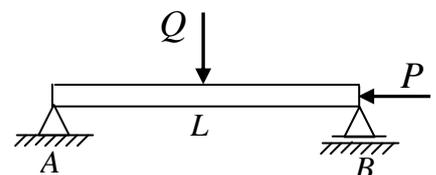


3. Deflection of beam-columns

Example 1: A simply supported beam-column of length L is subjected to a concentrated load Q at midspan and an axial compressive force P as shown in the figure below. Derive and solve the differential equation of the deflection curve. Then find the max. deflection and max. bending moment.

(Neglect self weight)



Solution:

$$EI \cdot \frac{d^2 y}{dx^2} = -M_x.$$

But $M_x = \frac{Q}{2}x + Py,$

$$\therefore EI \cdot \frac{d^2 y}{dx^2} = -\left[\frac{Q}{2}x + Py\right],$$

$$\Rightarrow \frac{d^2 y}{dx^2} + \frac{P}{EI}y = -\frac{Q}{2EI}x.$$

Let $\beta^2 = \frac{P}{EI} \Rightarrow \frac{d^2 y}{dx^2} + \beta^2 y = -\frac{Q}{2EI}x,$

or $(D^2 + \beta^2)y = -\frac{Q}{2EI}x \Rightarrow m^2 + \beta^2 = 0,$

$$\Rightarrow m^2 = -\beta^2 \Rightarrow m_{1,2} = \pm \beta i,$$

$$\therefore y_c = C_1 \cos \beta x + C_2 \sin \beta x.$$

Let $y_p = A_0 + A_1 x \Rightarrow y'_p = A_1 \Rightarrow y''_p = 0.$

Substituting,

$$0 + \beta^2(A_0 + A_1 x) = -\frac{Q}{2EI}x,$$

$$\therefore \beta^2 A_0 = 0 \Rightarrow A_0 = 0,$$

$$\beta^2 A_1 = -\frac{Q}{2EI} \Rightarrow A_1 = -\frac{Q}{2\beta^2 EI} = -\frac{Q}{2P},$$

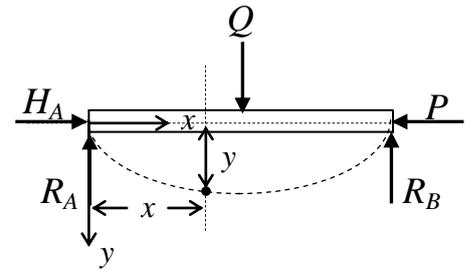
$$\therefore y_p = -\frac{Q}{2P}x.$$

$$y = y_c + y_p \Rightarrow y = C_1 \cos \beta x + C_2 \sin \beta x - \frac{Q}{2P}x. \quad (0 \leq x \leq L/2) \quad (\text{G.S})$$

Boundary conditions,

$$1. y(0) = 0 \Rightarrow 0 = C_1 + 0 \Rightarrow C_1 = 0.$$

$$2. y'(L/2) = 0, \quad y' = -\beta C_1 \sin \beta x + \beta C_2 \cos \beta x - \frac{Q}{2P},$$



To determine M_x :

From left;

$$M_x = R_A \cdot x + H_A \cdot y. \quad (0 \leq x \leq L/2)$$

$$\sum F_x = 0,$$

$$H_A - P = 0 \Rightarrow H_A = P.$$

$$\sum (M)_B = 0,$$

$$Q \cdot \frac{L}{2} - R_A \cdot L = 0 \Rightarrow R_A = \frac{Q}{2}.$$

$$\therefore M_x = \frac{Q}{2}x + Py.$$

$$\Rightarrow 0 = 0 + \beta C_2 \cos \beta L/2 - \frac{Q}{2P} \Rightarrow C_2 = \frac{Q}{2P\beta \cos \beta L/2}$$

$$\therefore y = \frac{Q}{2P\beta \cos \beta L/2} \sin \beta x - \frac{Q}{2P} x \quad \text{or} \quad y = \frac{Q}{2P} \left(\frac{\sin \beta x}{\beta \cos \beta L/2} - x \right) \quad (\text{P.S})$$

From symmetry, max. deflection occurs at midspan (i.e. at $x = L/2$),

$$\therefore y_{\max} = \frac{Q}{2P} \left(\frac{\sin \beta L/2}{\beta \cos \beta L/2} - \frac{L}{2} \right) = \frac{QL}{4P} \left(\frac{\tan \beta L/2}{\beta L/2} - 1 \right)$$

Since $M_x = \frac{Q}{2}x + Py$, thus max. B.M occurs at midspan (i.e. at $x = L/2$),

$$M_{\max} = \frac{Q}{2} \cdot \frac{L}{2} + Py_{\max} = \frac{QL}{4} + P \left[\frac{QL}{4P} \left(\frac{\tan \beta L/2}{\beta L/2} - 1 \right) \right] = \frac{Q \tan \beta L/2}{2\beta}$$

Example 2: Find the deflection equation.

Solution :

$$EI \cdot \frac{d^2 y}{dx^2} = -M_x$$

Let the deflection at the free end is (d).

$$M_x = -P(d - y) - \frac{w}{2}(L - x)^2,$$

$$\therefore EI \cdot \frac{d^2 y}{dx^2} = -[-P(d - y) - \frac{w}{2}(L - x)^2],$$

$$\Rightarrow \frac{d^2 y}{dx^2} + \frac{P}{EI} y = \frac{Pd}{EI} + \frac{w}{2EI} (L - x)^2$$

$$\text{Let } \beta^2 = \frac{P}{EI} \Rightarrow \frac{d^2 y}{dx^2} + \beta^2 y = \beta^2 d + \frac{w}{2EI} (L - x)^2,$$

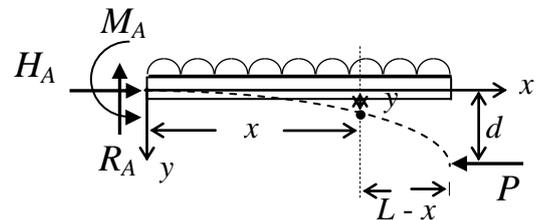
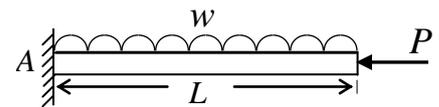
$$\text{or } (D^2 + \beta^2)y = \beta^2 d + \frac{w}{2EI} (L - x)^2,$$

$$\Rightarrow m^2 + \beta^2 = 0 \Rightarrow m^2 = -\beta^2 \Rightarrow m_{1,2} = \pm \beta i,$$

$$\therefore y_c = C_1 \cos \beta x + C_2 \sin \beta x$$

$$\text{Let } y_p = A_0 + A_1 x + A_2 x^2 \Rightarrow y'_p = A_1 + 2A_2 x \Rightarrow y''_p = 2A_2$$

Substituting,



To determine M_x :
From right;
$$M_x = -P(d - y) - \frac{w}{2}(L - x)^2$$

$$2A_2 + \beta^2(A_0 + A_1x + A_2x^2) = \beta^2d + \frac{w}{2EI}(L^2 - 2Lx + x^2),$$

$$\therefore \beta^2A_2 = \frac{w}{2EI} \Rightarrow A_2 = \frac{w}{2\beta^2EI} = \frac{w}{2P},$$

$$\beta^2A_1 = -\frac{wL}{EI} \Rightarrow A_1 = -\frac{wL}{\beta^2EI} = -\frac{wL}{P},$$

$$2A_2 + \beta^2A_0 = \beta^2d + \frac{wL^2}{2EI} \Rightarrow A_0 = d + \frac{wL^2}{2\beta^2EI} - \frac{2A_2}{\beta^2} \Rightarrow A_0 = d + \frac{wL^2}{2P} - \frac{w}{\beta^2P}.$$

$$\therefore y_p = d + \frac{wL^2}{2P} - \frac{w}{\beta^2P} - \frac{wLx}{P} + \frac{wx^2}{2P}$$

$$= d + \frac{w}{2P}(L^2 - 2Lx + x^2) - \frac{w}{\beta^2P} = d + \frac{w}{2P}(L-x)^2 - \frac{w}{\beta^2P}.$$

$$y = y_c + y_p \Rightarrow y = C_1 \cos \beta x + C_2 \sin \beta x + d + \frac{w}{2P}(L-x)^2 - \frac{w}{\beta^2P}. \quad (\text{G.S})$$

Boundary conditions,

$$1. y(0) = 0 \Rightarrow 0 = C_1 + 0 + d + \frac{wL^2}{2P} - \frac{w}{\beta^2P} \Rightarrow C_1 = -d - \frac{wL^2}{2P} + \frac{w}{\beta^2P}.$$

$$2. y'(0) = 0, \quad y' = -\beta C_1 \sin \beta x + \beta C_2 \cos \beta x - \frac{w}{P}(L-x),$$

$$\Rightarrow 0 = 0 + \beta C_2 - \frac{wL}{P} \Rightarrow C_2 = \frac{wL}{\beta P}.$$

$$\therefore y = \left(-d - \frac{wL^2}{2P} + \frac{w}{\beta^2P} \right) \cos \beta x + \left(\frac{wL}{\beta P} \right) \sin \beta x + d + \frac{w}{2P}(L-x)^2 - \frac{w}{\beta^2P}. \quad (\text{P.S})$$

To find the max. deflection (i.e. deflection at the free end):

$$y(L) = d \Rightarrow d = \left(-d - \frac{wL^2}{2P} + \frac{w}{\beta^2P} \right) \cos \beta L + \left(\frac{wL}{\beta P} \right) \sin \beta L + d + 0 - \frac{w}{\beta^2P},$$

$$\Rightarrow d \cos \beta L = \left(-\frac{wL^2}{2P} \right) \cos \beta L + \left(\frac{w}{\beta^2P} \right) (\cos \beta L - 1) + \left(\frac{wL}{\beta P} \right) \sin \beta L,$$

$$\Rightarrow d = \left(\frac{w}{\beta^2P} \right) (1 - \sec \beta L) + \left(\frac{wL}{\beta P} \right) \tan \beta L - \frac{wL^2}{2P},$$

$$\Rightarrow d = \frac{wL^2}{P} \left(\frac{1 - \sec \beta L}{(\beta L)^2} + \frac{\tan \beta L}{\beta L} - \frac{1}{2} \right).$$

Example 3: For the shown beam-column, derive and solve the differential equation of the deflection curve. (Neglect self weight)

Solution:

$$EI \cdot \frac{d^2 y}{dx^2} = -M_x.$$

But $M_x = \frac{qLx}{6} - Py - \frac{qx^3}{6L},$

$$\therefore EI \cdot \frac{d^2 y}{dx^2} = -\left[\frac{qLx}{6} - Py - \frac{qx^3}{6L}\right],$$

$$\Rightarrow \frac{d^2 y}{dx^2} - \frac{P}{EI} y = \frac{qx^3}{6LEI} - \frac{qLx}{6EI}.$$

Let $\beta^2 = \frac{P}{EI} \Rightarrow \frac{d^2 y}{dx^2} - \beta^2 y = \frac{qx^3}{6LEI} - \frac{qLx}{6EI},$

or $(D^2 - \beta^2)y = \frac{qx^3}{6LEI} - \frac{qLx}{6EI} \Rightarrow m^2 - \beta^2 = 0,$

$$\Rightarrow m^2 = \beta^2 \Rightarrow m_{1,2} = \pm\beta,$$

$$\therefore y_c = C_1 e^{\beta x} + C_2 e^{-\beta x}.$$

Let $y_p = A_0 + A_1 x + A_2 x^2 + A_3 x^3,$

$$\Rightarrow y'_p = A_1 + 2A_2 x + 3A_3 x^2 \Rightarrow y''_p = 2A_2 + 6A_3 x.$$

Substituting,

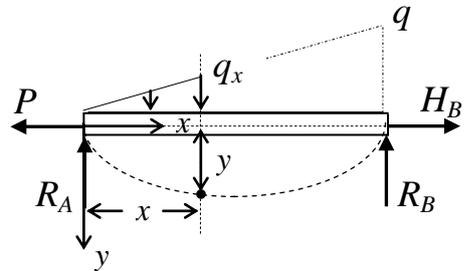
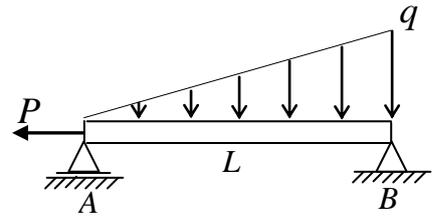
$$2A_2 + 6A_3 x - \beta^2(A_0 + A_1 x + A_2 x^2 + A_3 x^3) = \frac{qx^3}{6LEI} - \frac{qLx}{6EI},$$

$$\therefore -\beta^2 A_3 = \frac{q}{6LEI} \Rightarrow A_3 = -\frac{q}{6L\beta^2 EI} = -\frac{q}{6LP},$$

$$-\beta^2 A_2 = 0 \Rightarrow A_2 = 0,$$

$$6A_3 - \beta^2 A_1 = -\frac{qL}{6EI} \Rightarrow A_1 = \frac{qL}{6\beta^2 EI} + \frac{6A_3}{\beta^2} = \frac{qL}{6P} - \frac{q}{L\beta^2 P},$$

$$2A_2 - \beta^2 A_0 = 0 \Rightarrow A_0 = \frac{2A_2}{\beta^2} = 0.$$



To determine M_x :

From left;

$$M_x = R_A x - Py - \frac{1}{2} q_x \cdot x \cdot \frac{x}{3}.$$

$$\sum (M)_B = 0,$$

$$\frac{1}{2} qL \frac{L}{3} - R_A \cdot L = 0 \Rightarrow R_A = \frac{qL}{6}.$$

$$\frac{q}{L} = \frac{q_x}{x} \Rightarrow q_x = \frac{q \cdot x}{L}.$$

$$\therefore M_x = \frac{qL}{6} \cdot x - Py - \frac{1}{2} \cdot \frac{q \cdot x}{L} \cdot x \cdot \frac{x}{3}$$

$$= \frac{qLx}{6} - Py - \frac{qx^3}{6L}.$$

$$\therefore y_p = \left(\frac{qL}{6P} - \frac{q}{L\beta^2 P} \right) x - \frac{q}{6LP} x^3.$$

$$y = y_c + y_p \Rightarrow y = C_1 e^{\beta x} + C_2 e^{-\beta x} + \left(\frac{qL}{6P} - \frac{q}{L\beta^2 P} \right) x - \frac{q}{6LP} x^3. \quad (\text{G.S})$$

Boundary conditions,

$$1. y(0) = 0 \Rightarrow 0 = C_1 + C_2 + 0 \Rightarrow C_2 = -C_1.$$

$$2. y(L) = 0 \Rightarrow 0 = C_1 e^{\beta L} + C_2 e^{-\beta L} + \frac{qL^2}{6P} - \frac{q}{\beta^2 P} - \frac{qL^2}{6P},$$

$$\Rightarrow C_1 (e^{\beta L} - e^{-\beta L}) = \frac{q}{\beta^2 P} \Rightarrow C_1 = \frac{q}{\beta^2 P (e^{\beta L} - e^{-\beta L})}.$$

$$\begin{aligned} \therefore y &= \frac{q}{\beta^2 P (e^{\beta L} - e^{-\beta L})} e^{\beta x} - \frac{q}{\beta^2 P (e^{\beta L} - e^{-\beta L})} e^{-\beta x} + \left(\frac{qL}{6P} - \frac{q}{L\beta^2 P} \right) x - \frac{q}{6LP} x^3 \\ &= \frac{q(e^{\beta x} - e^{-\beta x})}{\beta^2 P (e^{\beta L} - e^{-\beta L})} + \left(\frac{qL}{6P} - \frac{q}{L\beta^2 P} \right) x - \frac{q}{6LP} x^3 \\ &= \frac{q}{\beta^2 P} \left(\frac{\sinh \beta x}{\sinh \beta L} + \frac{\beta^2 L x}{6} - \frac{x}{L} - \frac{\beta^2 x^3}{6L} \right). \end{aligned} \quad (\text{P.S})$$