6- Simultaneous Linear Ordinary Differential Equations

Example 1: Solve the following differential equations

$$\frac{dy}{dt}+\frac{dx}{dt}-3t=0,$$

$$2\frac{dx}{dt}+y=0.$$

Solution :

Using D-operator gives,

$$Dy+Dx=3t, \qquad \text{.........} (1)$$

$$y+2Dx=0. \qquad \text{.........} (2)$$

In matrix form:

$$\begin{bmatrix} D & D \\ 1 & 2D \end{bmatrix}\begin{Bmatrix} y \\ x \end{Bmatrix}=\begin{bmatrix} 3t \\ 0 \end{bmatrix}.$$

Using Cramer's rule to solve the above matrix, gives

$$y=\frac{\begin{vmatrix} 3t & D \\ 0 & 2D \end{vmatrix}}{\begin{vmatrix} D & D \\ 1 & 2D \end{vmatrix}}=\frac{2D(3t)-D(0)}{2D(D)-1(D)}=\frac{6}{2D^2-D},$$

or $(2D^2-D)y=6.$ (Non-homogeneous linear ODE with constant coefficients)

$$2m^2-m=0 \quad \Rightarrow \quad m(2m-1)=0 \quad \Rightarrow \quad m_1=0 \quad \text{and} \quad m_2=1/2.$$

$$\therefore \quad y_c=C_1+C_2e^{t/2}.$$

Let $y_p=A_o.t \quad \Rightarrow \quad y'_p=A_o \quad \Rightarrow \quad y''_p=0.$

Substituting,

$2(0) - A_o = 6 \quad \Rightarrow \quad A_o = -6 \quad \Rightarrow \quad y_p = -6t$.

$y = y_c + y_p \quad \Rightarrow \quad y = C_1 + C_2 e^{t/2} - 6t$.

Similarly,

$$x = \frac{\begin{vmatrix} D & 3t \\ 1 & 0 \end{vmatrix}}{\begin{vmatrix} D & D \\ 1 & 2D \end{vmatrix}} = \frac{D(0) - 3t(1)}{2D(D) - 1(D)} = \frac{-3t}{2D^2 - D},$$

or $(2D^2 - D)x = -3t$. (Non-homogeneous linear ODE with constant coefficients)

$2m^2 - m = 0 \quad \Rightarrow \quad m(2m-1) = 0 \quad \Rightarrow \quad m_1 = 0$ and $m_2 = 1/2$.

$\therefore \quad x_c = C_3 + C_4 e^{t/2}$.

Let $x_p = (B_o + B_1 t)t = B_o t + B_1 t^2 \quad \Rightarrow \quad x'_p = B_o + 2B_1 t \quad \Rightarrow \quad x''_p = 2B_1$.

Substituting,

$2(2B_1) - (B_o + 2B_1 t) = -3t$.

$\therefore \quad -2B_1 = -3 \quad \Rightarrow \quad B_1 = 3/2$,

$4B_1 - B_o = 0 \quad \Rightarrow \quad B_o = 4B_1 = 4(3/2) = 6$.

$\therefore \quad x_p = 6t + \frac{3}{2}t^2$.

$x = x_c + x_p \quad \Rightarrow \quad x = C_3 + C_4 e^{t/2} + 6t + \frac{3}{2}t^2$.

Substituting x and y in Eq. (2) gives,

$C_1 + C_2 e^{t/2} - 6t + 2(\frac{1}{2}C_4 e^{t/2} + 6 + 3t) = 0$,

$C_1 + 12 + (C_2 + C_4)e^{t/2} = 0$,

$\therefore \quad C_1 + 12 = 0 \quad \Rightarrow \quad C_1 = -12$,

$C_2 + C_4 = 0 \quad \Rightarrow \quad C_2 = -C_4$.

$\therefore \quad y = -12 - C_4 e^{t/2} - 6t$ and $x = C_3 + C_4 e^{t/2} + 6t + \frac{3}{2}t^2$,

or $y = -12 - Ae^{t/2} - 6t$ and $x = B + Ae^{t/2} + 6t + \frac{3}{2}t^2$. (G.S)

Example 2: Solve

$$\frac{dy}{dt}+\frac{dx}{dt}+3y+5x-e^{-t}=0,$$

$$2\frac{dx}{dt}+\frac{dy}{dt}+x+y-3=0.$$

Solution :

Using D-operator gives,

$$Dx+5x+Dy+3y=e^{-t} \quad \Rightarrow \quad (D+5)x+(D+3)y=e^{-t}, \qquad \dots\dots\dots (1)$$

$$2Dx+x+Dy+y=3 \quad \Rightarrow \quad (2D+1)x+(D+1)y=3, \qquad \dots\dots\dots (2)$$

In matrix form:

$$\begin{bmatrix} D+5 & D+3 \\ 2D+1 & D+1 \end{bmatrix}\begin{Bmatrix} x \\ y \end{Bmatrix}=\begin{bmatrix} e^{-t} \\ 3 \end{bmatrix}.$$

Using Cramer's rule to solve the above matrix, gives

$$x=\frac{\begin{vmatrix} e^{-t} & D+3 \\ 3 & D+1 \end{vmatrix}}{\begin{vmatrix} D+5 & D+3 \\ 2D+1 & D+1 \end{vmatrix}}=\frac{(D+1)e^{-t}-(D+3)(3)}{(D+5)(D+1)-(2D+1)(D+3)}$$

$$=\frac{-e^{-t}+e^{-t}-0-9}{D^2+D+5D+5-(2D^2+6D+D+3)}=\frac{-9}{-D^2-D+2}=\frac{9}{D^2+D-2},$$

or $\quad (D^2+D-2)x=9$. (Non-homogeneous linear ODE with constant coefficients)

$$m^2+m-2=0 \quad \Rightarrow \quad (m+2)(m-1)=0 \quad \Rightarrow \quad m_1=-2 \ \text{ and } \ m_2=1.$$

$$\therefore \quad x_c=C_1e^{-2t}+C_2e^{t}.$$

Let $\quad x_p=A_o \quad \Rightarrow \quad x'_p=0=x''_p$.

Substituting,

$$0+0-2A_o=9 \quad \Rightarrow \quad A_o=-9/2 \quad \Rightarrow \quad x_p=-9/2.$$

$$x=x_c+x_p \quad \Rightarrow \quad x=C_1e^{-2t}+C_2e^{t}-\frac{9}{2}.$$

Similarly,

$$y=\frac{\begin{vmatrix} D+5 & e^{-t} \\ 2D+1 & 3 \end{vmatrix}}{\begin{vmatrix} D+5 & D+3 \\ 2D+1 & D+1 \end{vmatrix}}=\frac{(D+5)(3)-(2D+1)e^{-t}}{-D^2-D+2}$$

$$=\frac{0+15+2e^{-t}-e^{-t}}{-D^2-D+2}=\frac{15+e^{-t}}{-D^2-D+2}=\frac{-15-e^{-t}}{D^2+D-2},$$

Or $(D^2+D-2)y=-e^{-t}-15$. (Non-homogeneous LODE with constant coeffs.)

$$m^2+m-2=0 \quad \Rightarrow \quad (m+2)(m-1)=0 \quad \Rightarrow \quad m_1=-2 \quad \text{and} \quad m_2=1.$$

$$\therefore \quad y_c=C_3e^{-2t}+C_4e^{t}.$$

Let $y_p=A_1+A_2e^{-t} \quad \Rightarrow \quad y'_p=-A_2e^{-t} \quad \Rightarrow \quad y''_p=A_2e^{-t}$.

Substituting,

$$A_2e^{-t}-A_2e^{-t}-2(A_1+A_2e^{-t})=-e^{-t}-15 \quad \Rightarrow \quad -2A_1-2A_2e^{-t}=-e^{-t}-15.$$

$$\therefore \quad -2A_1=-15 \quad \Rightarrow \quad A_1=15/2,$$

$$-2A_2=-1 \quad \Rightarrow \quad A_2=1/2,$$

$$\therefore \quad y_p=\frac{1}{2}e^{-t}+\frac{15}{2}.$$

$$y=y_c+y_p \quad \Rightarrow \quad y=C_3e^{-2t}+C_4e^{t}+\frac{1}{2}e^{-t}+\frac{15}{2}.$$

Substituting *x* and *y* in Eq. (1) gives,

$$-2C_1e^{-2t}+C_2e^{t}+5(C_1e^{-2t}+C_2e^{t}-\frac{9}{2})-2C_3e^{-2t}+C_4e^{t}-\frac{1}{2}e^{-t}+3(C_3e^{-2t}+$$

$$C_4e^{t}+\frac{1}{2}e^{-t}+\frac{15}{2})=e^{-t}.$$

$$(3C_1+C_3)e^{-2t}+(6C_2+4C_4)e^{t}-\frac{45}{2}+\frac{45}{2}+e^{-t}=e^{-t}.$$

$$\therefore \quad 3C_1+C_3=0 \quad \Rightarrow \quad C_3=-3C_1,$$

$$6C_2+4C_4=0 \quad \Rightarrow \quad C_4=-\frac{3}{2}C_2.$$

$$\therefore \quad x=C_1e^{-2t}+C_2e^{t}-\frac{9}{2} \quad \text{and} \quad y=-3C_1e^{-2t}-\frac{3}{2}C_2e^{t}+\frac{1}{2}e^{-t}+\frac{15}{2}. \qquad \text{(G.S)}$$