4- Taylor Series

Introduction

Taylor series is the foundation of many numerical methods. Many of numerical techniques are derived directly from Taylor series, as are the estimates of the errors involved in employing these techniques.

Maclaurin series

Suppose that the value of the function $f(x)$, shown in the figure, and the values of all of its derivatives at $x=0$, i.e. $f(0), f'(0), f''(0), f'''(0),\ldots,$ are known and the value of this function at a point x is to be determined. One method is to approximate $f(x)$ by its tangent line at $x=0$, which has the equation:

$p(x)=c_0+c_1x.$ (polynomial of degree 1)

At $x=0$, $p(x)=p(0) \quad \Rightarrow \quad p(0)=c_0+c_1(0)$,

$\therefore \quad c_0=p(0)$, but $p(0)=f(0) \;\Rightarrow\; c_0=f(0)$.

$p'(x)=c_1$.

At $x=0$, $p'(x)=p'(0) \;\Rightarrow\; p'(0)=c_1$,

but $p'(0)=f'(0) \;\Rightarrow\; c_1=f'(0)$.

$\therefore \; p(x)=f(0)+xf'(0) \;\Rightarrow\; f(x)\approx f(0)+xf'(0)$.

The accuracy of the approximation will be better improved as the degree of the approximation polynomial is increased. If a polynomial of infinite degree is used, then the following approximation is obtained:

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+\frac{x^4}{4!}f^{IV}(0)+\ldots\ldots\ldots,$$

or simply $\quad f(x)=\sum_{k=0}^{\infty}\frac{x^k}{k!}f^k(0)$.

The above series (polynomial) is called the Maclaurin series (polynomial).

Taylor series

Maclaurin series gives an approximation of a function $f(x)$ in the vicinity of $x=0$. the more general case of approximating $f(x)$ in the vicinity of an arbitrary value $x=a$ is now considered. The basic idea is the same as before. Thus, if a polynomial of infinite degree is used to approximate a function $f(x)$ which its value and all its derivatives' values are known at $x=a$, then the following polynomial will obtained:

$$f(x)=f(a)+(x-a)f'(a)+\frac{(x-a)^2}{2!}f''(a)+\frac{(x-a)^3}{3!}f'''(a)+\frac{(x-a)^4}{4!}f^{IV}(0)+\ldots\ldots,$$

or simply $f(x)=\sum_{k=0}^{\infty}\frac{(x-a)^k}{k!}f^k(a)$.

The above series (polynomial) is called the Taylor series expansion for the function f about $x=a$. It is obvious that Maclaurin series is a special case of Taylor series when the point of expansion is $x=0$ (i.e. $a=0$).

Another used formula of Taylor series expansion of a function f about x, where its value and all its derivatives' values are known at the point x, is

$$f(x+h)=f(x)+hf'(x)+\frac{h^2}{2!}f''(x)+\frac{h^3}{3!}f'''(x)+\frac{h^4}{4!}f^{IV}(x)+\ldots\ldots .$$

Order of error

The error in the value of $f(x)$ which refers to the error resulted from omitting terms beyond the term contains the n^{th} derivative is denoted as $O(x-a)^{n+1}$.

If we take one term of Taylor series, then

$$f(x)=f(a)+O(x-a),$$

if we take two terms, then

$$f(x)=f(a)+(x-a)f'(a)+O(x-a)^2,$$

and if we take n terms, then

$$f(x)=f(a)+(x-a)f'(a)+\frac{(x-a)^2}{2!}f''(a)+\ldots+\frac{(x-a)^{n-1}}{(n-1)!}f^{n-1}(a)+O(x-a)^n.$$

It is obvious that the error decreases as its order increases, i.e;

$$O(x-a)^{n+1}<O(x-a)^n.$$

Error in truncated Taylor series

The difference $R_n(x)$ (also called the error or remainder) between the exact value of the function $f(x)$ and the value obtained from the n[th] Taylor series $T_n(x)$ is

$R_n(x) = f(x) - T_n(x)$, which is known as the n[th] remainder, where

$$T_n(x) = \sum_{k=0}^{n} \frac{(x-a)^k}{k!} f^k(a) \qquad \text{and} \qquad R_n(x) = \sum_{k=n+1}^{\infty} \frac{(x-a)^k}{k!} f^k(a).$$

Thus the value of $f(x)$ can be written as:

$f(x) = T_n(x) + R_n(x)$, which is called Taylor formula with remainder.

The upper bound of the remainder in a truncated series can be estimated by:

$$R(x) \le \left| \frac{(x-a)^r}{r!} f^r_{\max} \right| . \qquad \text{(Lagrange's form)}$$

where r is the power of $(x-a)$ in the first truncated term and the maximum value of the derivative f^r occurs at some point c lies in the interval $[x,a]$.

Notes:

1- Let the power series is $\sum_{k=0}^{\infty} U_k$. If the limit $f(x) = \lim_{k\to\infty} \left| \frac{U_{k+1}}{U_k} \right| = L$ exists, then

i- The series converges when $L < 1$.
ii- The series diverges when $L > 1$.
iii- The test fails when $L = 1$.

2- The number of terms of a given power series $\sum_{k=0}^{\infty} U_k$, that are required to compute x correct to a given accuracy ε, cab be estimated from $|U_k| < x.\varepsilon$.

Example 1: Find the Taylor series expansion for the function $f(x) = e^x$ about $x = 0$. Then use it to find $f(x) = e^{0.5}$ to an error of order $O(x)^3$ and compare it with the exact value.

Solution:

When the expansion is about $x = 0$, then Taylor series reduces to Maclaurin series.

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \ldots\ldots\ldots ,$$

$$f(x) = e^x \;\Rightarrow\; f'(x) = f''(x) = f^n(x) = e^x \;\Rightarrow\; f(0) = f'(0) = f''(0) = f^n(0) = e^0 = 1,$$

$\therefore\ f(x) = e^x = 1 + x.(1) + \frac{x^2}{2!}.(1) + \frac{x^3}{3!}.(1) + ...$, or $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ...$.

To compute $e^{0.5} \quad \Rightarrow \quad e^x = e^{0.5} \quad \Rightarrow \quad x = 0.5$,

$\therefore\ e^{0.5} = 1 + 0.5 + \frac{(0.5)^2}{2!} + O(0.5)^3 \quad \Rightarrow \quad e^{0.5} = 1 + 0.5 + \frac{(0.5)^2}{2} = 1.625$.

The (exact) value is $e^{0.5} = 1.648721$ (from the scientific calculator).

The percent relative error $P = \left|\frac{exact - approx}{exact}\right| \times 100 = \left|\frac{1.648721 - 1.625}{1.648721}\right| \times 100 = 1.44\%$.

Example 2: Find the Taylor series expansion of $f(x) = e^{\sqrt{x}}$ about $x = 0$.

Solution:

When the expansion is about $x = 0$, then Taylor series reduces to Maclaurin series.

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + ... ,$$

$$f(x) = e^{\sqrt{x}} \quad \Rightarrow \quad f(0) = e^{\sqrt{0}} = 1,$$

$$f'(x) = e^{\sqrt{x}} . \frac{1}{2} x^{-1/2} \quad \Rightarrow \quad f'(x) = \frac{e^{\sqrt{x}}}{2\sqrt{x}} \quad \Rightarrow \quad f'(0) = \frac{e^{\sqrt{0}}}{2\sqrt{0}} = \frac{1}{0}. \text{ (undefined)}$$

$\because$ Since $f'(0)$ does not exit (undefined) $\Rightarrow \therefore\ f(x) = e^{\sqrt{x}}$ can not be expanded about $x = 0$, or the Taylor series expansion of $f(x) = e^{\sqrt{x}}$ about $x = 0$ does not exist.

Example 3: Use Taylor series to determine the square root of 13 to an error of order $O(x)^3$. Estimate the error and compare with the exact value.

Solution:

Taylor series is $f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + O(x-a)^3$.

$\because$ The required is a square root $\Rightarrow \therefore$ Let $f(x) = \sqrt{x}$.

Since the nearest number, of known square root, to 13 is the number 16, so we choose $x = 16$ to determine the square root of any number (i.e. find Taylor series expansion of $f(x) = \sqrt{x}$ about $x = 16$) and then to compute the required square root.

$$f(x) = \sqrt{x} \quad \Rightarrow \quad f(a) = f(16) = \sqrt{16} = 4,$$

$$f'(x) = \frac{1}{2\sqrt{x}} \quad \Rightarrow \quad f'(16) = \frac{1}{2\sqrt{16}} = \frac{1}{2(4)} = \frac{1}{8} ,$$

$$f''(x)=\frac{-1}{4\sqrt{x^3}} \quad\Rightarrow\quad f''(16)=\frac{-1}{4\sqrt{16^3}}=\frac{-1}{4\sqrt{16^2.(16)}}=\frac{-1}{4(16)(4)}=\frac{-1}{256}\ ,$$

$$\therefore\quad f(x)=\sqrt{x}=4+(x-16).\frac{1}{8}+\frac{(x-16)^2}{2}.\frac{-1}{256}+O(x-a)^3.$$

Now to compute $\sqrt{x} \quad\Rightarrow\quad \sqrt{x}=\sqrt{13} \quad\Rightarrow\quad x=13,$

$$\therefore\quad \sqrt{13}=4+(13-16).\frac{1}{8}+\frac{(13-16)^2}{2}.\frac{-1}{256}=3.607422.$$

The error can be estimated from $R(x)\le\left|\frac{(x-a)^r}{r!}\,f^{r}_{\max}\right|.$

Since the first truncated term in Taylor series contains the 3rd derivative (i.e. r= 3),

$$\therefore\ R(x)\le\left|\frac{(13-16)^3}{3!}\,f'''_{\max}\right|,$$

$$f'''(x)=\frac{-1}{4}.\frac{-3}{2}x^{-5/2}=\frac{3}{8\sqrt{x^5}}\ ,$$

$$f'''(x)=f'''(13)=\frac{3}{8\sqrt{13^5}}=6.15\times10^{-4} \ \text{ and } \ f'''(a)=f'''(16)=\frac{3}{8\sqrt{16^5}}=3.66\times10^{-4},$$

$$\therefore\ f'''_{\max}=6.15\times10^{-4}\ \Rightarrow\ R(x)\le\left|\frac{(13-16)^3}{6}.6.15\times10^{-4}\right|\ \Rightarrow\ R(x)\le2.78\times10^{-3}.$$

The (exact) value is $\sqrt{13}=3.605551$ (from the hand calculator).

The absolute error $\Delta=|exact-approx|=|3.60551-3.607422|=1.87\times10^{-3}<2.78\times10^{-3}$.