2- Variation of parameters method

In this method, the particular solution is assumed by replacing the arbitrary constants $C_1, C_2, \ldots\ldots, C_n$, in the complementary solution, by functions of *x*, say $u_1, u_2, \ldots\ldots, u_n$ to be determined later, that is

$$y_p = u_1 y_1 + u_2 y_2 + \ldots\ldots\ldots + u_n y_n ,$$

where *n* is the order of the non-homogeneous linear DE to be solved. Then, the assumed particular solution is substituted into the DE, and imposing conditions on the resulting equation leads to the following equations:

$$u_1' y_1 + u_2' y_2 + \ldots\ldots\ldots + u_n' y_n = 0,$$

$$u_1' y_1' + u_2' y_2' + \ldots\ldots\ldots + u_n' y_n' = 0,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$u_1' y_1^{n-1} + u_2' y_2^{n-1} + \ldots\ldots\ldots + u_n' y_n^{n-1} = g(x),$$

or in matrix form:

$$\begin{bmatrix} y_1 & y_2 & . & . & y_n \\ y_1' & y_2' & . & . & y_n' \\ . & . & . & . & . \\ . & . & . & . & . \\ y_1^{n-1} & y_2^{n-1} & . & . & y_n^{n-1} \end{bmatrix} \begin{Bmatrix} u_1' \\ u_2' \\ . \\ . \\ u_n' \end{Bmatrix} = \begin{bmatrix} 0 \\ 0 \\ . \\ . \\ g(x) \end{bmatrix}.$$

The individual derivatives $u_1', u_2', \ldots\ldots, u_n'$ are found by solving the above matrix, then ,by integration, the required functions $u_1, u_2, \ldots\ldots, u_n$ are determined.

Example 1: Solve $y'' - y = e^{x}$.

Solution :

$$(D^2 - 1)y = 0 \quad \Rightarrow \quad m^2 - 1 = 0 \quad \Rightarrow \quad m_{1,2} = \pm 1,$$

$$\therefore \quad y_c = C_1 e^{x} + C_2 e^{-x}. \qquad (y_1 = e^{x} \text{ and } y_2 = e^{-x})$$

Let $y_p = u_1 e^{x} + u_2 e^{-x}$.

$$\begin{bmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{bmatrix} \begin{Bmatrix} u'_1 \\ u'_2 \end{Bmatrix} = \begin{bmatrix} 0 \\ g(x) \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} e^{x} & e^{-x} \\ e^{x} & -e^{-x} \end{bmatrix} \begin{Bmatrix} u'_1 \\ u'_2 \end{Bmatrix} = \begin{bmatrix} 0 \\ e^{x} \end{bmatrix}.$$

Using Cramer's rule to solve the above matrix, gives

$$u'_1 = \frac{\begin{vmatrix} 0 & e^{-x} \\ e^{x} & -e^{-x} \end{vmatrix}}{\begin{vmatrix} e^{x} & e^{-x} \\ e^{x} & -e^{-x} \end{vmatrix}} \quad \Rightarrow \quad u'_1 = \frac{(0)(-e^{-x}) - (e^{-x})(e^{x})}{(e^{x})(-e^{-x}) - (e^{-x})(e^{x})} = \frac{-1}{-1-1} = \frac{-1}{-2} = \frac{1}{2},$$

$$\therefore \quad u_1 = \int \frac{1}{2} dx = \frac{1}{2} x.$$

Similarly,

$$u'_2 = \frac{\begin{vmatrix} e^{x} & 0 \\ e^{x} & e^{x} \end{vmatrix}}{\begin{vmatrix} e^{x} & e^{-x} \\ e^{x} & -e^{-x} \end{vmatrix}} \quad \Rightarrow \quad u'_2 = \frac{(e^{x})(e^{x}) - (0)(e^{x})}{-2} = \frac{e^{2x}}{-2} = -\frac{1}{2} e^{2x},$$

$$\therefore \quad u_2 = \int -\frac{1}{2} e^{2x} dx = -\frac{1}{4} e^{2x}.$$

$$\therefore \quad y_p = (\frac{1}{2} x) e^{x} + (-\frac{1}{4} e^{2x}) e^{-x} = \frac{1}{2} x e^{x} - \frac{1}{4} e^{x}.$$

$$y = y_c + y_p,$$

$$y = C_1 e^{x} + C_2 e^{-x} + \frac{1}{2} x e^{x} - \frac{1}{4} e^{x},$$

or $\quad y = (C_1 - \frac{1}{4} + \frac{1}{2} x) e^{x} + C_2 e^{-x} = (C + \frac{1}{2} x) e^{x} + C_2 e^{-x}. \qquad [C = C_1 - \frac{1}{4}] \qquad$ (G.S)

Example 2: Solve $(D^2+1)y=\sec x$.

Solution :

$$m^2+1=0 \quad \Rightarrow \quad m^2=-1 \quad \Rightarrow \quad m_{1,2}=\pm i, \qquad (\alpha=0 \text{ and } \beta=1)$$

$$\therefore \quad y_c=C_1\cos x+C_2\sin x. \qquad (y_1=\cos x \text{ and } y_2=\sin x)$$

Let $y_p=u_1\cos x+u_2\sin x$.

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}\begin{Bmatrix} u_1' \\ u_2' \end{Bmatrix}=\begin{bmatrix} 0 \\ g(x) \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix}\begin{Bmatrix} u_1' \\ u_2' \end{Bmatrix}=\begin{bmatrix} 0 \\ \sec x \end{bmatrix}.$$

Using Cramer's rule to solve the above matrix, gives

$$u_1'=\frac{\begin{vmatrix} 0 & \sin x \\ \sec x & \cos x \end{vmatrix}}{\begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}},$$

$$\Rightarrow \quad u_1'=\frac{(0)(\cos x)-(\sin x)(\sec x)}{(\cos x)(\cos x)-(\sin x)(-\sin x)}=\frac{-(\sin x)(\frac{1}{\cos x})}{\cos^2 x+\sin^2 x}=\frac{-\frac{\sin x}{\cos x}}{1}=-\frac{\sin x}{\cos x},$$

$$\therefore \quad u_1=\int -\frac{\sin x}{\cos x}dx=\ln\cos x.$$

Similarly,

$$u_2'=\frac{\begin{vmatrix} \cos x & 0 \\ -\sin x & \sec x \end{vmatrix}}{\begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}},$$

$$\Rightarrow \quad u_2'=\frac{(\cos x)(\sec x)-(0)(-\sin x)}{1}=\frac{(\cos x)(\frac{1}{\cos x})}{1}=1,$$

$$\therefore \quad u_2=\int(1)dx=x.$$

$$\therefore \quad y_p=(\ln\cos x)\cos x+(x)\sin x=\cos x\ln\cos x+x\sin x.$$

$$y=y_c+y_p,$$

$$y=C_1\cos x+C_2\sin x+\cos x\ln\cos x+x\sin x,$$

or $\quad y=(C_1+\ln\cos x)\cos x+(C_2+x)\sin x. \qquad$ (G.S)

Solution of some linear DE with variable coefficients

There are some important linear DE with variable coefficients which can be always reduced, by a suitable substitution, to linear DE with constant coefficients.

1- Euler-Cauchy equation

The general form of Euler-Cauchy equation is

$$a_n x^n \frac{d^n y}{dx^n} + a_{n-1} x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + ...+ a_1 x \frac{dy}{dx} + a_o y = g(x),$$

where $a_n, a_{n-1}, ..., a_o$ are constants.

For example, the DE $(x^3 y''' + 3x^2 y'' + xy' + 4y = \cos x)$ is an Euler-Cauchy equation. Euler-Cauchy equations can be always reduced to linear DE with constant coefficients by the suitable substitution: $z = \ln x$ or $x = e^z$.

Example 1: Solve $x^2 \frac{d^2 y}{dx^2} - 2y = x$.

Solution :

Let $z = \ln x$ (i.e. $x = e^z$) $\Rightarrow$ $\frac{dz}{dx} = \frac{1}{x}$,

$$\frac{dy}{dx} = \frac{dy}{dz}.\frac{dz}{dx} = \frac{1}{x}.\frac{dy}{dz},$$

$$\frac{d^2 y}{dx^2} = \frac{d}{dx}(\frac{1}{x}.\frac{dy}{dz}) = \frac{1}{x}(\frac{d^2 y}{dz^2}.\frac{dz}{dx}) + \frac{dy}{dz}(\frac{-1}{x^2}) = \frac{1}{x^2}(\frac{d^2 y}{dz^2} - \frac{dy}{dz}),$$

Substituting,

$$x^2[\frac{1}{x^2}(\frac{d^2 y}{dz^2} - \frac{dy}{dz}] - 2y = e^z,$$

$$\Rightarrow \frac{d^2 y}{dz^2} - \frac{dy}{dz} - 2y = e^z.$$ (Non-homogeneous linear DE with constant coefficients)

$(D^2 - D - 2)y = 0 \quad \Rightarrow \quad m^2 - m - 2 = 0,$

$(m+1)(m-2) = 0 \quad \Rightarrow \quad m_1 = -1$ and $m_2 = 2,$

$$\therefore \quad y_c = C_1 e^{-z} + C_2 e^{2z}.$$

Let $\quad y_p = Ae^z \quad \Rightarrow \quad y'_p = Ae^z = y''_p.$

Substituting, $\quad Ae^z - Ae^z - 2Ae^z = e^z \quad \Rightarrow \quad -2Ae^z = e^z \quad \Rightarrow \quad A = -\frac{1}{2},$

$$\therefore \quad y_p = -\frac{1}{2}e^z.$$

$$y = y_c + y_p \quad \Rightarrow \quad y = C_1 e^{-z} + C_2 e^{2z} - \frac{1}{2}e^z.$$

But $\quad z = \ln x \quad \Rightarrow \quad y = C_1 e^{-\ln x} + C_2 e^{2\ln x} - \frac{1}{2}e^{\ln x},$

$$\Rightarrow \quad y = C_1 x^{-1} + C_2 x^2 - \frac{1}{2}x. \qquad \text{(G.S)}$$

Example 2: Solve $\quad y''' + \frac{3}{x}.y'' = \frac{6}{x}.$

Solution :

The given DE is linear DE with variable coefficients. Multiplying it by x^3 gives

$$x^3 y''' + 3x^2 y'' = 6x^2. \qquad \text{(Euler-Cauchy equation)}$$

Let $\quad z = \ln x \quad$ (i.e. $x = e^z$) $\quad \Rightarrow \quad \frac{dz}{dx} = \frac{1}{x},$

$$\frac{dy}{dx} = \frac{dy}{dz}.\frac{dz}{dx} = \frac{1}{x}.\frac{dy}{dz},$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(\frac{1}{x}.\frac{dy}{dz}) = \frac{1}{x}(\frac{d^2y}{dz^2}.\frac{dz}{dx}) + \frac{dy}{dz}(\frac{-1}{x^2}) = \frac{1}{x^2}(\frac{d^2y}{dz^2} - \frac{dy}{dz}),$$

$$\frac{d^3y}{dx^3} = \frac{d}{dx}[\frac{1}{x^2}(\frac{d^2y}{dz^2} - \frac{dy}{dz})] = \frac{1}{x^2}(\frac{d^3y}{dz^3}.\frac{dz}{dx} - \frac{d^2y}{dz^2}.\frac{dz}{dx}) + (\frac{d^2y}{dz^2} - \frac{dy}{dz})(\frac{-2}{x^3})$$

$$= \frac{1}{x^3}(\frac{d^3y}{dz^3} - 3\frac{d^2y}{dz^2} + 2\frac{dy}{dz})$$

Substituting,

$$x^3[\frac{1}{x^3}(\frac{d^3y}{dz^3} - 3\frac{d^2y}{dz^2} + 2\frac{dy}{dz})] + 3x^2[\frac{1}{x^2}(\frac{d^2y}{dz^2} - \frac{dy}{dz})] = 6(e^z)^2,$$

$\Rightarrow \dfrac{d^3y}{dz^3} - \dfrac{dy}{dz} = 6e^{2z}$. (Non-homogeneous linear DE with constant coefficients)

$(D^3 - D)y = 0 \;\Rightarrow\; m^3 - m = 0 \;\Rightarrow\; m(m^2 - 1) = 0 \;\Rightarrow\; m_1 = 0$ and $m_{2,3} = \pm 1$,

$\therefore \; y_c = C_1 + C_2 e^{z} + C_3 e^{-z}$.

Let $y_p = Ae^{2z} \;\Rightarrow\; y'_p = 2Ae^{2z} \;\Rightarrow\; y''_p = 4Ae^{2z} \;\Rightarrow\; y'''_p = 8Ae^{2z}$.

Substituting, $8Ae^{2z} - 2Ae^{2z} = 6e^{2z} \;\Rightarrow\; 6Ae^{2z} = 6e^{2z} \;\Rightarrow\; A = 1$,

$\therefore \; y_p = e^{2z}$.

$y = y_c + y_p \;\Rightarrow\; y = C_1 + C_2 e^{z} + C_3 e^{-z} + e^{2z}$.

But $z = \ln x \;\Rightarrow\; y = C_1 + C_2 e^{\ln x} + C_3 e^{-\ln x} + e^{2\ln x}$,

$\Rightarrow \; y = C_1 + C_2 x + C_3 x^{-1} + x^2$. (G.S)

2- Legender equation

The general form of Legender equation is,

$$a_n (Ax+B)^n \frac{d^n y}{dx^n} + a_{n-1} (Ax+B)^{n-1} \frac{d^{n-1}y}{dx^{n-1}} + \dots\dots\dots + a_1 (Ax+B)\frac{dy}{dx} + a_o y = g(x),$$

where $a_n, a_{n-1}, \dots\dots, a_o$ are constants.

For example, the DE $((2x+1)^3 y''' + 2(2x+1)y' + 4y = \ln x)$ is a Legender equation.

Legender equations can be always reduced to linear DE with constant coefficients by the suitable substitution: $z = \ln(Ax+B)$ or $Ax + B = e^z$.

Example 1: Solve $(x-2)^2 y'' + 2(x-2)y' - 6y = 0$.

Solution :

Let $z = \ln(x-2)$ (i.e. $x - 2 = e^z$) $\Rightarrow \dfrac{dz}{dx} = \dfrac{1}{x-2}$,

$\dfrac{dy}{dx} = \dfrac{dy}{dz}.\dfrac{dz}{dx} = \dfrac{1}{x-2}.\dfrac{dy}{dz}$,

$$\frac{d^2y}{dx^2}=\frac{d}{dx}(\frac{1}{x-2}.\frac{dy}{dz})=\frac{1}{x-2}(\frac{d^2y}{dz^2}.\frac{dz}{dx})+\frac{dy}{dz}(\frac{-1}{(x-2)^2})=\frac{1}{(x-2)^2}(\frac{d^2y}{dz^2}-\frac{dy}{dz}),$$

Substituting,

$$(x-2)^2[\frac{1}{(x-2)^2}(\frac{d^2y}{dz^2}-\frac{dy}{dz})]+2(x-2)[\frac{1}{x-2}.\frac{dy}{dz}]-6y=0,$$

$$\Rightarrow \quad \frac{d^2y}{dz^2}+\frac{dy}{dz}-6y=0.$$ (Homogeneous linear DE with constant coefficients)

$$(D^2+D-6)y=0 \quad \Rightarrow \quad m^2+m-6=0,$$

$$(m+3)(m-2)=0 \quad \Rightarrow \quad m_1=-3 \quad \text{and} \quad m_2=2,$$

$$\therefore \quad y=C_1e^{-3z}+C_2e^{2z}.$$

But $z=\ln(x-2) \quad \Rightarrow \quad y=C_1e^{-3\ln(x-2)}+C_2e^{2\ln(x-2)},$

$$\Rightarrow \quad y=C_1(x-2)^{-3}+C_2(x-2)^2. \qquad \text{(G.S)}$$