4. Simple vibration

Any system having mass and elasticity can vibrate when it is subjected to an exciting force. The study of a vibrated system requires determination of displacement of each point in that system. In continuous bodies, there is an infinite number of these points, thus the analysis is very complicated. So that, simplifications are used by considering only a limited number of these points, each point may have motion in one, two, or three directions and rotation about one, two, or three axes. Each component of these motions is known as the degree of freedom 'DOF'. So that, each point may have one to six DOF. The DOF of the whole system is the sum of the degrees of freedom for all considered points.

* Free vibration: is the vibration that occurs in the absence of exciting forces. This vibration is usually caused by an initial displacement and/or initial velocity.
* Forced vibration: is the vibration that occurs due to the effect of an exciting force.
* Undamped vibration: When the motion is not subjected to a counter effect, such as friction or air resistance. In this case, the kinetic energy is constant and the amplitude of motion remains unchanged.
* Damped vibration: When the motion is subjected to a counter effect. In this case, the kinetic energy is dissipated during motion and the amplitude is reduced with time. Actually, all systems must have some damping.

Free vibration

Undamped free vibration

Systems having single DOF can be represented by a spring-mass system as shown in the figure below.

Case I: Static;

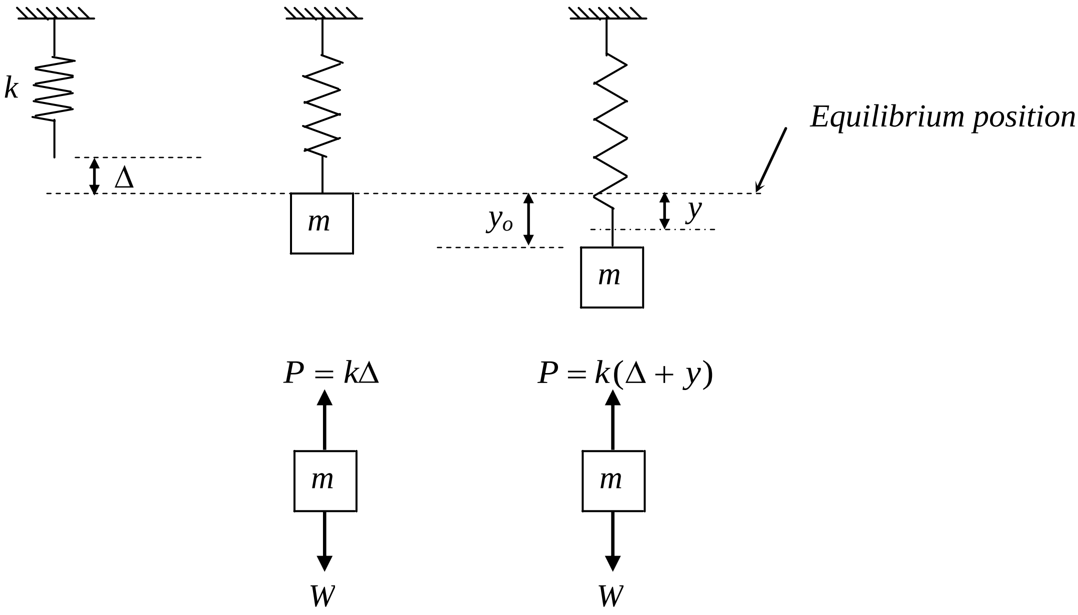
A mass *m* is attached to a spring of stiffness *k.* At equilibrium:

$$\sum F_y = 0 \quad \Rightarrow \quad W - P = 0 \quad \Rightarrow \quad W = P \quad \Rightarrow \quad W = k.\Delta .$$

$$\therefore \quad k = \frac{W}{\Delta} \quad \Rightarrow \quad k = \frac{mg}{\Delta} .$$

Case II: Dynamic;

If the mass oscillates due to an initial displacement or velocity, then at any time *t*:



Case I: Static   *Case II: Dynamic*

$$\sum F_y = m.a \quad \Rightarrow \quad W - P = m.a \quad \Rightarrow \quad W - k(\Delta + y) = m.\ddot{y},$$

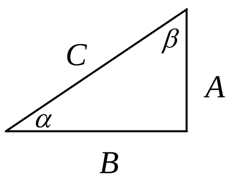
$$\Rightarrow \quad W - k\Delta - ky = m.\ddot{y} \quad \Rightarrow \quad m.\ddot{y} + ky = 0,$$

$$\Rightarrow \quad \ddot{y} + \frac{k}{m} y = 0.$$

Let $\omega^2 = \dfrac{k}{m}$ $\quad \Rightarrow \quad \ddot{y} + \omega^2 y = 0$ or $(D^2 + \omega^2)y = 0$,

$$\Rightarrow \quad r^2 + \omega^2 = 0 \quad \Rightarrow \quad r^2 = -\omega^2 \quad \Rightarrow \quad r_{1,2} = \pm\omega i,$$

$\therefore \quad y = A\cos\omega t + B\sin\omega t.$ (Simple harmonic motion)



Initial conditions,

At $t = 0$, $y = y_o$ and $v = \dot{y} = v_o$.

To find the amplitude of motion,

Let $A^2 + B^2 = C^2$.

$$y = C\left(\frac{A}{C}\cos\omega t + \frac{B}{C}\sin\omega t\right),$$

$$\Rightarrow \quad y = C(\sin\alpha\cos\omega t + \cos\alpha\sin\omega t) \quad \Rightarrow \quad y = C\sin(\omega t + \alpha),$$

$\cos\alpha = B/C$, $\sin\alpha = A/C$, $\alpha = \tan^{-1}(A/B)$, $\beta = \tan^{-1}(B/A)$.

or $\quad y = C(\dfrac{A}{C}\cos\omega t + \dfrac{B}{C}\sin\omega t)$,

$$\Rightarrow \quad y = C(\cos\beta\cos\omega t + \sin\beta\sin\omega t) \quad \Rightarrow \quad y = C\cos(\omega t - \beta),$$

where,

C is the amplitude of motion. (m)

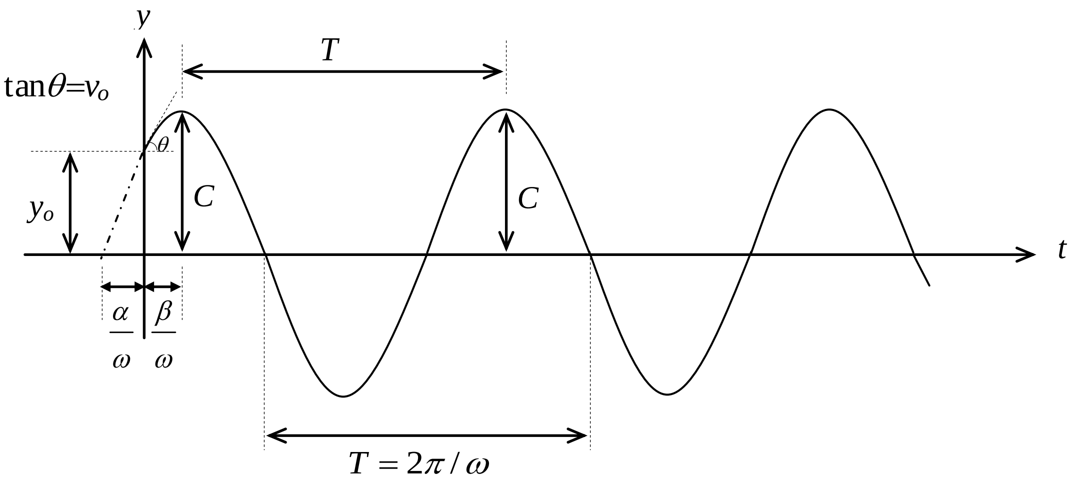
α and β are the phase angles (shifting angles) $(\alpha+\beta=\pi/2)$. (rad)

ω is the circular or angular frequency $\Rightarrow \quad \omega=\sqrt{\frac{k}{m}}$. (rad/s)

f is the natural frequency $\Rightarrow \quad f=\frac{\omega}{2\pi}$. $(cycle/s=hertz)$

T is the period of motion (i.e. the time required to complete one circle of motion)

$$\Rightarrow \quad T=\frac{2\pi}{\omega}=\frac{1}{f}. \quad (s)$$



Example: A mass of 4 *kg* is attached to a spring of 1.6 *kN/m* stiffness. The mass is pulled down with a velocity of 0.6 *m/s* and released at 4 *cm* below the equilibrium position. Find the equation of motion, angular frequency, natural frequency, period of motion, and amplitude of motion (maximum displacement). Then, find the minimum time at which the mass passes through the equilibrium position.

Solution:

The case is undamped free vibration,

$$\therefore \quad m.\ddot{y}+ky=0 \quad \Rightarrow \quad \ddot{y}+\frac{k}{m}y=0.$$

Let $\omega^2=\frac{k}{m} \quad \Rightarrow \quad \omega^2=\frac{1.6\times10^3}{4}=400 \quad \Rightarrow \quad \omega=20 rad/s$. (Angular frequency)

$$\therefore \quad \ddot{y}+400y=0 \quad \text{or} \quad (D^2+400)y=0,$$

$\Rightarrow \quad r^2 + 400 = 0 \quad \Rightarrow \quad r^2 = -400 \quad \Rightarrow \quad r_{1,2} = \pm 20i$,

$\therefore \quad y = A\cos 20t + B\sin 20t$. (G.S)

Initial conditions,

1. $y(0) = +0.04m \quad \Rightarrow \quad 0.04 = A + 0 \quad \Rightarrow \quad A = 0.04$.
2. $v(0) = \dot{y}(0) = +0.6m/s$, $\quad \dot{y} = -20A\sin 20t + 20B\cos 20t$,

$\Rightarrow \quad 0.6 = 0 + 20B \quad \Rightarrow \quad B = 0.03$.

$\therefore \quad y = 0.04\cos 20t + 0.03\sin 20t$. (P.S)

$$f = \frac{\omega}{2\pi} = \frac{20}{2\pi} = 3.183 \ Hz. \qquad \text{(Natural frequency)}$$

$$T = \frac{2\pi}{\omega} = \frac{1}{f} = 0.314 \, s. \qquad \text{(Period of motion)}$$

$$C = \sqrt{A^2 + B^2} = \sqrt{0.04^2 + 0.03^2} = 0.05 \ m. \qquad \text{(Amplitude of motion)}$$

The mass passes through the equilibrium position when $y = 0$,

$$\alpha = \tan^{-1}(A/B) = \tan^{-1}(0.04/0.03) = 0.9273 \ rad,$$

$$\therefore y = C\sin(\omega t + \alpha) = 0.05\sin(20t + 0.9273).$$

At $\quad y = 0 \quad \Rightarrow \quad 0 = 0.05\sin(20t + 0.9273) \quad \Rightarrow \quad \sin(20t + 0.9273) = 0$,

either $\quad 20t + 0.9273 = 0 \quad \Rightarrow \quad t = -0.046$, (neglected)

or $\quad 20t + 0.9273 = \pi \quad \Rightarrow \quad t = 0.1107 \ s$.

Damped free vibration

In all vibrating system there is some energy dissipation (damping). So that, the amplitude of motion decreases with time until vanishes. Viscous damping is assumed to be proportional with velocity,

$\therefore$ Viscous damping $D \propto v \quad \Rightarrow \quad D = cv \quad \Rightarrow \quad D = c\dot{y}$,

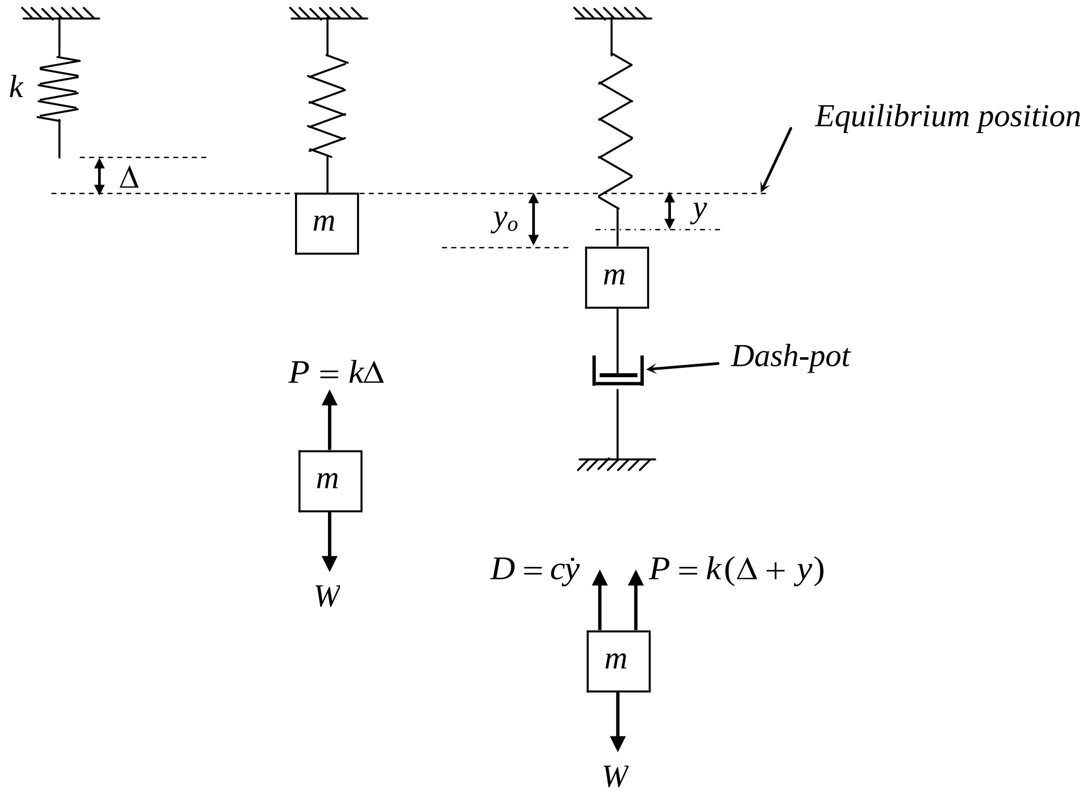
where c is the damping constant or the damping coefficient (*N.s/m*).

In the dynamic case;

$$\sum F_y = m.a \quad \Rightarrow \quad W - P - D = m.a \quad \Rightarrow \quad W - k(\Delta + y) - c\dot{y} = m.\ddot{y},$$

$$\Rightarrow \quad W - k\Delta - ky - c\dot{y} = m.\ddot{y}.$$

But, from static case $\quad W = k\Delta \quad \Rightarrow \quad -ky - c\dot{y} = m.\ddot{y}$,



$$\therefore \quad m.\ddot{y} + c\dot{y} + ky = 0 \qquad \text{or} \qquad (mD^2 + cD + k)y = 0,$$

$$\therefore \quad mr^2 + cr + k = 0 \qquad \Rightarrow \qquad r_{1,2} = \frac{-c \pm \sqrt{c^2 - 4km}}{2m}.$$

Case 1; When $c^2 - 4km = 0 \quad \Rightarrow \quad c = 2\sqrt{km} = c_{cr}$, (Critical damping)

$$r_1 = r_2 = \frac{-c}{2m} \qquad \Rightarrow \qquad y = Ae^{-ct/2m} + Bte^{-ct/2m}.$$ (No oscillations)

Case 2; When $c^2 - 4km > 0 \quad \Rightarrow \quad c > 2\sqrt{km}$ (i.e. $c > c_{cr}$), (Over damping)

$$r_1 \neq r_2 \qquad \Rightarrow \qquad y = Ae^{r_1 t} + Be^{r_2 t}.$$ (No oscillations)

Case 3; When $c^2 - 4km < 0 \quad \Rightarrow \quad c < 2\sqrt{km}$ (i.e. $c < c_{cr}$), (Under damping)

$$r_{1,2} = \frac{-c \pm \sqrt{4km - c^2}\,i}{2m} = -\frac{c}{2m} \pm \sqrt{\frac{4km - c^2}{4m^2}}\,i,$$

$$\therefore \quad y = e^{-ct/2m}(A\cos\omega_D t + B\sin\omega_D t),$$ (Oscillations occur)

or $y = Ce^{-ct/2m}\sin(\omega_D t + \alpha)$,

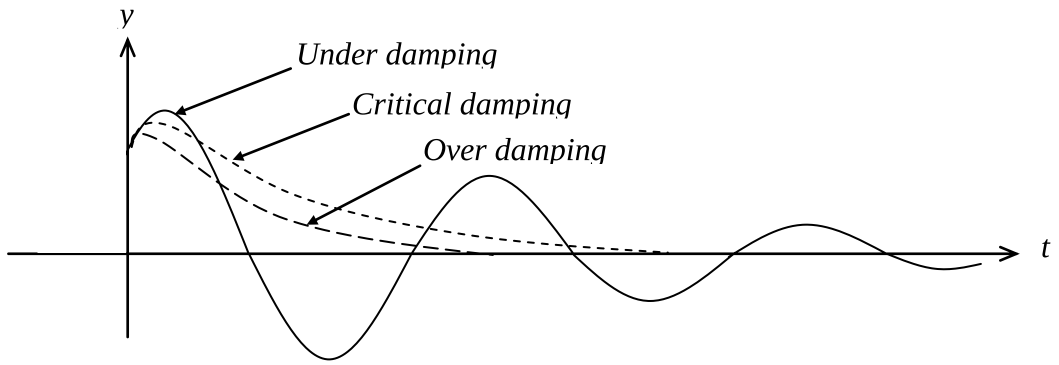
or $y = Ce^{-ct/2m}\cos(\omega_D t - \beta)$, where, $\omega_D = \sqrt{\frac{4km - c^2}{4m^2}} = \sqrt{\frac{k}{m} - \frac{c^2}{4m^2}} = \sqrt{\frac{k}{m}}\sqrt{1 - \frac{c^2}{4km}} = \omega\sqrt{1 - \frac{c^2}{(2\sqrt{km})^2}}$

$$= \omega\sqrt{1 - \frac{c^2}{(c_{cr})^2}} = \omega\sqrt{1-\lambda^2},$$

where, $\lambda = \frac{c}{c_{cr}}$ is the damping ratio.

***** In almost all cases, the state is under damping.

***** For structures, $c = (0.02 - 0.2)c_{cr}$.



Example: A 10 *kg* mass is attached to a 1.5 *m* long spring. At equilibrium, the spring measures 2.481 *m*. If the mass is pushed up and released from rest at 0.2 *m* above the equilibrium position, find the displacement as a function of time, critical damping constant, natural frequency, and period of motion. (Assume the damping coefficient $c = 20$ *N.s/m*).

Solution:

The case is damped free vibration,

$$\therefore \quad m.\ddot{y} + c\dot{y} + ky = 0,$$

$$k = \frac{W}{\Delta} = \frac{mg}{\Delta} = \frac{10 \times 9.81}{(2.481 - 1.5)} = 100 \ N/m,$$

$$\therefore \quad 10\ddot{y} + 20\dot{y} + 100y = 0 \qquad \text{or} \qquad (10D^2 + 20D + 100)y = 0,$$

$$\therefore \quad 10r^2 + 20r + 100 = 0 \qquad \Rightarrow \qquad r_{1,2} = \frac{-20 \pm \sqrt{20^2 - 4(10)(100)}}{2(10)} = -1 \pm 3i,$$

$$\therefore \quad y = e^{-t}(A\cos 3t + B\sin 3t). \qquad \text{(Under damping)} \qquad \text{(G.S)}$$

Initial conditions,

1. $y(0) = -0.2m \quad \Rightarrow \quad -0.2 = A + 0 \quad \Rightarrow \quad A = -0.2$.
2. $v(0) = \dot{y}(0) = 0, \quad \dot{y} = e^{-t}(-3A\sin 3t + 3B\cos 3t) - e^{-t}(A\cos 3t + B\sin 3t)$,

$\Rightarrow \quad 0 = 0 + 3B - (A + 0) \quad \Rightarrow \quad B = A/3 = -0.2/3.$

$$\therefore \; y = e^{-t}(-0.2\cos 3t - \frac{0.2}{3}\sin 3t) \quad \text{or} \quad y = -\frac{1}{15}e^{-t}(3\cos 3t + \sin 3t). \qquad \text{(P.S)}$$

$$c_{cr} = 2\sqrt{km} = 2\sqrt{100 \times 10} = 63.25 \; N.s/m. \qquad \text{(Critical damping)}$$

$$f = \frac{\omega_D}{2\pi} = \frac{3}{2\pi} = 0.477 \; Hz. \qquad \text{(Natural frequency)}$$

$$T = \frac{2\pi}{\omega_D} = \frac{1}{f} = 2.09 \, s. \qquad \text{(Period of motion)}$$