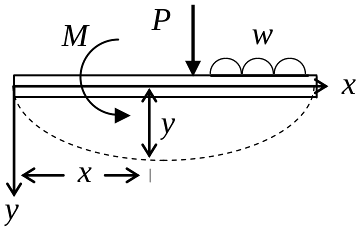
# 5- Applications on Second and Higher Order Linear ODE

## 1. Deflection of beams

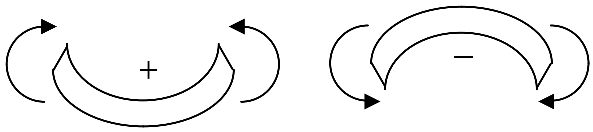
Differential equations of deflected shapes:




$$EI.\frac{d^2y}{dx^2} = -M_x,$$

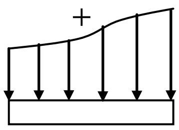
$$\because\ V = \frac{dM}{dx} \quad \Rightarrow \quad \therefore EI.\frac{d^3y}{dx^3} = -V_x,$$

$$\because\ w = -\frac{dV}{dx} \quad \Rightarrow \quad \therefore EI.\frac{d^4y}{dx^4} = w_x,$$




*B.M*


*S.F*




*U.D.L*

where;

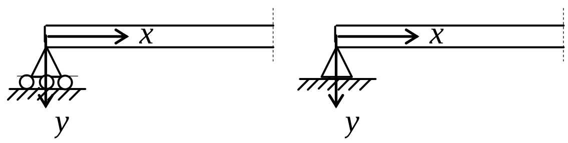
*y* is the deflection, $M_x$, $V_x$, and $w_x$ are the bending moment (*B.M*), shear force (*S.F*), and uniformly distributed load (*U.D.L*) at a section at distance *x*, respectively, and *EI* is the rigidity of the cross section of the beam.

The following boundary conditions are stated according to the supporting type,

***** Hinge or pin or roller,

No deflection $\Rightarrow$ $y(0) = 0$.

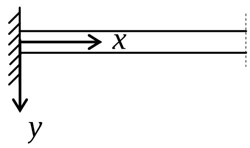
No bending moment $\Rightarrow$ $y''(0) = 0$.




***** Fixed or clamped,

No deflection $\Rightarrow$ $y(0) = 0$.

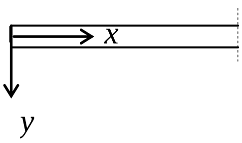
No rotation $\Rightarrow$ $y'(0) = 0$.



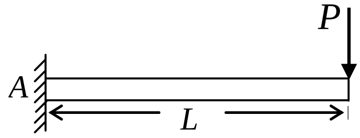

***** Free end,

No bending moment $\Rightarrow$ $y''(0) = 0$.

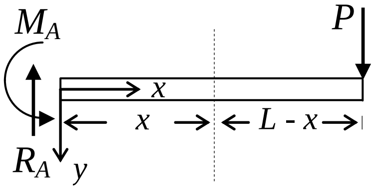
No shear force $\Rightarrow$ $y'''(0) = 0$.



**Example 1:** A cantilever beam of length $L$ is subjected to a concentrated load $P$ at the free end. Derive and solve the differential equation of the deflection curve of the beam and also find the maximum deflection.




**Solution:**




$$EI.\frac{d^2y}{dx^2} = -M_x.$$

$$M_x = -P(L-x),$$

$$\therefore \quad EI.\frac{d^2y}{dx^2} = -[-P(L-x)],$$

or $$\frac{d^2y}{dx^2} = \frac{P}{EI}(L-x).$$

To determine $M_x$:
Either from left;
$M_x = R_A.x - M_A$.
$\sum F_y = 0$,
$P - R_A = 0 \quad \Rightarrow \quad R_A = P$.
$\sum (M)_A = 0$,
$M_A - P.L = 0 \Rightarrow M_A = PL$.
$\therefore M_x = P.x - PL$
$= -P(L-x)$.
Or from right;
$M_x = -P(L-x)$.

*Method I,*

Since the right side terms are functions of $x$ only, we can solve the DE by integrating both sides directly,

$$\frac{dy}{dx} = \frac{P}{EI}(Lx - \frac{x^2}{2}) + C_1,$$

$$y = \frac{P}{EI}(\frac{Lx^2}{2} - \frac{x^3}{6}) + C_1x + C_2. \qquad \text{(G.S)}$$

*Method II,*

We can solve the DE as a non-homogeneous linear DE,

$$D^2y = 0 \quad \Rightarrow \quad m^2 = 0 \quad \Rightarrow \quad m_{1,2} = 0,$$

$$\therefore y_c = C_1x + C_2.$$

Let $$y_p = (A_o + A_1x)x^2 \quad \Rightarrow \quad y_p = A_ox^2 + A_1x^3,$$

$$y'_p = 2A_ox + 3A_1x^2 \quad \Rightarrow \quad y''_p = 2A_o + 6A_1x,$$

Substituting,

$$2A_o + 6A_1x = \frac{P}{EI}(L-x) \quad \Rightarrow \quad 2A_o + 6A_1x = \frac{PL}{EI} - \frac{Px}{EI},$$

$$\therefore \quad 2A_o = \frac{PL}{EI} \quad \Rightarrow \quad A_o = \frac{PL}{2EI} \, ,$$

$$6A_1 = -\frac{P}{EI} \quad \Rightarrow \quad A_1 = -\frac{P}{6EI} \, ,$$

$$\therefore \quad y_p = \frac{PL}{2EI}x^2 - \frac{P}{6EI}x^3 = \frac{P}{EI}\left(\frac{Lx^2}{2} - \frac{x^3}{6}\right).$$

$$y = y_c + y_p \quad \Rightarrow \quad y = C_1 x + C_2 + \frac{P}{EI}\left(\frac{Lx^2}{2} - \frac{x^3}{6}\right). \qquad \text{(G.S)}$$

*Boundary conditions,*

1- $y(0) = 0 \quad \Rightarrow \quad 0 = 0 + C_2 + 0 \quad \Rightarrow \quad C_2 = 0$.

2- $y'(0) = 0, \quad y' = C_1 + \frac{P}{EI}\left(Lx - \frac{x^2}{2}\right) \quad \Rightarrow \quad 0 = C_1 + 0 \quad \Rightarrow \quad C_1 = 0$.

$$\therefore \quad y = \frac{P}{EI}\left(\frac{Lx^2}{2} - \frac{x^3}{6}\right) \quad \text{or} \quad y = \frac{Px^2}{6EI}(3L - x). \qquad \text{(P.S)}$$

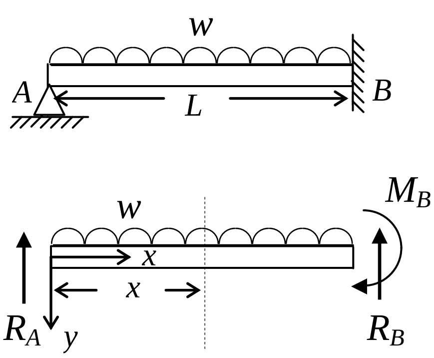
Maximum deflection of cantilever beams occurs at the free end, that is at $x = L$,

$$y_{max} = \frac{P(L)^2}{6EI}(3L - L) = \frac{PL^3}{3EI}.$$

**Example 2:** Find the deflection curve and the reactions for the beam shown below.

**Solution:**

$$EI.\frac{d^4y}{dx^4} = w_x .$$



(Statically indeterminate)

Here $w_x = w \quad \Rightarrow \quad \therefore \quad EI.\frac{d^4y}{dx^4} = w,$

or $\frac{d^4y}{dx^4} = \frac{w}{EI}.$

Integrating both sides directly gives,

$$\frac{d^3y}{dx^3} = \frac{wx}{EI} + C_1 \quad \Rightarrow \quad \frac{d^2y}{dx^2} = \frac{wx^2}{2EI} + C_1 x + C_2 \, ,$$

$$\frac{dy}{dx} = \frac{wx^3}{6EI} + \frac{C_1x^2}{2} + C_2 x + C_3 \quad \Rightarrow \quad y = \frac{wx^4}{24EI} + \frac{C_1x^3}{6} + \frac{C_2x^2}{2} + C_3 x + C_4 . \qquad \text{(G.S)}$$

*Boundary conditions,*

1- $y(0)=0 \quad \Rightarrow \quad 0=0+C_4 \quad \Rightarrow \quad C_4=0.$

2- $y''(0)=0 \quad \Rightarrow \quad 0=0+C_2 \quad \Rightarrow \quad C_2=0.$

3- $y(L)=0 \quad \Rightarrow \quad 0=\frac{wL^4}{24EI}+\frac{C_1L^3}{6}+C_3L.$ …….. (1)

4- $y'(L)=0 \quad \Rightarrow \quad 0=\frac{wL^3}{6EI}+\frac{C_1L^2}{2}+C_3.$ …….. (2)

Solving Eqs. (1) & (2) simultaneously yields,

$$C_1=-\frac{3wL}{8EI} \quad \text{and} \quad C_3=\frac{wL^3}{48EI}.$$

$$\therefore \quad y=\frac{wx^4}{24EI}-\frac{wLx^3}{16EI}+\frac{wL^3x}{48EI}. \qquad \text{(P.S)}$$

$$R_A=-EI(y''')_A=-EIy'''(0)=-EI(-\frac{3wL}{8EI})=\frac{3wL}{8}. \qquad (\uparrow)$$

$$R_B=-EI(y''')_B=-EIy'''(L)=-EI(\frac{wL}{EI}-\frac{3wL}{8EI})=-\frac{5wL}{8}. \qquad (\uparrow)$$

$$M_B=-EI(y'')_B=-EIy''(L)=-EI(\frac{wL^2}{2EI}-\frac{3wL^2}{8EI})=-\frac{wL^2}{8}. \qquad (\circlearrowright)$$

*Note,*

We can solve the above 4th order DE as a non-homogeneous linear DE, as follows

$$D^4y=0 \quad \Rightarrow \quad m^4=0 \quad \Rightarrow \quad m_{1,2,3,4}=0,$$

$$\therefore y_c=C_1+C_2x+C_3x^2+C_3x^3.$$

Let $y_p=A_o(x^4) \quad \Rightarrow \quad y_p=A_ox^4,$

$$y'_p=4A_ox^3 \quad \Rightarrow \quad y''_p=12A_ox^2 \quad \Rightarrow \quad y'''_p=24A_ox \quad \Rightarrow \quad y^{iv}_p=24A_o,$$

Substituting,

$$24A_o=\frac{w}{EI} \quad \Rightarrow \quad A_o=\frac{w}{24EI} \quad \Rightarrow \quad y_p=\frac{wx^4}{24EI}.$$

$$y=y_c+y_p \quad \Rightarrow \quad \therefore y=C_1+C_2x+C_3x^2+C_4x^3+\frac{wx^4}{24EI}. \qquad \text{(G.S)}$$

## 2. Buckling of columns

**Example 1:** Determine the critical buckling load of a hinged-hinged column.

**Solution:**

Consider a column of length $L$ , as shown in Fig.($a$) or Fig.($b$), hinged at both ends, and subjected to a compressive axial force $P$.

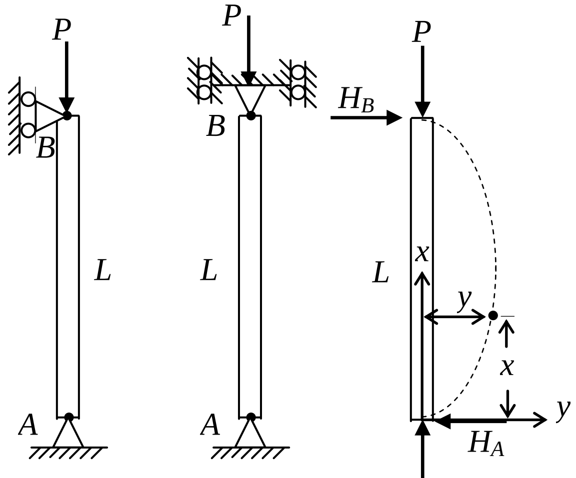


Fig.(a) Fig.(b)

$$EI.\frac{d^2y}{dx^2}=-M_x. \qquad \text{But} \qquad M_x=P.y,$$

To determine $M_x$:
Either from down (left);
$M_x=R_A.y+H_A.x$.
$\sum F_x=0$,
$R_A-P=0 \Rightarrow R_A=P$.
$\sum F_y=0$,
$H_B-H_A=0 \Rightarrow H_A=H_B$.
$\sum (M)_B=0$,
$H_A.L=0 \Rightarrow H_A=0$,
$\therefore H_B=0$.
$\therefore M_x=P.y$.
Or from up (right);
$M_x=P.y-H_B.(L-x)$
$=P.y$.

$$\therefore EI.\frac{d^2y}{dx^2}=-P.y \qquad \Rightarrow \qquad \frac{d^2y}{dx^2}+\frac{P}{EI}y=0.$$

Let $\beta^2=\dfrac{P}{EI}$ $\Rightarrow$ $\dfrac{d^2y}{dx^2}+\beta^2y=0$,

or $(D^2+\beta^2)y=0 \Rightarrow m^2+\beta^2=0$,

$\Rightarrow m^2=-\beta^2 \Rightarrow m_{1,2}=\pm\beta i$,

$\therefore \quad y=C_1\cos\beta x+C_2\sin\beta x.$ (G.S)

*Boundary conditions*,

1. $y(0)=0 \Rightarrow 0=C_1+0 \Rightarrow C_1=0$.

   $\therefore \quad y=C_2\sin\beta x$.

2. $y(L)=0 \Rightarrow 0=C_2\sin\beta L$.

   If $C_2=0 \Rightarrow y=0$.

   (i.e. the column remains straight)

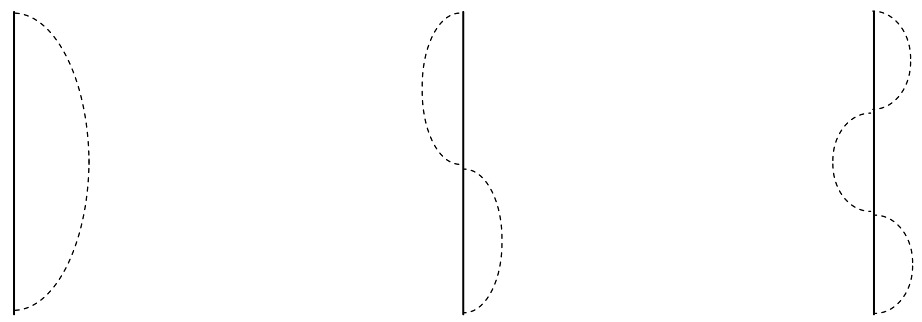
   $\therefore \quad C_2\neq 0 \Rightarrow \sin\beta L=0 \Rightarrow \beta L=0,\pi,2\pi,.......n\pi$,

   $\therefore \quad \beta L=n\pi \Rightarrow \beta=\dfrac{n\pi}{L}$. $(n=1,2,3,.....)$

But $\beta^2=\dfrac{P}{EI} \Rightarrow \dfrac{P}{EI}=\dfrac{n^2\pi^2}{L^2} \Rightarrow P=\dfrac{n^2\pi^2EI}{L^2}$.

For $n=1 \Rightarrow P_{cr}=\dfrac{\pi^2EL}{L^2}$, (Euler load or critical load)

and $y=C_2\sin\dfrac{\pi x}{L}$. ($C_2$ remains indeterminate, that is $y(\dfrac{L}{2})=C_2$)



$n=1$ $\qquad P_{cr}=\dfrac{\pi^2 EL}{L^2}$

$n=2$ $\qquad P_{cr}=\dfrac{4\pi^2 EL}{L^2}$

$n=3$ $\qquad P_{cr}=\dfrac{9\pi^2 EL}{L^2}$

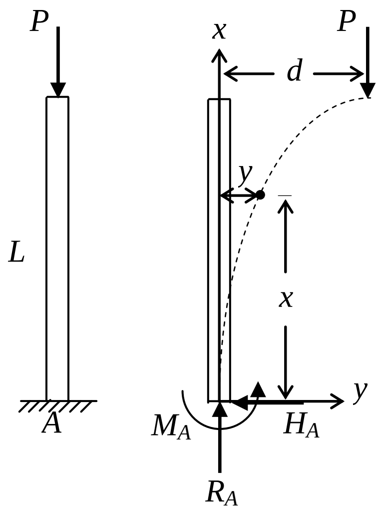
*Critical buckling load*: is the smallest value of axial load that can cause buckling.

* If $P < P_{cr}$, then the case is "*stable equilibrium*". In this case, no buckling would occur. If lateral deflection is produced, by a horizontal force, then this deflection vanishes when the horizontal force is removed.

* If $P = P_{cr}$, then the case is "*neutral equilibrium*". In this case, small and limited buckling may occur. If lateral deflection is produced, by a horizontal force, then this deflection remains constant even when the horizontal force is removed.

* If $P > P_{cr}$, then the case is "*unstable equilibrium*". In this case, large not-controlled buckling may occur. If lateral deflection is produced, by a horizontal force, then this deflection will be increased, and if not controlled, the column will collapse.

**Example 2:** Determine the critical buckling load of a fixed-free column.




**Solution:**

Let the buckling at the free end is ($d$).

$$EI.\frac{d^2y}{dx^2}=-M_x. \qquad \text{But} \quad M_x=-P(d-y),$$

$$\therefore EI.\frac{d^2y}{dx^2}=-[-P(d-y)] \quad \Rightarrow \quad \frac{d^2y}{dx^2}+\frac{P}{EI}y=\frac{P}{EI}d\,.$$

Let $\beta^2=\dfrac{P}{EI}$ $\qquad \Rightarrow \qquad \dfrac{d^2y}{dx^2}+\beta^2 y=\beta^2 d\,,$

or $(D^2+\beta^2)y=\beta^2 d \quad\Rightarrow\quad m^2+\beta^2=0$,

$\Rightarrow\quad m^2=-\beta^2 \quad\Rightarrow\quad m_{1,2}=\pm\beta i$,

$\therefore\quad y_c=C_1\cos\beta x+C_2\sin\beta x$.

Let $y_p=A \quad\Rightarrow\quad y_p'=y_p''=0$.

Substituting,

$$0+\beta^2 A=\beta^2 d \quad\Rightarrow\quad A=d \quad\Rightarrow\quad y_p=d.$$

$$y=y_c+y_p,$$

$$\therefore y=C_1\cos\beta x+C_2\sin\beta x+d. \qquad \text{(G.S)}$$

To determine $M_x$:
Either from up (right);
$M_x=-P(d-y)$.
Or from down (left);
$M_x=R_A.y+H_A.x-M_A$.
$\sum F_x=0$,
$R_A-P=0 \Rightarrow R_A=P$.
$\sum F_y=0$,
$-H_A=0 \Rightarrow H_A=0$.
$\sum(M)_A=0$,
$M_A-Pd=0 \Rightarrow M_A=Pd$.
$\therefore M_x=Py-Pd$
$=-P(d-y)$

*Boundary conditions*,

1. $y(0)=0 \quad\Rightarrow\quad 0=C_1+d \quad\Rightarrow\quad C_1=-d$.

   $\therefore y=-d\cos\beta x+C_2\sin\beta x+d$.

2. $y'(0)=0$, $\quad y'=-\beta C_1\sin\beta x+\beta C_2\cos\beta x \quad\Rightarrow\quad 0=0+\beta C_2 \quad\Rightarrow\quad C_2=0$.

   $\therefore y=-d\cos\beta x+d$ or $y=d(1-\cos\beta x)$.

3. $y(L)=d \quad\Rightarrow\quad d=d(1-\cos\beta L) \quad\Rightarrow\quad 1=1-\cos\beta L$,

   $\Rightarrow\quad \cos\beta L=0 \quad\Rightarrow\quad \beta L=\frac{\pi}{2},\frac{3\pi}{2},\ldots\ldots,\frac{(2n-1)\pi}{2}$,

   $\therefore\quad \beta L=\frac{(2n-1)\pi}{2} \quad\Rightarrow\quad \beta=\frac{(2n-1)\pi}{2L}. \qquad (n=1,2,3,\ldots..)$

But $\beta^2=\frac{P}{EI} \quad\Rightarrow\quad \frac{P}{EI}=\frac{(2n-1)^2\pi^2}{4L^2} \quad\Rightarrow\quad P_{cr}=\frac{(2n-1)^2\pi^2 EL}{4L^2}$.

For $n=1 \quad\Rightarrow\quad P_{cr}=\frac{\pi^2 EL}{4L^2}$. (i.e. $\frac{1}{4}$ times the critical load for hinged-hinged case)