_1 = 0.2$,

At $x=0 \Rightarrow j=0$, $j+1 = x + h_1 = 0 + 0.2 = 0.2$.

$$f'(0) \approx \frac{-f(0)+f(0.2)}{0.2} \Rightarrow f'(0) \approx \frac{-(\sqrt{0}+7(0)) + (\sqrt{0.2}+7(0.2))}{0.2} \approx 9.236.$$

2nd iteration: Take $h_2 = \frac{h_1}{2} = \frac{0.2}{2} = 0.1$.

The calculations must be repeated as in the 1st iteration and continued until $\Delta \le \varepsilon$.

No. of Iteration (i)	h_i	f'_i	$\Delta_i = \left\| f'_i - f'_{i-1} \right\|$
1	0.2	9.236068	----
2	0.1	10.162278	0.92621
3	0.05	11.472136	1.309858
4	0.025	13.324555	1.852419 (divergence)

$\therefore$ $f'(0)$ is undefined (does not exist).

Check: $f'(x) = \frac{1}{2\sqrt{x}} + 7 \Rightarrow f'(0) = \frac{1}{2\sqrt{0}} + 7 = \frac{1}{0} + 7$. (undefined)

Example 4: Find $f'(0)$, $f'(2)$, $f'(4)$, and $f''(0)$ with error of $O(h)^2$ for the function of the following equally spaced data:

x	0	1	2	3	4
$f(x)$	30	33	28	12	- 22

Solution:

* At $x=0$, forward differences must be used with $O(h)^2$,

$$f'_j = \frac{-3f_j + 4f_{j+1} - f_{j+2}}{2h} + O(h)^2,$$

$$\therefore f'(0) = \frac{-3f(0) + 4f(1) - f(2)}{2(1)} = \frac{-3(30) + 4(33) - (28)}{2} = 7.$$

$$f_j'' = \frac{2f_j - 5f_{j+1} + 4f_{j+2} - f_{j+3}}{h^2} + O(h)^2,$$

$$\therefore f''(0) = \frac{2f(0) - 5f(1) + 4f(2) - f(3)}{(1)^2} = \frac{2(30) - 5(33) + 4(28) - (12)}{1} = -5.$$

* At $x = 2$, use central differences with $O(h)^2$,

$$f_j' = \frac{-f_{j-1} + f_{j-1}}{2h} + O(h)^2,$$

$$\therefore f'(2) = \frac{-4f(1) + f(3)}{2(1)} = \frac{-(33) + (12)}{2} = -10.5.$$

* At $x = 4$, backward differences must be used with $O(h)^2$,

$$f_j' = \frac{3f_j - 4f_{j-1} + f_{j-2}}{2h} + O(h)^2,$$

$$\therefore f'(4) = \frac{3(4) - 4f(3) + f(2)}{2(1)} = \frac{3(-22) - 4(12) + (28)}{2} = -43.$$

<u>Example 5</u>: The following data represent a polynomial. Find its equation.

x	0	1	2	3	4	5
$f(x)$	1.0	0.5	8.0	35.5	95.0	198.5

Solution:

The forward differences can be calculated as shown in the table below:

x	$f(x)$	Δf	$\Delta^2 f$	$\Delta^3 f$	$\Delta^4 f$
0	1.0	- 0.5	8	12	0
1	0.5	7.5	20	12	0
2	8.0	27.5	32	12	
3	35.5	59.5	44		
4	95.0	103.5			
5	198.5				

Since the 3rd difference (which is equivalent to the 3rd derivative) is constant, then the polynomial is of 3rd degree. The forward difference representation of the 3rd derivative is: $\dfrac{d^3 f}{dx^3} = \dfrac{\Delta^3 f}{h^3} + O(h)$.

However, for a 3rd degree polynomial, this expression is exact (i.e. $O(h) = 0$).

$$\frac{d^3 f}{dx^3} = \frac{\Delta^3 f}{h^3} = \frac{12}{1^3} = 12 \quad \Rightarrow \quad \frac{d^2 f}{dx^2} = 12x + A,$$

$$\frac{df}{dx} = 6x^2 + Ax + B \quad \Rightarrow \quad f(x) = 2x^3 + \frac{A}{2}x^2 + Bx + C.$$

We have 3 unknown constants, so we need 3 points. Substituting the three points $(0,1)$, $(1,0.5)$, and $(2,8)$ into the above equation gives respectively:

$$2(0)^3 + \frac{A}{2}(0)^2 + B(0) + C = 1 \quad \Rightarrow \quad C = 1$$

$$2(1)^3 + \frac{A}{2}(1)^2 + B(1) + 1 = 0.5 \quad \Rightarrow \quad A + 2B = -5 \qquad \text{……….. (I)}$$

$$2(2)^3 + \frac{A}{2}(2)^2 + B(2) + 1 = 8 \quad \Rightarrow \quad 2A + 2B = -9 \qquad \text{………. (II)}$$

Solving Eqs. I and II simultaneously gives:

$A = -4$ and $B = -1/2$

$$\therefore f(x) = 2x^3 + \frac{-4}{2}x^2 + \frac{-1}{2}x + 1 \quad \Rightarrow \quad f(x) = 2x^3 - 2x^2 - x/2 + 1.$$

Example 6: The deflections at selected locations in a beam, of $EI = 4 \times 10^6$ N.m^2 and $L = 4$ m, are:

Location (m)	0	0.5	1	1.5	2	2.5	3	3.5	4
Deflection (mm)	0	12.7	23.1	30.8	33.3	29.9	22.6	11.8	0

Determine, as accurate as possible, the slope and shear force at both ends and the bending moment at midspan.

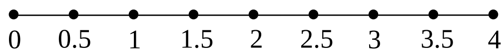
Solution:

Let x represents the location and y represents the deflection, then,

The slope $\theta = \dfrac{dy}{dx}$, shear force $V = -EI.\dfrac{d^3 y}{dx^3}$, and bending moment $M = -EI.\dfrac{d^2 y}{dx^2}$.

* At $x = 0$ m, forward differences must be used and we choose it with $O(h)^2$,

The slope $\theta_j = f_j' = \dfrac{-3f_j + 4f_{j+1} - f_{j+2}}{2h}$,



$$\therefore \theta_0 = \frac{-3y(0) + 4y(0.5) - y(1)}{2h} = \frac{-3(0) + 4(12.7) - (23.1)}{2(0.5)(1000)} = 27.7 \times 10^{-3}.$$

The shear force $V_j = -EI.f_j''' = -EI.\dfrac{-5f_j + 18f_{j+1} - 24f_{j+2} + 14f_{j+3} - 3f_{j+4}}{2h^3}$,

$$V_0 = -EI.\frac{-5y(0) + 18y(0.5) - 24y(1) + 14y(1.5) - 3y(2)}{2h^3},$$

$$V_0 = -4 \times 10^6.\frac{-5(0) + 18(12.7) - 24(23.1) + 14(30.8) - 3(33.3)}{2(0.5)^3(1000)} = -88000\text{N.} \quad (\uparrow)$$

* At $x = 4$ m, backward difference must be used and we choose it with $O(h)^2$,

The slope $\theta_j = f_j' = \dfrac{3f_j - 4f_{j-1} + f_{j-2}}{2h}$,

$$\therefore \theta_4 = \frac{3y(4) - 4y(3.5) + y(3)}{2h} = \frac{3(0) - 4(11.8) + (22.6)}{2(0.5)(1000)} = -24.6 \times 10^{-3}.$$

The shear force $V_j = -EI.f_j''' = -EI.\dfrac{5f_j - 18f_{j+1} + 24f_{j+2} - 14f_{j+3} + 3f_{j+4}}{2h^3}$,

$$V_4 = -EI.\frac{5y(4) - 18y(3.5) + 24y(3) - 14y(2.5) + 3y(2)}{2h^3},$$

$$V_4 = -4 \times 10^6.\frac{5(0) - 18(11.8) + 24(22.6) - 14(29.9) + 3(33.3)}{2(0.5)^3(1000)} = -180800\text{N.} \quad (\uparrow)$$

* At $x = 2$ m (midspan), using central difference and we choose it with $O(h)^4$,

The shear force $M_j = -EI.f_j'' = -EI.\dfrac{-f_{j-2} + 16f_{j-1} - 30f_j + 16f_{j+1} - f_{j+1}}{12h^2}$,

$$M_2 = -EI.\frac{-y(1) + 16y(1.5) - 30y(2) + 16y(2.5) - y(3)}{12h^2},$$

$$M_2 = -4 \times 10^6.\frac{-(23.1) + 16(30.8) - 30(33.3) + 16(29.9) - (22.6)}{12(0.5)^2(1000)} = 98000\text{N.m.} \quad (\circlearrowright)$$