2- Fixed point Method

A fixed point of a function $g(x)$ is a real number p such that $p = g(p)$. Graphically, fixed points of a function $y = g(x)$ are the points of intersection of $y = g(x)$ and $y = x$.

Fixed point method is used to determine roots of a function $f(x)$ as follows:

1- Rearrange the equation $f(x) = 0$ in the form $x = g(x)$ (so that x is on the left hand side of the equation).

2- Estimate an initial value to the root x_i and substitute it into $g(x)$ to get $g(x_i)$.

3- An improved estimation of the root is determined from $x_{i+1} = g(x_i)$ and so on.

Notes:

1- Fixed point method has very slow convergence.

2- For determining an expected root, lies in the interval (a, b), a certain expression of $x = g(x)$ seems to converge to this root if the absolute value of the slope of $g(x)$ is less than the slope of $y = x$, that is $|g'(x)| \le 1$ for all $x \in (a,b)$.

3- A certain expression of $x = g(x)$ may converge to one root at more.

4- If we can not get an expression of the form $x = g(x)$, then we could add x to both sides. For example, we can rewrite the equation $\sin x = 0$ in the form $x = \sin x + x$.

Example 1: Find the maximum value of the function $y = x^3/3 - 1.1x^2 - 3.1x$ correct to three decimals.

Solution:

Maximum value of the function y occurs when $y' = 0$,

$y' = x^2 - 2.2x - 3.1$,

Put $y' = 0 \quad \Rightarrow \quad x^2 - 2.2x - 3.1 = 0 \quad \Rightarrow \quad f(x) = 0$ (Root finding problem)

So we must find the root(s) of $f(x)$ where $f(x) = x^2 - 2.2x - 3.1$.

Check the sign of $f(x)$ at different values of *x*: (not necessary)

x	- 2	- 1	0	1	2	3	4
$f(x)$	5.3	0.1	- 3.1	- 4.3	- 1.3	- 0.7	4.1

There are two roots: The first root lies between *x* = - 1 and *x* = 0 and the second root lies between *x* = 3 and *x* = 4.

By using fixed point method, rearrange the equation $f(x) = 0$ in the form $x = g(x)$:

$x^2 - 2.2x - 3.1 = 0$,

either $x^2 = 2.2x + 3.1 \quad \Rightarrow \quad x = \sqrt{2.2x + 3.1}$ (the first expression)

or $x.(x - 2.2) = 3.1 \quad \Rightarrow \quad x = \dfrac{3.1}{x - 2.2}$ (the second expression)

or $2.2x = x^2 - 3.1 \quad \Rightarrow \quad x = \dfrac{x^2 - 3.1}{2.2}$ (the third expression)

* For the first expression $x = \sqrt{2.2x + 3.1}$,

Convergence test: (not necessary)

$$g(x) = \sqrt{2.2x + 3.1} \quad \Rightarrow \quad g'(x) = \frac{1.1}{\sqrt{2.2x + 3.1}},$$

- For the first root which $\in (-1, 0)$,

$$|g'(-1)| = \left| \frac{1.1}{\sqrt{2.2(-1) + 3.1}} \right| = 1.16 > 1 \text{ Not Ok}, \qquad |g'(0)| = \left| \frac{1.1}{\sqrt{2.2(0) + 3.1}} \right| = 0.62 \le 1 \text{ Ok}.$$

Thus, this expression will not converge to this root.

- For the second root which $\in (3, 4)$,

$$|g'(3)| = \left| \frac{1.1}{\sqrt{2.2(3) + 3.1}} \right| = 0.35 \le 1 \text{ Ok}, \qquad |g'(4)| = \left| \frac{1.1}{\sqrt{2.2(4) + 3.1}} \right| = 0.32 \le 1 \text{ Ok}.$$

Thus, this expression will converge to this root.

<u>1st iteration:</u> Let $x_o = 3 \Rightarrow x_1 = g(x_o) \Rightarrow x_1 = g(3) = \sqrt{2.2(3)+3.1} = 3.114482$.

<u>2nd iteration:</u> $x_1 = 3.114482 \Rightarrow x_2 = g(3.114482) = \sqrt{2.2(3.114482)+3.1} = 3.154657$.

The calculations must be repeated and continued until $\Delta \le \varepsilon$.

i	x_i	$x_{i+1} = g(x_i)$	$\Delta_i = \lvert x_{i+1} - x_i \rvert$
0	3	3.114482	0.11….
1	3.114482	3.154657	0.04….
2	3.154657	3.168635	0.01….
3	3.168635	3.173483	8.2×10^{-3}
4	3.173483	3.175163	1.68×10^{-3}
5	3.175163	3.175745	$5.8\times10^{-4} < \varepsilon$

The root is $x_{root} \approx 3.175745$.

$$y(3.175745) = (3.175745)^3/3 - 1.1(3.175745)^2 - 3.1(3.175745) = -4.924442.$$

<u>*Notes:*</u>

1- Another arrangement for the above table of calculations may be used as below:

i	x_i	$\Delta_i = \lvert x_i - x_{i-1} \rvert$
0	3	-
1	3.114482	0.11….
2	3.154657	0.04….
3	3.168635	0.01….
4	3.173483	8.2×10^{-3}
5	3.175163	1.68×10^{-3}
6	3.175745	$5.8\times10^{-4} < \varepsilon$

1- If we choose another initial values to the root, this expression will always converge to this root which lies in the interval (3,4), for example:

i	x_i	$\Delta = \lvert x_i - x_{i-1} \rvert$
0	- 1	-
1	0.948683	1.94….
2	2.277521	1.32….
3	2.847902	0.57….
4	3.060292	0.21….
5	3.135704	0.07….
6	3.162048	0.02….
7	3.171199	9.15×10^{-3}
8	3.174372	3.17×10^{-3}
9	3.175471	1.1×10^{-3}
10	3.175852	$3.81\times10^{-4} < \varepsilon$

i	x_i	$\Delta = \lvert x_i - x_{i-1} \rvert$
0	7	-
1	4.301163	2.69….
2	3.544370	0.75….
3	3.301153	0.24….
4	3.219089	0.08….
5	3.190924	0.02….
6	3.181200	9.7×10^{-3}
7	3.177836	3.3×10^{-3}
8	3.176671	1.1×10^{-3}
9	3.176268	$4\times10^{-4} < \varepsilon$

* For the second expression $x = \dfrac{3.1}{x-2.2}$,

Convergence test: (not necessary)

$$g(x) = \frac{3.1}{x-2.2} \quad \Rightarrow \quad g'(x) = \frac{-3.1}{(x-2.2)^2},$$

- For the first root which $\in (-1,0)$,

$$\left|g'(-1)\right| = \left|\frac{-3.1}{(-1-2.2)^2}\right| = 0.3 \le 1 \text{ Ok}, \qquad \left|g'(0)\right| = \left|\frac{-3.1}{(0-2.2)^2}\right| = 0.64 \le 1 \text{ Ok}.$$

Thus, this expression will converge to this root.

- For the second root which $\in (3,4)$,

$$\left|g'(3)\right| = \left|\frac{-3.1}{(3-2.2)^2}\right| = 4.8 > 1 \text{ Not Ok}, \qquad \left|g'(4)\right| = \left|\frac{-3.1}{(4-2.2)^2}\right| = 0.96 \le 1 \text{ Ok}.$$

Thus, this expression will not converge to this root.

1st iteration: Let $x_o = -1 \Rightarrow x_1 = g(x_o) \Rightarrow x_1 = g(-1) = \dfrac{3.1}{(-1)-2.2} = -0.96875$.

2nd iteration: $x_1 = -0.96875 \Rightarrow x_2 = g(-0.96875) = \dfrac{3.1}{(-0.96875)-2.2} = -0.978304$.

The calculations must be repeated and continued until $\Delta \le \varepsilon$.

i	x_i	$x_{i+1} = g(x_i)$	$\Delta_i = \lvert x_{i+1} - x_i \rvert$
0	- 1	- 0.96875	0.031….
1	- 0.96875	- 0.978304	9.5×10^{-3}
2	- 0.978304	- 0.975363	2.9×10^{-3}
3	- 0.975363	- 0.976266	$9\times10^{-4} < \varepsilon$

The root is $x_{root} \approx -0.976266$.

$$y(-0.976266) = (-0.976266)^3/3 - 1.1(-0.976266)^2 - 3.1(-0.976266) = 1.667862.$$

Thus, the maximum value of y is 1.667862, approximately.

Note:

If we want to know which root would the third expression $x = \dfrac{x^2 - 3.1}{2.2}$ converge to, then we could use the convergence test:

$$g(x) = \frac{x^2 - 3.1}{2.2} \quad \Rightarrow \quad g'(x) = \frac{x}{1.1},$$

- For the first root which $\in (-1,0)$,

$$|g'(-1)| = \left|\frac{-1}{1.1}\right| = 0.91 \le 1 \text{ Ok}, \qquad |g'(0)| = \left|\frac{0}{1.1}\right| = 0 \le 1 \text{ Ok}.$$

Thus, this expression will converge to this root.

- For the second root which $\in (3,4)$,

$$|g'(3)| = \left|\frac{3}{1.1}\right| = 2.7 > 1 \text{ Not Ok}, \qquad |g'(4)| = \left|\frac{4}{1.1}\right| = 3.6 > 1 \text{ Not Ok}.$$

Thus, this expression will not converge to this root.

Example 2: Find the value of x which makes the function $f(x) = (2-x)e^{-x/4}$ equal to 1. ($\varepsilon = 1\times10^{-3}$)

Solution:

$$f(x) = 1 \quad \Rightarrow \quad (2-x)e^{-x/4} = 1,$$

$$\therefore (2-x)e^{-x/4} - 1 = 0 \quad \Rightarrow \quad h(x) = 0. \text{ (Root finding problem)}$$

So we must find the root(s) of $h(x)$ where $h(x) = (2-x)e^{-x/4} - 1$.

Check the sign of $h(x)$ at different values of x: (not necessary)

x	- 2	- 1	0	1	2	3
$h(x)$	1.4	1.3	1	- 0.22	- 1	- 1.47

Thus, there is a root lies between x = 0 and x = 1.

By using fixed point method, rearrange the equation $h(x)=0$ in the form $x=g(x)$:

$$(2-x)e^{-x/4}-1=0 \;\Rightarrow\; (2-x)e^{-x/4}=1 \;\Rightarrow\; 2-x=\frac{1}{e^{-x/4}},$$

$$2-x=e^{x/4} \;\Rightarrow\; x=2-e^{x/4}. \quad \text{(in this expression } g(x)=2-e^{x/4}\text{)}$$

<u>1st iteration:</u> Let $x_o=1 \;\Rightarrow\; x_1=g(x_o) \;\Rightarrow\; x_1=g(1)=2-e^{(1)/4}=0.715975$.

<u>2nd iteration:</u> $x_1=0.715975 \;\Rightarrow\; x_2=g(0.715975)=2-e^{(0.715975)/4}=0.803987$.

The calculations must be repeated and continued until $\Delta\le\varepsilon$.

i	x_i	$x_{i+1}=g(x_i)$	$\Delta_i=\lvert x_{i+1}-x_i\rvert$
0	1	0.715975	0.28….
1	0.715975	0.803987	0.08….
2	0.803987	0.777379	0.02….
3	0.777379	0.785485	8.1×10^{-3}
4	0.785485	0.783021	2.4×10^{-3}
5	0.783021	0.783771	$7.5\times10^{-4}<\varepsilon$

The root is $x_{root}\approx 0.783771$.

<u>Note:</u>

If we start with another possible expression of $x=g(x)$ like:

$$(2-x)e^{-x/4}-1=0 \;\Rightarrow\; e^{-x/4}=\frac{1}{2-x} \;\Rightarrow\; e^{x/4}=2-x,$$

$$\frac{x}{4}=\ln|2-x| \;\Rightarrow\; x=4\ln|2-x|. \quad \text{(In this expression } g(x)=4\ln|2-x|\text{)}$$

Then, we get the following results:

i	x_i	$x_{i+1}=g(x_i)$	$\Delta_i=\lvert x_{i+1}-x_i\rvert$
0	1	0	1
1	0	2.772589	2.77….
2	2.772589	-1.032034	3.03….
3	-1.032034	4.436934	5.46…. (divergence)

Thus, this expression does not converge to the required root. Therefore we must search for another expression of $x=g(x)$.