3- Newton-Raphson Method

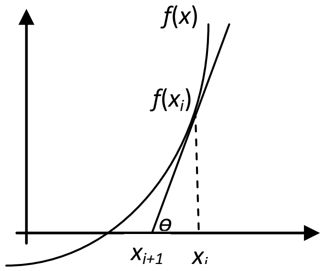
This is one of the more popular methods used for solving non-linear algebraic equations. It is also known as Newton's method or the tangent method. It is convergent faster than the previous methods. The formula of this method can be derived as follows.

Let x_i be an estimation to the required root of a given function $f(x)$. A better estimation x_{i+1} can be obtained by using the zero of the tangent to the function at x_i. The tangent line passes the x-axis at the improved root x_{i+1}. The value of x_{i+1} can be determined as follows:

$f'(x_i) = \tan\theta$, but from the shown figure:

$$\tan\theta = \frac{f(x_i)}{x_i - x_{i+1}} \quad \Rightarrow \quad \therefore f'(x_i) = \frac{f(x_i)}{x_i - x_{i+1}},$$

or $$x_i - x_{i+1} = \frac{f(x_i)}{f'(x_i)} \quad \Rightarrow \quad x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}.$$



Notes:

1. Newton-Raphson method has slow convergence in regions of multiple roots.
2. Near the maxima and minima points, Newton-Raphson method is either convergent to these points or convergent to a non-required root or divergent.

Example 1: Find the positive root of ($x^2 - 4\sin x$) to an accuracy of $\varepsilon = 1\times10^{-6}$.

Solution:

Let $f(x) = x^2 - 4\sin x$, and check the sign of *f(x)*: (not necessary)

x	0	1	2	3
$f(x)$	0	- 2.366	0.363	8.4

There is a positive root lies between x = 1 and x = 2 and it is closer to x = 2.

To find this root by using Newton-Raphson method,

1st iteration: Let $x_o = 2$,

$$f(x) = x^2 - 4\sin x \quad \Rightarrow \quad f(x_o) = f(2) = (2)^2 - 4\sin 2 = 0.362810,$$

$$f'(x)=2x-4\cos x \quad \Rightarrow \quad f'(x_o)=f'(2)=2(2)-4\cos 2=5.664587,$$

$$x_1=x_o-\frac{f(x_o)}{f'(x_o)} \quad \Rightarrow \quad x_1=2-\frac{0.362810}{5.664587}=1.935951.$$

2nd iteration: $x_1=1.935951.$

The calculations must be repeated as in the 1st iteration and continued until $\Delta \le \varepsilon$.

i	x_i	$f(x_i)$	$f'(x_i)$	$x_{i+1}=x_i-\frac{f(x_i)}{f'(x_i)}$	$\Delta_i=\lvert x_{i+1}-x_i\rvert$
0	2	0.362810	5.664587	1.935951	0.064….
1	1.935951	0.011623	5.300277	1.933756	2.2×10^{-3}
2	1.933756	1.18×10^{-5}	5.287682	1.933754	2.2×10^{-6}
3	1.933754	1.25×10^{-6}	5.287671	1.933754	$\approx 2.3\times10^{-7}<\varepsilon$

After 4 iterations the positive root is $x_{root}\approx 1.933754$.

Note:

Another arrangement for the above table of calculations may be used as below:

i	x_i	$x_{i+1}=x_i-\frac{f(x_i)}{f'(x_i)}=x_i-\frac{x_i^2-4\sin x_i}{2x_i-4\cos x_i}$	$\Delta_i=\lvert x_{i+1}-x_i\rvert$
0	2	1.935951	0.064….
1	1.935951	1.933756	2.2×10^{-3}
2	1.933756	1.933754	2.2×10^{-6}
3	1.933754	1.933754	$\approx 2.3\times10^{-7}<\varepsilon$

Example 2: Find the root of $f(x)=(2-x)e^{-x/4}-1$ such that $\lvert f(x)\rvert<1\times10^{-6}$.

Solution:

By using Newton-Raphson method,

$$f(x)=(2-x)e^{-x/4}-1,$$

$$f'(x)=(2-x)e^{-x/4}(-1/4)+e^{-x/4}(-1) \quad \Rightarrow \quad f'(x)=(\frac{x}{4}-\frac{3}{2})e^{-x/4}.$$

1st iteration: Let $x_o=3$, (chosen arbitrary)

$$f(x_o)=f(3)=(2-3)e^{-3/4}-1=-1.472366,$$

$$f'(x_o)=f'(3)=(\frac{3}{4}-\frac{3}{2})e^{-3/4}=-0.354275,$$

$$x_1 = x_o - \frac{f(x_o)}{f'(x_o)} \quad \Rightarrow \quad x_1 = 3 - \frac{-1.472366}{-0.354275} = -1.156000.$$

2nd iteration: $x_1 = -1.156000$.

The calculations must be repeated as above and continued until $|f(x)| < 1\times10^{-6}$.

i	x_i	$f(x_i)$	$f'(x_i)$	$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$
0	3	- 1.472366	- 0.354275	- 1.156000
1	- 1.156000	3.213550	- 2.388479	0.189438
2	0.189438	0.726814	- 1.385448	0.714043
3	0.714043	0.075722	- 1.105445	0.782542
4	0.782542	0.001130	- 1.072594	0.783596
5	0.783596	$3.4\times10^{-8} < 1\times10^{-6}$		

Hence the root is $x_{root} \approx 0.783596$.

Note:
If we choose $x_o = 8 \Rightarrow x_1 = 34.778112 \Rightarrow x_2 = 869.152844$. (divergence)

4- Modified Newton Method

To find the roots of a function $f(x)$, define a new function $u(x)$ given by

$$u(x) = \frac{f(x)}{f'(x)} \qquad \text{.................................} \ (1)$$

The function $u(x)$ has the same roots as does $f(x)$, since $u(x)$ becomes zero everywhere that $f(x)$ is zero. If $f(x)$ has a multiple root at $x = c$ of multiplicity r (this could occur, for example, if $f(x)$ contained a factor $(x-c)^r$). The $u(x)$ may be readily shown to have a single root at $x = c$.

$$f(x) = (x-c)^r \quad \Rightarrow \quad f'(x) = r(x-c)^{r-1},$$

$$u(x) = \frac{f(x)}{f'(x)} \quad \Rightarrow \quad u(x) = \frac{(x-c)^r}{r(x-c)^{r-1}} \quad \Rightarrow \quad u(x) = \frac{(x-c)}{r}.$$

Since Newton-Raphson method is effective for simple roots, we can apply this method to $u(x)$ instead of $f(x)$,

$$x_{i+1} = x_i - \frac{u(x_i)}{u'(x_i)}.$$

From Eq.(1), $u'(x) = \dfrac{[f'(x)]^2 - f(x).f''(x)}{[f'(x)]^2} \quad \Rightarrow \quad u'(x) = 1 - \dfrac{f(x).f''(x)}{[f'(x)]^2}.$

The advantage of this method over the conventional Newton's method is in finding multiple roots with a faster convergence.

Example 1: Find the root(s) of the function $f(x) = x^2 - 2.5x + 1.5625$ to $\varepsilon = 1\times10^{-6}$.

Solution:

Check the sign of $f(x)$ at different values of *x*:

x	- 1	0	1	2	3	4
f(x)	5.06..	1.56..	0.06..	0.56..	3.06..	7.56..

There is an expected root(s) lies between *x* = 1 and *x* = 2.

To find this expected root (if any): by using modified Newton method,

1st iteration: Let $x_o = 1$,

$$f(x) = x^2 - 2.5x + 1.5625 \Rightarrow f'(x) = 2x - 2.5 \Rightarrow f''(x) = 2,$$

$$f(x_o) = f(1) = 0.0625 \Rightarrow f'(x_o) = f'(1) = -0.5 \Rightarrow f''(x_o) = f''(1) = 2,$$

$$u(x) = \frac{f(x)}{f'(x)} \Rightarrow u(x) = \frac{0.0625}{-0.5} = -0.125,$$

$$u'(x) = 1 - \frac{f(x).f''(x)}{[f'(x)]^2} \Rightarrow u'(x) = 1 - \frac{(0.0625)(2)}{(-0.5)^2} = 0.5,$$

$$x_1 = x_o - \frac{u(x_o)}{u'(x_o)} \Rightarrow x_1 = 1 - \frac{-0.125}{0.5} = 1.25.$$

2nd iteration: $x_1 = 1.25$.

The calculations must be repeated as in the 1st iteration and continued until $\Delta \le \varepsilon$.

i	x_i	$u(x_i)$	$u'(x_i)$	$x_{i+1} = x_i - \frac{u(x_i)}{u'(x_i)}$	$\Delta_i = \lvert x_{i+1} - x_i \rvert$
0	1	- 0.125	0.5	1.25	0.25
1	1.25	0		1.25	$0 < \varepsilon$

After 2 iterations the root is $x_{root} = 1.25$.

Check for multiple root, $f'(x_{root}) = f'(1.25) = 0 \Rightarrow x_{root} = 1.25$ is a multiple root.

Note:

If we use Newton-Raphson method to find the above root, with the same initial value, then we will need more than 15 iterations to get the required accuracy. Thus, Newton-Raphson method has a very slow convergence in determining multiple roots.

Example 2: Find the smallest positive root of the function

$$f(x) = x^4 - 8.6x^3 - 35.51x^2 + 464.4x - 998.46. \qquad (\varepsilon = 1\times10^{-6})$$

Solution:

Check the sign of $f(x)$ at different values of *x*:

x	0	1	2	3	4	5	6	7	8
$f(x)$	- 998.46	- 577.17	- 264.5	- 76.05	- 3.42	- 14.21	- 52.02	- 36.45	136.9

There is a root lies between *x* = 7 and *x* = 8, but there is an expected root(s) lies between *x* = 4 and *x* = 5.

To find this expected root, if any, by using modified Newton method,

<u>1st iteration:</u> Let $x_o = 4$,

$$f(x) = x^4 - 8.6x^3 - 35.51x^2 + 464.4x - 998.46 \quad \Rightarrow \quad f(x_0) = f(4) = -3.42,$$

$$f'(x) = 4x^3 - 25.8x^2 - 71.02x + 464.4 \quad \Rightarrow \quad f'(x_0) = f'(4) = 23.52,$$

$$f''(x) = 12x^2 - 51.6x - 71.02 \quad \Rightarrow \quad f''(x_0) = f''(4) = -85.42,$$

$$u(x) = \frac{f(x)}{f'(x)} \quad \Rightarrow \quad u(x) = \frac{-3.42}{23.52} = -0.145408,$$

$$u'(x) = 1 - \frac{f(x).f''(x)}{[f'(x)]^2} \quad \Rightarrow \quad u'(x) = 1 - \frac{(-3.42)(-85.42)}{(23.52)^2} = 0.471906,$$

$$x_1 = x_o - \frac{u(x_o)}{u'(x_o)} \quad \Rightarrow \quad x_1 = 4 - \frac{-0.145408}{0.471906} = 4.308129.$$

<u>2nd iteration:</u> $x_1 = 4.308129$.

The calculations must be repeated as in the 1st iteration and continued until $\Delta \le \varepsilon$.

i	x_i	$u(x_i)$	$u'(x_i)$	$x_{i+1} = x_i - \frac{u(x_i)}{u'(x_i)}$	$\Delta_i = \lvert x_{i+1} - x_i \rvert$
0	4	- 0.145408	0.4719062	4.308129	0.308129
1	4.308129	4.0687×10^{-3}	0.5009915	4.300008	8.123×10^{-3}
2	4.300008	4.0315×10^{-6}	0.5000996	4.300000	$8\times10^{-6} < \varepsilon$

After 3 iterations the positive root is $x_{root} = 4.3$.

Check for multiple root, $f'(x_{root}) = f'(4.3) = 0 \quad \Rightarrow \quad x_{root} = 4.3$ is a multiple root.