Numerical Analysis

Syllabus

1- Introduction.
2- Numerical Solution of Algebraic Equations (Roots of equations).
3- Numerical Solution of Set of Algebraic Equations.
4- Taylor Series.
5- Numerical Differentiation (Finite Differences Calculus).
6- Numerical Integration.
7- Numerical Solution of Ordinary Differential Equations.
8- Curve Fitting.
9- Interpolation And Extrapolation.

References

- Numerical Methods in Engineering Practice,
  by A. W. Al-Khafaji and J. R. Tooley.

- Numerical Methods,
  by R. W. Hornbeck.

- Numerical Methods Using MATLAB,
  by J. H. Mathew and K. D. Fink.

- Numerical Analysis,
  by R. L. Burden and J. D. Faires.

1-Introduction

Numerical methods

Numerical methods are a class of techniques used for solving a wide variety of mathematical problems in terms of numbers using only arithmetic and logic operations. The main advantage of numerical methods is their ability to solve problems that can not be treated using classical analytical mathematics, such as non-linearity and complex geometries. The disadvantage is that the solutions using numerical methods are iterative, approximate, and not exact as those obtained by analytical methods.

Errors in numerical computations

1- Errors from the method of solution, since all numerical methods are only approximate.
2- Errors from solution truncation, since numerical methods are iterative and the iterations can not continue infinite times.
3- Errors from numbers round off.
4- Errors from the mathematical model of the physical problems. For example in flexural formula the dx^2 term is neglected , also usually $\sin\theta$ is approximated to θ for small values of the later.

Error calculation

If x_{approx} is an approximate to x_{exact} then:

1- The absolute error is E or $\Delta = \left| x_{exact} - x_{approx} \right|$.

2- The relative error is $R = \left| \dfrac{x_{exact} - x_{approx}}{x_{exact}} \right|$.

3- The percent relative error is $P = \left| \dfrac{x_{exact} - x_{approx}}{x_{exact}} \right| \times 100$.

2- Numerical Solution of Algebraic Equations (Roots of Equations)

Introduction

A problem commonly encountered in engineering is that of determining the roots of an equation of the form $y = f(x)$. Finding the roots of an equation is equivalent to finding the values of x for which $f(x) = 0$. For this reason the roots of equation are often called the zeros of the equation. Different techniques of varying degrees of accuracy and rates of convergence were developed to determine these roots.

Root solving problem consists of finding the values of the independent variable which satisfy relationships, such as:

$$Ax^3 + Bx^2 = Cx + D.$$

The procedure for finding the roots will always be to collect all terms on one side of the equality sign, for example (for the above equation):

$$Ax^3 + Bx^2 - Cx - D = 0.$$

For any values of x other than the roots, this equation will not be satisfied. So in general:

$$f(x) = Ax^3 + Bx^2 - Cx - D.$$

Now, finding the roots of the above equation is now equivalent to finding the values of x for which $f(x)$ is zero, i.e:

$$f(x) = 0.$$

Single and multiple roots

1- x_1 is a single (simple) root if $f(x_1) = 0$ and $f'(x_1) \neq 0$.

2- x_1 is a multiple root of :

multiplicity 2 if $f(x_1) = f'(x_1) = 0$ and $f''(x_1) \neq 0$,

multiplicity 3 if $f(x_1) = f'(x_1) = f''(x_1) = 0$ and $f'''(x_1) \neq 0$, and so on.

Accuracy in roots determination

All roots determination numerical methods are iterative, hence roots of different degrees of accuracy can be obtained depending on the method used and the number of iteration performed. The iteration should be continued until one (or more) of such following conditions are satisfied:

1- $\Delta = \left| x_i - x_{i-1} \right| \le \varepsilon$, where ε is the allowed absolute difference between two successive trials (iterations).

2- $\left| f(x_i) \right| \le E$, where E is the allowed absolute error in the value of the function.

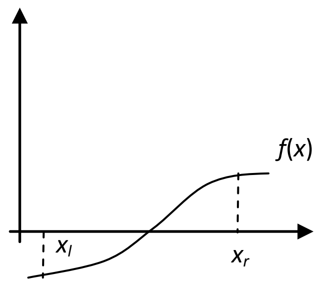
Solution of algebraic equations (determining roots of equations)

1- Bisection Method

This method, which is also known as interval halving method, is too inefficient for hand computation but is ideally suited to machine computation.

To find a real root of a given function *f(x)*, the following steps will be used:

1- Estimate two approximations of the root x_l and x_r such that $f(x_l) < 0$ ($f(x_l)$ is negative) and $f(x_r) > 0$ ($f(x_r)$ is positive).



2- Bisect the interval (x_l, x_r) to find its midpoint

$x_m = \dfrac{x_r + x_l}{2}$ (which is considered as an improved approximation of root).

3- Check the sign of $f(x_m)$. If $f(x_m) < 0$, this mean that the root lies between x_m and x_r, then for the next iteration let $x_l = x_m$. If $f(x_m) > 0$, this mean that the root lies between x_l and x_m, then for the next iteration let $x_r = x_m$.

4- Repeat steps 2 and 3 until the required accuracy ε is achieved.

Notes:

1- For each iteration, the root is assumed to be the midpoint of the last interval found to contain it, i.e; the root is x_m.

2- For each iteration, the maximum absolute error Δ in the value of the root is no greater than one half the size of last interval found to contain the root, i.e;

$$\Delta = \frac{\left| x_r - x_l \right|}{2} \quad \text{or} \quad \Delta = \left| (x_m)_i - (x_m)_{i-1} \right|.$$

3- The maximum error Δ in the value of the root in a given iteration is one half its value in the previous iteration; $\Delta_i = \frac{1}{2}\Delta_{i-1}$. So this method has very slow convergence.

4- The bisection method cannot be used to find roots of functions that do not change their sign (from positive to negative or from negative to positive).

5- Since $\Delta_i = \frac{1}{2}\Delta_{i-1}$ and $\Delta = \frac{|x_r - x_l|}{2}$, so we can estimate the number of iterations *n* required to find a root to an accuracy of ε as follows:

$$\Delta \le \varepsilon \;\Rightarrow\; \frac{|x_r - x_l|}{2^n} \le \varepsilon \;\Rightarrow\; 2^n.\varepsilon \ge |x_r - x_l| \;\Rightarrow\; n\ln 2 + \ln\varepsilon \ge \ln|x_r - x_l|,$$

$$\therefore n \ge \frac{\ln|x_r - x_l| - \ln\varepsilon}{\ln 2} \quad \text{or} \quad n \ge \frac{\ln\left|\frac{x_r - x_l}{\varepsilon}\right|}{\ln 2}.$$

Example 1: Find the root(s) of the function $f(x) = x^3 - 5x^2 - 2x + 10$ using six iterations.

Solution:

Check the sign of $f(x)$ at different values of *x*:

x	- 2	- 1	0	1	2	3	4	5
$f(x)$	- 14	6	10	4	- 6	- 14	- 8	0

There are three roots: The first root lies between *x* = -2 and *x* = -1, the second root lies between *x* = 1 and *x* = 2, and the third root is *x* = 5 (exact value).

To find the first root by using the bisection method,

1st iteration: Let $x_l = -2$ and $x_r = -1$.

$$x_m = \frac{x_r + x_l}{2} \quad\Rightarrow\quad x_m = \frac{-1 + (-2)}{2} = -1.5$$

Check the sign of $f(x_m)$ $\Rightarrow$ $f(-1.5) = (-1.5)^3 - 5(-1.5)^2 - 2(-1.5) + 10 = -1.625$.

Since $f(x_m) < 0$, then for the next iteration $x_l = x_m = -1.5$ and $x_r = -1$ (unchanged)

2nd iteration: $x_l = -1.5$ and $x_r = -1$.

The calculations must be repeated as in the 1st iteration and continued until the required number of iterations is reached.

It is more preferred to put the calculations in a table as below:

No. of Iteration (i)	x_l	x_r	$x_m = \frac{x_r + x_l}{2}$	$f(x_m)$	$\Delta = \left\lvert \frac{x_r - x_l}{2} \right\rvert$
1	- 2	- 1	- 1.5	- 1.625	0.5
2	- 1.5	- 1	- 1.25	2.73….	0.25
3	- 1.5	- 1.25	- 1.375	0.69….	0.125
4	- 1.5	- 1.375	- 1.4375	- 0.42….	0.0625
5	- 1.4375	- 1.375	- 1.40625	0.14….	0.03125
6	- 1.4375	- 1.40625	- 1.421875	- 0.13….	0.015625

After six iterations the first approximate root is $x_{root} \approx -1.421875$.

To find the second root:

By using the bisection method,

<u>1st iteration:</u> Let $x_l = 2$ and $x_r = 1$.

$$x_m = \frac{x_r + x_l}{2} \quad \Rightarrow \quad x_m = \frac{1+2}{2} = 1.5$$

Check the sign of $f(x_m) \quad \Rightarrow \quad f(1.5) = (1.5)^3 - 5(1.5)^2 - 2(1.5) + 10 = -0.875$.

Since $f(x_m) < 0$, then for the next iteration $x_l = x_m = 1.5$ and $x_r = 1$ (unchanged).

<u>2nd iteration:</u> $x_l = 1.5$ and $x_r = 1$. And so on.

i	x_l	x_r	$x_m = \frac{x_r + x_l}{2}$	$f(x_m)$	$\Delta = \left\lvert \frac{x_r - x_l}{2} \right\rvert$
1	2	1	1.5	- 0.875	0.5
2	1.5	1	1.25	1.64….	0.25
3	1.5	1.25	1.375	0.39….	0.125
4	1.5	1.375	1.4375	- 0.42….	0.0625
5	1.4375	1.375	1.40625	0.05….	0.03125
6	1.4375	1.40625	1.421875	- 0.07….	0.015625

After six iterations the second approximate root is $x_{root} \approx 1.421875$.

Example 2: Find the point(s) of intersection of $y = \ln x$ and $y = 2x - 3$ accurately to three decimal places (i.e; $\varepsilon = 1 \times 10^{-3}$).

Solution:

To find the point(s) of intersection we put $y_1 = y_2 \Rightarrow \ln x = 2x - 3$,

$$\therefore \ln x - 2x + 3 = 0 \quad \Rightarrow \quad f(x) = 0 \text{ (Root finding problem)}$$

So we must find the root(s) of $f(x)$ where $f(x) = \ln x - 2x + 3$.

Check the sign of $f(x)$ at different values of x:

x	0.01	1	2	3	4	5
$f(x)$	- 1.625	1	- 0.306	- 1.9	- 3.6	- 5.39

There are two roots: The first root lies between x = 1 and x = 2 and the second root lies between x = 0.01 and x = 1. To find the first root by using the bisection method,

<u>1st iteration:</u> Let $x_l = 2$ and $x_r = 1 \Rightarrow x_m = \frac{x_r + x_l}{2} \Rightarrow x_m = \frac{1+2}{2} = 1.5$.

Check the sign of $f(x_m) \Rightarrow f(1.5) = \ln(1.5) - 2(1.5) + 3 = 0.40547$.

Since $f(x_m) > 0$, then for the next iteration $x_l = 2$ (unchanged) and $x_r = x_m = 1.5$.

<u>2nd iteration:</u> $x_l = 2$ and $x_r = 1.5$.

The calculations must be repeated as in the 1st iteration and continued until $\Delta \le \varepsilon$.

i	x_l	x_r	$x_m = \frac{x_r + x_l}{2}$	$f(x_m)$	$\Delta = \left\lvert \frac{x_r - x_l}{2} \right\rvert$
1	2	1	1.5	0.405…	0.5
2	2	1.5	1.75	0.059....	0.25
3	2	1.75	1.875	- 0.12….	0.125
4	1.875	1.75	1.8125	- 0.03….	0.0625
5	1.8125	1.75	1.78125	0.14….	0.03125
6	1.8125	1.78125	1.79688	- 0.007….	0.015625
7	1.79688	1.78125	1.78906	0.003…	0.00782
8	1.79688	1.78906	1.79297	- 0.002…	0.00391
9	1.79297	1.78906	1.79101	- 0.0007..	0.00196
10	1.79101	1.78906	1.79003	0.002…	$0.00098 < \varepsilon$

After 10 iterations the first approximate root is $x_{root} \approx 1.79003$.

$y \approx \ln 1.79003 \approx 0.582232 \Rightarrow$ the first point of intersection is (1.79003,0.582232).

H.W:

The second point of intersection is (0.048673,-2.9026).

Note:

If we want to estimate the number of iterations required to find the above first root to the given accuracy, then:

$$n \geq \frac{\ln\left|\frac{x_r - x_l}{\varepsilon}\right|}{\ln 2} \quad \Rightarrow \quad n \geq \frac{\ln\left|\frac{1-2}{1\times 10^{-3}}\right|}{\ln 2} \quad \Rightarrow \quad n \geq 9.96.$$

So we need 10 iterations.