7- Matrices and Determinants for Solving Simultaneous Algebraic Equations

Introduction

The solution of set of algebraic equations is an important step in wide variety of engineering problems, such as the structural analysis, network analysis, numerical solution of differential equations, ….etc. There are various methods to solve a set of algebraic equations, such as Cramer's rule, Gauss elimination, Gauss-Jordan elimination, and matrix inverse.

1. Cramer's rule

$$C.X = R \qquad \Rightarrow \qquad X_k = \frac{|C_k|}{|C|}.$$

Example: Solve the following set of algebraic equations

$$2x + 4y - 2z - 2 = 0,$$

$$x + z - 3 = 0,$$

$$2x + y - z - 1 = 0.$$

Solution :

$$2x + 4y - 2z = 2,$$

$$x + z = 3,$$

$$2x + y - z = 1.$$

In matrix form:

$$\begin{bmatrix} 2 & 4 & -2 \\ 1 & 0 & 1 \\ 2 & 1 & -1 \end{bmatrix} \begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}.$$

Using Cramer's rule to solve the above matrix, gives

$$x = \frac{\begin{vmatrix} 2 & 4 & -2 \\ 3 & 0 & 1 \\ 1 & 1 & -1 \end{vmatrix}}{\begin{vmatrix} 2 & 4 & -2 \\ 1 & 0 & 1 \\ 2 & 1 & -1 \end{vmatrix}} = \frac{2\begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix} + (-1)(4)\begin{vmatrix} 3 & 1 \\ 1 & -1 \end{vmatrix} + (-2)\begin{vmatrix} 3 & 0 \\ 1 & 1 \end{vmatrix}}{2\begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix} + (-1)(4)\begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} + (-2)\begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix}}$$

$$= \frac{2[0(-1)-1(1)]-4[3(-1)-1(1)]-2[3(1)-0(1)]}{2[0(-1)-1(1)]-4[1(-1)-1(2)]-2[1(1)-0(2)]} = \frac{8}{8} = 1.$$

Similarly,

$$y = \frac{\begin{vmatrix} 2 & 2 & -2 \\ 1 & 3 & 1 \\ 2 & 1 & -1 \end{vmatrix}}{\begin{vmatrix} 2 & 4 & -2 \\ 1 & 0 & 1 \\ 2 & 1 & -1 \end{vmatrix}} = \frac{2\begin{vmatrix} 3 & 1 \\ 1 & -1 \end{vmatrix} + (-1)(2)\begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} + (-2)\begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix}}{8}$$

$$= \frac{2[3(-1)-1(1)]-2[1(-1)-1(2)]-2[1(1)-3(2)]}{8} = \frac{8}{8} = 1.$$

$$z = \frac{\begin{vmatrix} 2 & 4 & 2 \\ 1 & 0 & 3 \\ 2 & 1 & 1 \end{vmatrix}}{\begin{vmatrix} 2 & 4 & -2 \\ 1 & 0 & 1 \\ 2 & 1 & -1 \end{vmatrix}} = \frac{2\begin{vmatrix} 0 & 3 \\ 1 & 1 \end{vmatrix} + (-1)(4)\begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} + 2\begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix}}{8}$$

$$= \frac{2[0(1)-3(1)]-4[1(1)-3(2)]+2[1(1)-0(2)]}{8} = \frac{16}{8} = 2.$$

2. Gauss elimination

Example: Solve the following system of algebraic equations

$$2x + 4y - 2z = 2,$$

$$x + z = 3,$$

$$2x + y - z = 1.$$

Solution :

In matrix form:

$$\begin{bmatrix} 2 & 4 & -2 \\ 1 & 0 & 1 \\ 2 & 1 & -1 \end{bmatrix} \begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}.$$

Use Gauss elimination method to solve the above matrix:

Step 1;

$$\begin{matrix} R_1/2 \rightarrow \\ (-1)R_1 + R_2 \rightarrow \\ (-2)R_1 + R_3 \rightarrow \end{matrix} \begin{bmatrix} 1 & 2 & -1 \\ 0 & -2 & 2 \\ 0 & -3 & 1 \end{bmatrix} \begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}.$$

Step 2;

$$\begin{matrix} \\ R_2/(-2) \rightarrow \\ (3)R_2 + R_3 \rightarrow \end{matrix} \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & -2 \end{bmatrix} \begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -4 \end{bmatrix}.$$

Note: R_1, R_2 and R_3 mean the first, second and third row, respectively.

From $R_3 \Rightarrow \quad -2z = -4 \quad \Rightarrow \quad z = 4/2 \quad \Rightarrow \quad z = 2$.

Use back substitution:

From $R_2 \Rightarrow \quad y - z = -1 \quad \Rightarrow \quad y = -1 + z \quad \Rightarrow \quad y = -1 + 2 \quad \Rightarrow \quad y = 1$.

From $R_1 \Rightarrow \quad x + 2y - z = 1 \quad \Rightarrow \quad x = 1 - 2y + z \quad \Rightarrow \quad x = 1 - 2(1) + 2 \quad \Rightarrow \quad x = 1$.