Forced vibration

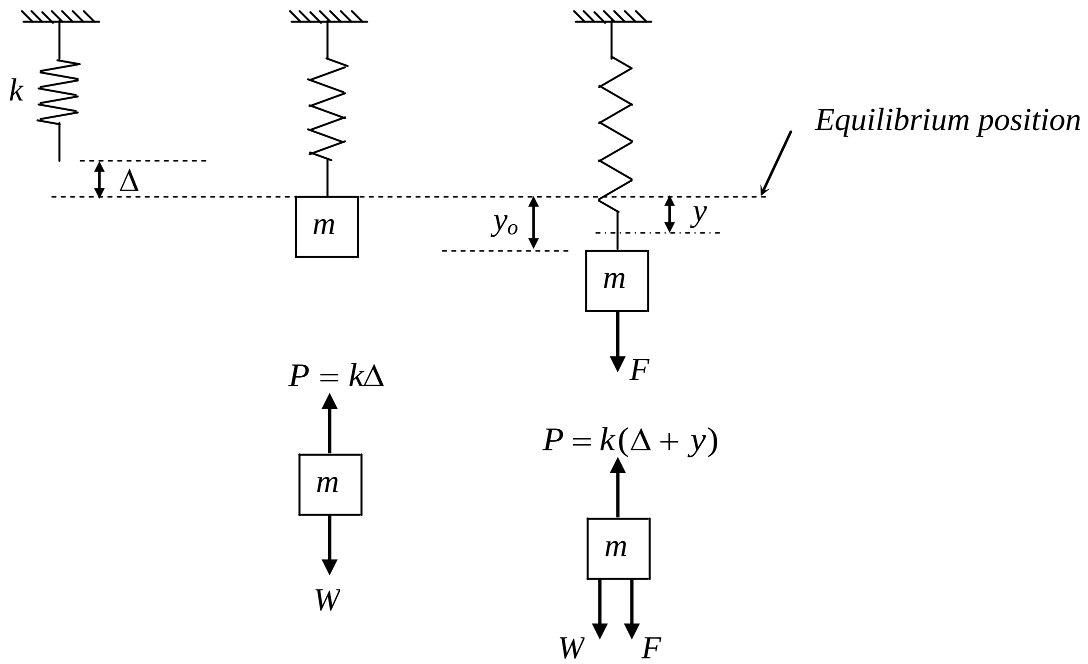
Undamped forced vibration

In the dynamic case;

$$\sum F_y = m.a \quad \Rightarrow \quad F + W - P = m.a \quad \Rightarrow \quad F + W - k(\Delta + y) = m.\ddot{y},$$

$$\Rightarrow \quad F + W - k\Delta - ky = m.\ddot{y}.$$

But, from static case $W = k\Delta \quad \Rightarrow \quad F - ky = m.\ddot{y} \quad \Rightarrow \quad m.\ddot{y} + ky = F$.



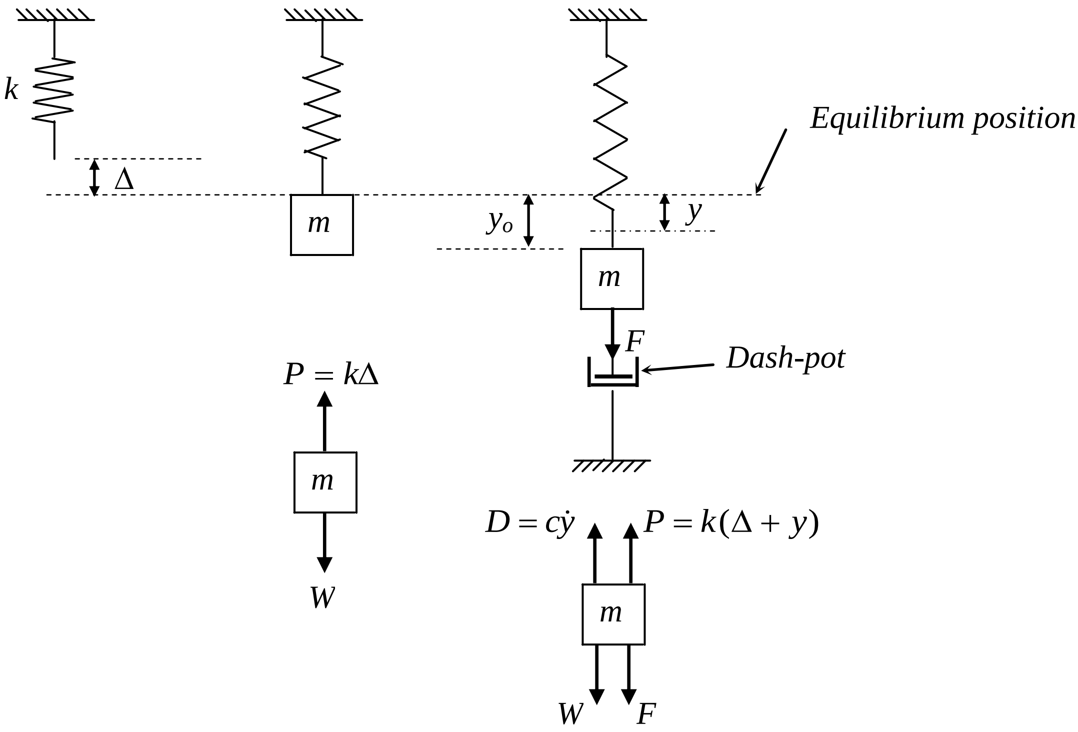
Damped forced vibration

In the dynamic case;

$$\sum F_y = m.a \quad \Rightarrow \quad F + W - P - D = m.a \quad \Rightarrow \quad F + W - k(\Delta + y) - c\dot{y} = m.\ddot{y},$$

$$\Rightarrow \quad F + W - k\Delta - ky - c\dot{y} = m.\ddot{y}.$$

But, from static case $W = k\Delta \quad \Rightarrow \quad F - ky - c\dot{y} = m.\ddot{y} \quad \Rightarrow \quad m.\ddot{y} + c\dot{y} + ky = F$.



Example 1: A 5 *kg* mass is connected to a spring of 8.5 *kN/m* stiffness and subjected to an exciting dynamic force of $50\cos 30t$ *kN*. Assuming the viscous damping coefficient is 24% of the critical damping, determine the equation of motion of the mass. (Initial displacement and velocity are zero)

Solution:

The case is damped forced vibration,

$$\therefore \quad m.\ddot{y} + c\dot{y} + ky = F,$$

$$c_{cr} = 2\sqrt{km} = 2\sqrt{8.5 \times 1000 \times 5} = 412.3 \ N.s/m,$$

$$c = 0.24c_{cr} = 0.24 \times 412.3 \approx 100 \ N.s/m,$$

$$\therefore \quad 5\ddot{y} + 100\dot{y} + 8500y = 50000\cos 30t \quad \text{or} \quad (5D^2 + 100D + 8500)y = 50000\cos 30t,$$

$$\therefore \quad 5r^2 + 100r + 8500 = 0 \quad \Rightarrow \quad r_{1,2} = \frac{-100 \pm \sqrt{100^2 - 4(5)(8500)}}{2(5)} = -10 \pm 40i,$$

$$\therefore \quad y_c = e^{-10t}(A\cos 40t + B\sin 40t).$$

Let $y_p = C_1\cos 30t + C_2\sin 30t$,

$$\Rightarrow \dot{y}_p = -30C_1\sin 30t + 30C_2\cos 30t \quad \Rightarrow \quad \ddot{y}_p = -900C_1\cos 30t - 900C_2\sin 30t .$$

Substituting,

$$5(-900C_1\cos 30t - 900C_2\sin 30t) + 100(-30C_1\sin 30t + 30C_2\cos 30t) +$$
$$8500(C_1\cos 30t + C_2\sin 30t) = 50000\cos 30t ,$$

$$(4000C_1 + 3000C_2)\cos 30t + (-3000C_1 + 4000C_2)\sin 30t = 50000\cos 30t ,$$

$$\therefore \quad 4000C_1 + 3000C_2 = 50000 \qquad \ldots\ldots\ldots\ (1)$$

$$-3000C_1 + 4000C_2 = 0 \qquad \ldots\ldots\ldots\ (2)$$

Solving Eqs. (1) & (2) simultaneously yields, $C_1 = 8$ and $C_2 = 6$.

$$\therefore \quad y_p = 8\cos 30t + 6\sin 30t .$$

$$y = y_c + y_p \quad \Rightarrow \quad y = e^{-10t}(A\cos 40t + B\sin 40t) + 8\cos 30t + 6\sin 30t . \qquad \text{(G.S)}$$

Initial conditions,

1. $y(0) = 0 \quad \Rightarrow \quad 0 = A + 0 + 8 + 0 \quad \Rightarrow \quad A = -8$.
2. $v(0) = \dot{y}(0) = 0$,

$$\dot{y} = e^{-10t}(-40A\sin 40t + 40B\cos 40t) - 10e^{-10t}(A\cos 40t + B\sin 40t) -$$
$$30(8)\sin 30t + 30(6)\cos 30t ,$$

$$\Rightarrow \quad 0 = 0 + 40B - 10(A + 0) - 0 + 180 \quad \Rightarrow \quad B = -6.5 .$$

$$\therefore \quad y = e^{-10t}(-8\cos 40t - 6.5\sin 40t) + 8\cos 30t + 6\sin 30t ,$$

$$\text{or} \quad y = -e^{-10t}(8\cos 40t + 6.5\sin 40t) + 8\cos 30t + 6\sin 30t . \qquad \text{(P.S)}$$

Example 2: Find the equation of motion for a system subjected to the external force $F\sin\omega_f t$. (Neglect damping)

Solution:

The case is undamped forced vibration,

$$\therefore \quad m.\ddot{y} + ky = F\sin w_f t \quad \Rightarrow \quad \ddot{y} + \frac{k}{m}y = \frac{F}{m}\sin\omega_f t .$$

Let $\omega^2 = \dfrac{k}{m} \quad \Rightarrow \quad \ddot{y} + \omega^2 y = \dfrac{F}{m}\sin\omega_f t$,

$$(D^2 + \omega^2)y = 0 \quad \Rightarrow \quad r^2 + \omega^2 = 0 \quad \Rightarrow \quad r^2 = -\omega^2 \quad \Rightarrow \quad r_{1,2} = \pm\omega i ,$$

$$\therefore \quad y_c = A\cos\omega t + B\sin\omega t .$$

Let $y_p = C_1 \cos\omega_f t + C_2 \sin\omega_f t$,

$$\Rightarrow \dot{y}_p = -\omega_f C_1 \sin\omega_f t + \omega_f C_2 \cos\omega_f t \quad \Rightarrow \quad \ddot{y}_p = -\omega_f^2 C_1 \cos\omega_f t - \omega_f^2 C_2 \sin\omega_f t .$$

Substituting,

$$-\omega_f^2 C_1 \cos\omega_f t - \omega_f^2 C_2 \sin\omega_f t + \omega^2 (C_1 \cos\omega_f t + C_2 \sin\omega_f t) = \frac{F}{m} \sin\omega_f t ,$$

$$(-\omega_f^2 C_1 + \omega^2 C_1)\cos\omega_f t + (-\omega_f^2 C_2 + \omega^2 C_2)\sin\omega_f t = \frac{F}{m}\sin\omega_f t ,$$

$$\therefore \ -\omega_f^2 C_1 + \omega^2 C_1 = 0 \quad \Rightarrow \quad (-\omega_f^2 + \omega^2) C_1 = 0 \quad \Rightarrow \quad C_1 = 0 ,$$

$$-\omega_f^2 C_2 + \omega^2 C_2 = \frac{F}{m} \quad \Rightarrow \quad (-\omega_f^2 + \omega^2) C_2 = \frac{F}{m} \quad \Rightarrow \quad C_2 = \frac{F}{m(\omega^2 - \omega_f^2)} ,$$

$$\therefore \ y_p = \frac{F}{m(\omega^2 - \omega_f^2)} \sin\omega_f t .$$

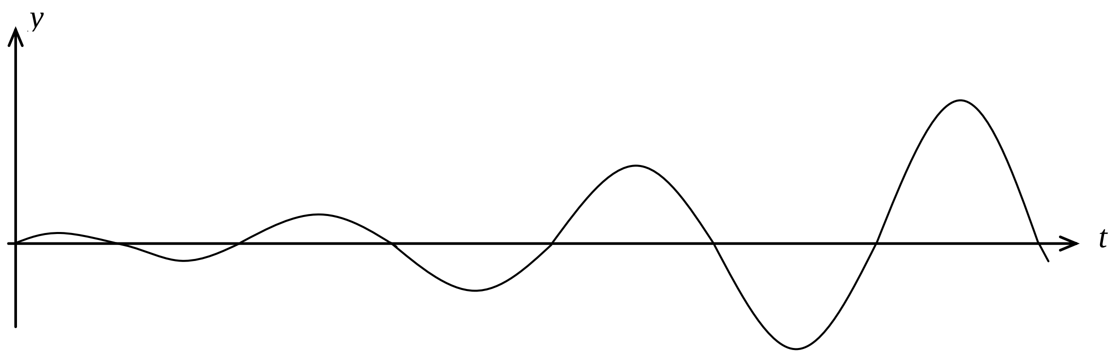
$$y = y_c + y_p \quad \Rightarrow \quad y = A\cos\omega t + B\sin\omega t + \frac{F}{m(\omega^2 - \omega_f^2)}\sin\omega_f t . \qquad \text{(G.S)}$$

Notes,

***** If $\omega_f \to \omega$, then $y \to \infty$. (Resonance)

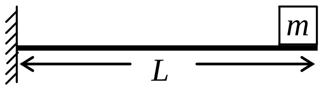
***** If $\omega_f = \omega$, then $y = A\cos\omega t + B\sin\omega t - \dfrac{F}{2m\omega} t\cos\omega t$.

(i.e. solution increases in amplitude as *t* increases)



Simple vibration of structures

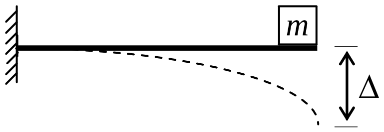
Example 1: A mass m is put on the end of a cantilever beam, of negligible mass, as shown below. Determine the natural frequency of this system.



Solution :

Let the deflection at the free end is Δ.

The deflection at the free end due to a tip concentrated load P is,



$$\Delta = \frac{PL^3}{3EI}.$$

The stiffness for a single DOF system,

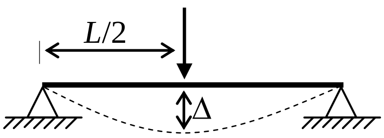
$$k = \frac{P}{\Delta} \quad \Rightarrow \quad k = \frac{P}{PL^3/3EI} \quad \Rightarrow \quad \therefore\ k = \frac{3EI}{L^3}.$$

$$\omega = \sqrt{\frac{k}{m}} \quad \Rightarrow \quad \omega = \sqrt{\frac{3EI}{mL^3}},$$

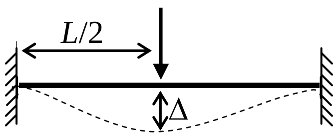
$$f = \frac{\omega}{2\pi} \quad \Rightarrow \quad f = \frac{1}{2\pi}\sqrt{\frac{3EI}{mL^3}}.$$

Notes,

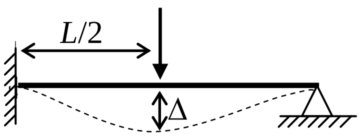
***** The values of k for various cases are as shown below.



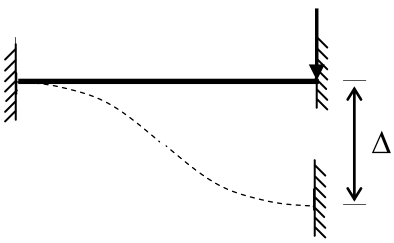
$k = 48EI/L^3$



$k = 192EI/L^3$



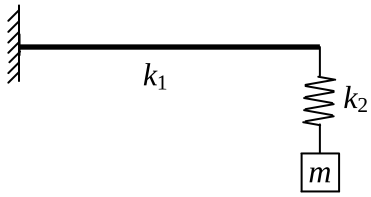
$k = 768EI/L^3$



$k = 12EI/L^3$

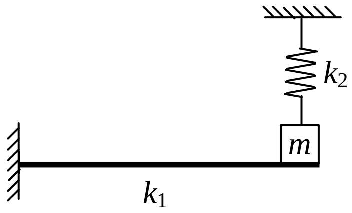
* The equivalent stiffness for "*in series*" connection is,

$$\frac{1}{k_{eq}} = \frac{1}{k_1} + \frac{1}{k_2}.$$

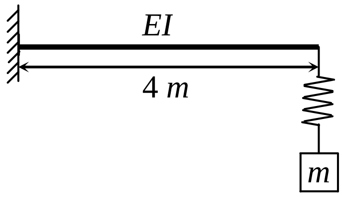


* The equivalent stiffness for "*in parallel*" connection is,

$$k_{eq} = k_1 + k_2.$$



Example 2: Determine the natural frequency of the system shown in the figure. The beam is of negligible mass. Given $m = 25\ kg$, $E = 200\ GPa$, $I = 5 \times 10^{-7}\ m^4$, and $k_{spring} = 5\ kN/m$.



Solution :

The stiffness for a cantilever beam subjected to a tip concentrated load is,

$$k_{beam} = \frac{3EI}{L^3} \quad \Rightarrow \quad k_{beam} = \frac{3(200\times10^9)(5\times10^{-7})}{4^3} = 4687.5\ N/m.$$

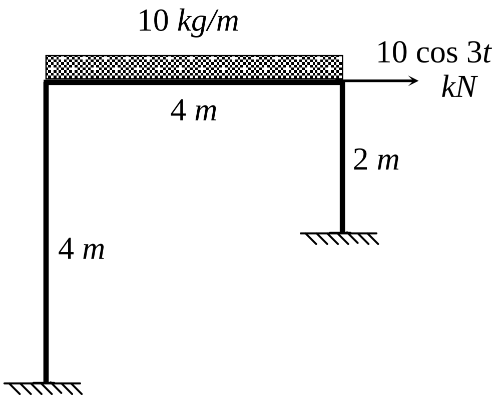
$$\frac{1}{k_{eq}} = \frac{1}{k_{beam}} + \frac{1}{k_{spring}} \quad \Rightarrow \quad \frac{1}{k_{eq}} = \frac{1}{4687.5} + \frac{1}{5000} \quad \Rightarrow \quad k_{eq} = 2419.4\ N/m.$$

$$\omega = \sqrt{\frac{k}{m}} \quad \Rightarrow \quad \omega = \sqrt{\frac{2419.4}{25}} = 9.837\ rad/s.$$

$$f = \frac{\omega}{2\pi} \quad \Rightarrow \quad f = \frac{9.837}{2\pi} = 1.566\ Hz.$$

Example 3: Assuming the damping ratio is equal to 0.1 and neglecting self weights of all members, find the equation of motion of the frame, shown in the figure, under the action of the exciting dynamic force. The initial displacement and velocity are zero. For columns, take $EI = 60\ kN/m^2$ and $k = 3EI/L^3$.

Solution :



The case is damped forced vibration,

$$\therefore \quad m.\ddot{y} + c\dot{y} + ky = F \ .$$

$$m = 10 \times 4 = 40 \ kg.$$

$$k_{eq} = k_{column1} + k_{column2}$$

$$= \left(\frac{3EI}{L^3}\right)_{column1} + \left(\frac{3EI}{L^3}\right)_{column2}$$

$$= \frac{3(60\times 10^3)}{4^3} + \frac{3(60\times 10^3)}{2^3} = 25312.5 \ N/m.$$

$$c_{cr} = 2\sqrt{km} = 2\sqrt{25312.5 \times 40} = 2012.46 \ N.s/m.$$

But, $\lambda = \dfrac{c}{c_{cr}} = 0.1 \quad \Rightarrow \quad c = 0.1 c_{cr} = 0.1 \times 2012.46 = 201.246 \ N.s/m.$

$$\therefore \quad 40\ddot{y} + 201.246\dot{y} + 25312.5y = 10 \times 10^3 \cos 3t ,$$

or $\ddot{y} + 5.03\dot{y} + 632.813y = 250\cos 3t \quad \Rightarrow \quad (D^2 + 5.03D + 632.813)y = 250\cos 3t$.

$$\therefore r^2 + 5.03r + 632.813 = 0 \Rightarrow r_{1,2} = \frac{-5.03 \pm \sqrt{5.03^2 - 4(632.813)}}{2} = -2.515 \pm 25.03i ,$$

$$\therefore \quad y_c = e^{-2.515t}(A\cos 25.03t + B\sin 25.03t).$$

Let $y_p = C_1 \cos 3t + C_2 \sin 3t$,

$$\Rightarrow \quad \dot{y}_p = -3C_1 \sin 3t + 3C_2 \cos 3t \quad \Rightarrow \quad \ddot{y}_p = -9C_1 \cos 3t - 9C_2 \sin 3t \ .$$

Substituting,

$$-9C_1 \cos 3t - 9C_2 \sin 3t + 5.03(-3C_1 \sin 3t + 3C_2 \cos 3t) +$$
$$632.813(C_1 \cos 3t + C_2 \sin 3t) = 250\cos 3t ,$$

$$(623.813C_1 + 15.09C_2)\cos 3t + (-15.09C_1 + 623.813C_2)\sin 3t = 250\cos 3t ,$$

$$\therefore \quad 623.813C_1 + 15.09C_2 = 250 \qquad \text{.......... (1)}$$

$$-15.09C_1 + 623.813C_2 = 0 \qquad \text{.......... (2)}$$

Solving Eqs. (1) & (2) simultaneously yields, $C_1 \approx 0.4$ and $C_2 \approx 0.01$.

$\therefore \quad y_p = 0.4\cos 3t + 0.01\sin 3t$.

$y = y_c + y_p \Rightarrow y = e^{-2.515t}(A\cos 25.03t + B\sin 25.03t) + 0.4\cos 3t + 0.01\sin 3t$. (G.S)

Initial conditions,

1. $y(0) = 0 \quad \Rightarrow \quad 0 = A + 0 + 0.4 + 0 \quad \Rightarrow \quad A = -0.4$.
2. $v(0) = \dot{y}(0) = 0$,

$$\dot{y} = e^{-2.515t}(-25.03A\sin 25.03t + 25.03B\cos 25.03t) - 2.515e^{-2.515t}(A\cos 25.03t + B\sin 25.03t) - 3(0.4)\sin 3t + 3(0.01)\cos 3t,$$

$\Rightarrow \quad 0 = 0 + 25.03B - 2.515(A + 0) - 0 + 3(0.01) \quad \Rightarrow \quad B = -0.0414$.

$\therefore y = e^{-2.515t}(-0.4\cos 25.03t - 0.0414\sin 25.03t) + 0.4\cos 3t + 0.01\sin 3t$. (P.S)