3- Numerical Solution of Set of Algebraic Equations

Introduction

The solution of set of algebraic equations is an important step in wide variety of engineering problems, such as the numerical solution of differential equations, the structural analysis, network analysis, ….etc.

Iterative methods

In the these methods an initial set of values of the unknowns are assumed to determine improved approximate values of these unknowns which in turn are used to determine better approximations and so on. This iteration continues until sufficiently values are obtained.

Solution of Set of linear algebraic equations

1- Jacobi iteration

The system of equations:

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \dots\dots + a_{1m}x_m &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \dots\dots + a_{2m}x_m &= b_2 \\
\dots\dots\dots\dots\dots\dots\dots\dots\dots & \qquad \dots\dots\dots\dots\dots \ (1) \\
a_{m1}x_1 + a_{m2}x_2 + \dots\dots + a_{mm}x_m &= b_m
\end{aligned}$$

Can be written as:

$$\begin{aligned}
x_1 &= [b_1 - (a_{12}x_2 + \dots\dots + a_{1m}x_m)]/a_{11} \\
x_2 &= [b_2 - (a_{21}x_1 + \dots\dots + a_{2m}x_m)]/a_{22} \\
\dots\dots\dots\dots\dots\dots\dots\dots\dots & \qquad \dots\dots\dots\dots\dots \ (2) \\
x_m &= [b_m - (a_{m1}x_1 + a_{m2}x_2 + \dots\dots)]/a_{mm}
\end{aligned}$$

In this method initial trial values are assumed which are substituted in the iterative equations (Eq.2) of the unknowns to obtain better approximations of the unknowns that are used to obtain new improved approximations. This method converges if :

$$\left|a_{ii}\right| > \sum_{j=1}^{n}\left|a_{ij}\right| \qquad j = 1,2,\dots\dots,n \text{ but } i \neq j \qquad \dots\dots\dots\dots\dots\dots \ (3)$$

i.e. the absolute value of the element located on the main diagonal in each row is greater than the sum of the absolute values of the other elements in that row. So the procedure of solution in Jacobi method is as follows:

1- The equations are rearranged for condition of convergence in Eq.3.
2- The resulting equations are written in the iterative expressions of Eq.2.
3- A set of initial values of the unknowns are assumed.
4- These values are substituted in the iterative equations to obtain new values.
5- Step 3 is repeated until the required accuracy is achieved.

Example 1: Solve the following set of equations:

$$4x - 8y + z + 21 = 0,$$
$$-2x + y + 5z - 15 = 0,$$
$$4x - y + z - 7 = 0.$$

Solution:.

Use Jacobi iteration,

Step 1: Rearrange the equations for convergence:

$$4x - y + z = 7,$$
$$4x - 8y + z = -21,$$
$$-2x + y + 5z = 15.$$

Step 2: Find the iterative equations:

$$x_{i+1} = (7 + y_i - z_i)/4,$$
$$y_{i+1} = (21 + 4x_i + z_i)/8,$$
$$z_{i+1} = (15 + 2x_i - y_i)/5.$$

Step 3: Assume initial values:

$$x_o = y_o = z_o = 1.$$

Step 4: Substitute the initial values into the iterative equations to get new values:

1st iteration:

$$x_1 = (7 + 1 - 1)/4 = 1.75,$$
$$y_1 = (21 + 4(1) + 1)/8 = 3.25,$$
$$z_1 = (15 + 2(1) - 1)/5 = 3.2.$$

2nd iteration: $x_1 = 1.75$, $y_1 = 3.25$, and $z_1 = 3.2$.

The calculations must be repeated as in the 1st iteration and continued until the required accuracy (if any) is achieved.

No. of Iteration (i)	x_i	y_i	z_i
0	1	1	1
1	1.75	3.25	3.2
2	1.7625	3.9	3.05
3	1.9625	3.8875	2.925
4	1.990625	3.971875	3.0075
5	1.99109	3.99625	3.001875
…..	…..	…..	…..
…..	…..	…..	…..
	→2	→4	→3

2- Gauss-Seidel iteration

As $(x_1)_{i+1}$ is expected to be a better approximation than $(x_1)_i$,then it appears more advantageous to use the value of $(x_1)_{i+1}$ in determining $(x_2)_{i+1}$ rather than using $(x_1)_i$. Similarly, the value of $(x_1)_{i+1}$ and $(x_2)_{i+1}$ are used to determine the value of $(x_3)_{i+1}$, and so on. The using of this procedure will , in general, yield results that are more rapidly convergent than the conventional Jacobi iteration.

Example: Solve the following set of equations:

$$4x - 8y + z + 21 = 0,$$
$$-2x + y + 5z - 15 = 0,$$
$$4x - y + z - 7 = 0.$$

Solution:.

Use Gauss-Seidel iteration,

Step 1: Rearrange the equations for convergence:

$$4x - y + z = 7,$$
$$4x - 8y + z = -21,$$
$$-2x + y + 5z = 15.$$

Step 2: Find the iterative equations:

$$x_{i+1} = (7 + y_i - z_i)/4,$$
$$y_{i+1} = (21 + 4x_{i+1} + z_i)/8,$$
$$z_{i+1} = (15 + 2x_{i+1} - y_{i+1})/5.$$

Step 3: Assume initial values:

$$x_o = y_o = z_o = 1.$$

Step 4: Substitute the initial values into the iterative equations to get new values:

1st iteration:

$$x_1 = (7 + 1 - 1)/4 = 1.75,$$
$$y_1 = (21 + 4(1.75) + 1)/8 = 3.625,$$
$$z_1 = (15 + 2(1.75) - 3.625)/5 = 2.975.$$

2nd iteration: $x_1 = 1.75$, $y_1 = 3.625$, and $z_1 = 2.975$.

The calculations must be repeated as in the 1st iteration and continued until the required accuracy (if any) is achieved.

i	x_i	y_i	z_i
0	1	1	1
1	1.75	3.625	2.975
2	1.9125	3.953125	2.974375
3	1.994688	3.994141	2.999047
…..	…..	…..	…..
…..	…..	…..	…..
	→2	→4	→3

Solution of Set of nonlinear algebraic equations

These equations can be solved by the Gauss- Seidel iteration.

Example 1: Solve the following system:

$$x + 4y + z^2 - 18 = 0,$$
$$x^2 + y + 4z - 15 = 0,$$
$$4x + y^2 + z - 11 = 0.$$

Solution:

Use Gauss-Seidel iteration,

Step 1: Rearrange the equations for convergence:

$$4x + y^2 + z = 11,$$
$$x + 4y + z^2 = 18,$$
$$x^2 + y + 4z = 15.$$

Step 2: Find the iterative equations:

$$x_{i+1} = (11 - y^2{}_i - z_i)/4,$$
$$y_{i+1} = (18 - x_{i+1} - z^2{}_i)/4,$$
$$z_{i+1} = (15 - x^2{}_{i+1} - y_{i+1})/4.$$

Step 3: Assume initial values:

$$x_o = y_o = z_o = 1.$$

Step 4: Substitute the initial values into the iterative equations to get new values:

1st iteration:

$$x_1 = (11 - 1^2 - 1)/4 = 2.25,$$
$$y_1 = (18 - 2.25 - 1^2)/4 = 3.6875,$$
$$z_1 = (15 - 2.25^2 - 3.6875)/4 = 1.5625.$$

2nd iteration: $x_1 = 2.25$, $y_1 = 3.6875$, and $z_1 = 1.562$.

The calculations must be repeated as in the 1st iteration and continued until the required accuracy (if any) is achieved.

No. of Iteration (i)	x_i	y_i	z_i
0	1	1	1
1	2.25	3.6875	1.5625
…..	…..	…..	…..
…..	…..	…..	…..
…..	…..	…..	…..
	→1	→2	→3

Example 2: Solve:

$$x^2 + xy = 10,$$

$$y + 3xy^2 = 57.$$

Solution:

Use the concept of Gauss-Seidel iteration,

Find the iterative equations:

$$x_{i+1} = \sqrt{10 - x_i y_i},$$

$$y_{i+1} = \sqrt{\frac{57 - y_i}{3x_{i+1}}}.$$

Assume initial values:

$$x_o = y_o = 1.$$

Step 4: Substitute the initial values into the iterative equations to get new values:

1st iteration:

$$x_1 = \sqrt{10 - (1)(1)} = 3,$$

$$y_1 = \sqrt{\frac{57 - 1}{3(3)}} = 2.494438.$$

2nd iteration: $x_1 = 3,\ y_1 = 2.494438.$

The calculations must be repeated as in the 1st iteration and continued until the required accuracy (if any) is achieved.

No. of Iteration (*i*)	x_i	y_i
0	1	1
1	3	2.494438
2	1.586407	3.384172
3	2.152052	2.881771
4	1.948917	3.042387
5	2.017583	2.985723
…..	…..	…..
…..	…..	…..
	→2	→3

Note: Another expressions for the iterative equations must be used if divergence is occurred.