Example 4: Find the interval of convergence of Maclaurin expansion for $\sin x$. How many terms are needed to compute $\sin\frac{1}{2}$ accurately to 6 decimals?

Solution:.

Maclaurin series is $f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \ldots\ldots$,

$$f(x) = \sin x \quad \Rightarrow \quad f(0) = \sin 0 = 0,$$
$$f'(x) = \cos x \quad \Rightarrow \quad f'(0) = \cos 0 = 1,$$
$$f''(x) = -\sin x \quad \Rightarrow \quad f''(0) = -\sin 0 = 0,$$
$$f'''(x) = -\cos x \quad \Rightarrow \quad f'''(0) = -\cos 0 = -1,$$

$$\therefore \; f(x) = \sin x = 0 + x(1) + \frac{x^2}{2!}.(0) + \frac{x^3}{3!}.(-1) + \ldots\ldots \Rightarrow \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\ldots.$$

Convergence test: Check $\lim_{k\to\infty}\left|\frac{U_{k+1}}{U_k}\right|$,

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\ldots \quad \Rightarrow \quad \sin x = \sum_{k=0}^{\infty}(-1)^k.\frac{x^{2k+1}}{(2k+1)!},$$

$$\therefore \lim_{k\to\infty}\left|\frac{U_{k+1}}{U_k}\right| = \lim_{k\to\infty}\left|\frac{\dfrac{x^{2(k+1)+1}}{(2(k+1)+1)!}}{\dfrac{x^{2k+1}}{(2k+1)!}}\right| = \lim_{k\to\infty}\left|\frac{\dfrac{x^{2k+3}}{(2k+3)!}}{\dfrac{x^{2k+1}}{(2k+1)!}}\right|,$$

$$= \lim_{k\to\infty}\left|\frac{x^{2k+3}}{(2k+3)!}.\frac{(2k+1)!}{x^{2k+1}}\right| = \lim_{k\to\infty}\left|\frac{x^{2k+3}}{(2k+3)(2k+2)(2k+1)!}.\frac{(2k+1)!}{x^{2k+1}}\right|,$$

$$= \lim_{k\to\infty}\left|\frac{x^2}{(2k+3)(2k+2)}\right| = \left|\frac{x^2}{(2\infty+3)(2\infty+2)}\right| = \left|\frac{x^2}{\infty}\right| = 0 < 1.$$

$\therefore$ The series is convergent for all values of $x \in R$.

$\therefore$ The interval of convergence is $(-\infty,+\infty)$.

Estimation of terms No.: Use $|U_k| < x.\varepsilon$,

Here $\varepsilon = 1\times10^{-6}$ and $\sin x = \sin\frac{1}{2} \Rightarrow x = \frac{1}{2}$,

$$\therefore \quad \frac{\left(\frac{1}{2}\right)^{2k+1}}{(2k+1)!} < \frac{1}{2}.(1\times10^{-6}) \Rightarrow \frac{1}{2^{2k+1}(2k+1)!} < \frac{1}{2\times10^6},$$

$2^{2k+1}(2k+1)! > 2\times10^6 \Rightarrow$ By trial and error $k=4 \Rightarrow \therefore$ we need 5 terms.

Example 5: Check whether the Maclaurin expansion for $\frac{1}{1-x}$ is valid to compute 4^{-1} or not.

Solution:.

Maclaurin series is $f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \ldots\ldots$,

$$f(x) = \frac{1}{1-x} = (1-x)^{-1} \quad \Rightarrow \quad f(0) = (1-0)^{-1} = 1,$$

$$f'(x) = -1(1-x)^{-2}.(-1) = (1-x)^{-2} \quad \Rightarrow \quad f'(0) = (1-0)^{-2} = 1,$$

$$f''(x) = -2(1-x)^{-3}.(-1) = 2(1-x)^{-3} \quad \Rightarrow \quad f''(0) = 2(1-0)^{-3} = 2,$$

$$f'''(x) = 2.(-3)(1-x)^{-4}.(-1) = 6(1-x)^{-4} \quad \Rightarrow \quad f'''(0) = 6(1-0)^{-4} = 6,$$

$$\therefore \; f(x) = \frac{1}{1-x} = 1 + x(1) + \frac{x^2}{2!}.(2) + \frac{x^3}{3!}.(6) + \ldots\ldots \Rightarrow \quad \frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots\ldots.$$

<u>Convergence test</u>: Check $\lim\limits_{k\to\infty}\left|\frac{U_{k+1}}{U_k}\right|$,

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots\ldots \quad \Rightarrow \quad \frac{1}{1-x} = \sum_{k=0}^{\infty} x^k .$$

$$\therefore \; \lim_{k\to\infty}\left|\frac{U_{k+1}}{U_k}\right| = \lim_{k\to\infty}\left|\frac{x^{k+1}}{x^k}\right| = \lim_{k\to\infty} |x| = |x| .$$

$\therefore$ The series converges when $|x| < 1 \Rightarrow$ either $x < 1$ or $-x < 1 \Rightarrow x > -1$.

$\therefore$ The interval of convergence is $(-1,1)$.

To compute $4^{-1} \;\Rightarrow\; 4^{-1} = \frac{1}{1-x} \;\Rightarrow\; \frac{1}{4} = \frac{1}{1-x} \;\Rightarrow\; 1-x = 4 \;\Rightarrow\; x = -3 \notin (-1,1)$.

$\therefore$ Thus, the series is not valid to compute 4^{-1} (since it will diverge).

<u>**Example 6**</u>: After five seconds, the following information of a moving body is measured: position = 25 m, velocity = 10 m/s, and acceleration = 2 m/s^2. Using the principal of Taylor series, estimate the position after another five seconds.

Solution:.

Taylor series is $f(t) = f(a) + (t-a)\,f'(a) + \frac{(t-a)^2}{2!}\,f''(a) + \ldots\ldots$.

If the position is $f(t)$, then the velocity is $f'(t)$ and the acceleration is $f''(t)$.

Expanding the function $f(t)$ about t = 5 s yields:

$$f(t) = f(5) + (t-5)\,f'(5) + \frac{(t-5)^2}{2!}\,f''(5) + O(t)^3,$$

$$f(t) = 25 + (t-5)(10) + \frac{(t-5)^2}{2!}(2) + O(t)^3.$$

At t = 10 s $\Rightarrow f(10) = 25 + (10-5)(10) + \frac{(10-5)^2}{2!}(2) = 100$ m.