3. Gauss-Jordan elimination

Example: Solve the following system

$$2x+4y-2z=2,$$

$$x+z=3,$$

$$2x+y-z=1.$$

Solution :

In matrix form:

$$\begin{bmatrix} 2 & 4 & -2 \\ 1 & 0 & 1 \\ 2 & 1 & -1 \end{bmatrix} \begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}.$$

Use Gauss-Jordan elimination method to solve the above matrix:

Step 1;

$$\begin{matrix} R_1/2 \rightarrow \\ (-1)R_1+R_2 \rightarrow \\ (-2)R_1+R_3 \rightarrow \end{matrix} \begin{bmatrix} 1 & 2 & -1 \\ 0 & -2 & 2 \\ 0 & -3 & 1 \end{bmatrix} \begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}.$$

Step 2;

$$\begin{matrix} (-2)R_2+R_1 \rightarrow \\ R_2/(-2) \rightarrow \\ (3)R_2+R_3 \rightarrow \end{matrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & -2 \end{bmatrix} \begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = \begin{bmatrix} 3 \\ -1 \\ -4 \end{bmatrix}.$$

Step 3;

$$\begin{matrix} (-1)R_3+R_1 \rightarrow \\ (1)R_3+R_2 \rightarrow \\ R_3/(-2) \rightarrow \end{matrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}.$$

$$\therefore \quad \begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}.$$

4. Inverse of a matrix

$$C.X = R \qquad \Rightarrow \qquad C^{-1}.C.X = C^{-1}.R.$$

But, $C^{-1}.C = I \qquad \Rightarrow \qquad X = C^{-1}.R.$

Example: Solve the following system

$$2x + 4y - 2z = 2,$$

$$x + z = 3,$$

$$2x + y - z = 1.$$

Solution :

In matrix form:

$$\begin{bmatrix} 2 & 4 & -2 \\ 1 & 0 & 1 \\ 2 & 1 & -1 \end{bmatrix} \begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}.$$

Use matrix inverse method to solve the above matrix:

To find C^{-1};

$$\left[\begin{matrix} 2 & 4 & -2 \\ 1 & 0 & 1 \\ 2 & 1 & -1 \end{matrix}\right]\left[\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}\right].$$

$$\begin{matrix} R_1/2 \rightarrow \\ (-1)R_1 + R_2 \rightarrow \\ (-2)R_1 + R_3 \rightarrow \end{matrix} \left[\begin{matrix} 1 & 2 & -1 \\ 0 & -2 & 2 \\ 0 & -3 & 1 \end{matrix}\right]\left[\begin{matrix} 1/2 & 0 & 0 \\ -1/2 & 1 & 0 \\ -1 & 0 & 1 \end{matrix}\right].$$

$$\begin{matrix} (-2)R_2 + R_1 \rightarrow \\ R_2/(-2) \rightarrow \\ (3)R_2 + R_3 \rightarrow \end{matrix} \left[\begin{matrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & -2 \end{matrix}\right]\left[\begin{matrix} 0 & 1 & 0 \\ 1/4 & -1/2 & 0 \\ -1/4 & -3/2 & 1 \end{matrix}\right].$$

$$\begin{matrix} (-1)R_3 + R_1 \rightarrow \\ (1)R_3 + R_2 \rightarrow \\ R_3/(-2) \rightarrow \end{matrix} \left[\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}\right]\left[\begin{matrix} -1/8 & 1/4 & 1/2 \\ 3/8 & 1/4 & -1/2 \\ 1/8 & 3/4 & -1/2 \end{matrix}\right].$$

$$\therefore \qquad C^{-1}=\begin{bmatrix}-1/8 & 1/4 & 1/2\\ 3/8 & 1/4 & -1/2\\ 1/8 & 3/4 & -1/2\end{bmatrix}.$$

$$X=C^{-1}.R \qquad \Rightarrow \qquad \begin{Bmatrix}x\\ y\\ z\end{Bmatrix}=\begin{bmatrix}-1/8 & 1/4 & 1/2\\ 3/8 & 1/4 & -1/2\\ 1/8 & 3/4 & -1/2\end{bmatrix}\begin{bmatrix}2\\ 3\\ 1\end{bmatrix}.$$

$$\therefore \quad x=-\frac{1}{8}\times 2+\frac{1}{4}\times 3+\frac{1}{2}\times 1=-\frac{1}{4}+\frac{3}{4}+\frac{1}{2}=1,$$

$$y=\frac{3}{8}\times 2+\frac{1}{4}\times 3-\frac{1}{2}\times 1=\frac{3}{4}+\frac{3}{4}-\frac{1}{2}=1,$$

$$z=\frac{1}{8}\times 2+\frac{3}{4}\times 3-\frac{1}{2}\times 1=\frac{1}{4}+\frac{9}{4}-\frac{1}{2}=2.$$

Note; we can find C^{-1} by another procedure, as follows,

$$C^{-1}=\frac{adjointC}{|C|}, \qquad \text{where,} \qquad adjointC=[cofactorC]^T.$$

$$|C|=\begin{vmatrix}2 & 4 & -2\\ 1 & 0 & 1\\ 2 & 1 & -1\end{vmatrix}=2\begin{vmatrix}0 & 1\\ 1 & -1\end{vmatrix}+(-1)(4)\begin{vmatrix}1 & 1\\ 2 & -1\end{vmatrix}+(-2)\begin{vmatrix}1 & 0\\ 2 & 1\end{vmatrix}$$

$$=2[0(-1)-1(1)]-4[1(-1)-1(2)]-2[1(1)-0(2)]=-2+12-2=8.$$

$$cofactorC=\begin{bmatrix}+\begin{vmatrix}0 & 1\\ 1 & -1\end{vmatrix} & -\begin{vmatrix}1 & 1\\ 2 & -1\end{vmatrix} & +\begin{vmatrix}1 & 0\\ 2 & 1\end{vmatrix}\\ -\begin{vmatrix}4 & -2\\ 1 & -1\end{vmatrix} & +\begin{vmatrix}2 & -2\\ 2 & -1\end{vmatrix} & -\begin{vmatrix}2 & 4\\ 2 & 1\end{vmatrix}\\ +\begin{vmatrix}4 & -2\\ 0 & 1\end{vmatrix} & -\begin{vmatrix}2 & -2\\ 1 & 1\end{vmatrix} & +\begin{vmatrix}2 & 4\\ 1 & 0\end{vmatrix}\end{bmatrix}=\begin{bmatrix}-1 & 3 & 1\\ 2 & 2 & 6\\ 4 & -4 & -4\end{bmatrix}.$$

$$adjointC=[cofactorC]^T=\begin{bmatrix}-1 & 2 & 4\\ 3 & 2 & -4\\ 1 & 6 & -4\end{bmatrix}.$$

$$\therefore \qquad C^{-1}=\frac{1}{8}\times\begin{bmatrix}-1 & 2 & 4\\ 3 & 2 & -4\\ 1 & 6 & -4\end{bmatrix}=\begin{bmatrix}-1/8 & 1/4 & 1/2\\ 3/8 & 1/4 & -1/2\\ 1/8 & 3/4 & -1/2\end{bmatrix}.$$