Example 4: Solve $2y''' - y'' + 36y' - 18y = 0$.

Solution :

$$(2D^3 - D^2 + 36D - 18)y = 0 \quad \Rightarrow \quad 2m^3 - m^2 + 36m - 18 = 0,$$

$$2m(m^2 + 18) - (m^2 + 18) = 0 \quad \Rightarrow \quad (m^2 + 18)(2m - 1) = 0 \ ,$$

Either $2m - 1 = 0 \quad \Rightarrow \quad m_1 = \frac{1}{2}$,

or $m^2 + 18 = 0 \quad \Rightarrow \quad m^2 = -18 \quad \Rightarrow \quad m_{2,3} = \pm\sqrt{-18} = \pm\sqrt{18}i = \pm 3\sqrt{2}i$,

$$\therefore \quad y = C_1 e^{x/2} + e^{(0)x}(C_2 \cos 3\sqrt{2}x + C_3 \sin 3\sqrt{2}x),$$

or $y = C_1 e^{x/2} + C_2 \cos 3\sqrt{2}x + C_3 \sin 3\sqrt{2}x$. (G.S)

Example 5: Solve $y''' + 7y'' + 11y' + 5y = 0$.

Solution :

$$(D^3 + 7D^2 + 11D + 5)y = 0 \quad \Rightarrow \quad m^3 + 7m^2 + 11m + 5 = 0.$$

By trial & error, if $m = -1$, then $(-1)^3 + 7(-1)^2 + 11(-1) + 5 = 0$,

$\therefore \; m = -1$ is a root $\Rightarrow$ $(m+1)$ is a factor.

To find the other factor we use long division.

$$\begin{array}{r|l} & m^2 + 6m + 5 \\ \hline m+1 & m^3 + 7m^2 + 11m + 5 \\ & m^3 + m^2 \\ \hline & \quad 6m^2 + 11m + 5 \\ & \quad 6m^2 + 6m \\ \hline & \qquad 5m + 5 \\ & \qquad 5m + 5 \\ \hline & \qquad\quad 0 \end{array}$$

$\therefore \quad (m+1)(m^2 + 6m + 5) = 0$,

$\Rightarrow \quad (m+1)(m+1)(m+5) = 0$,

$\therefore \quad m_1 = -1, \; m_2 = -1, \text{ and } m_3 = -5$,

$\therefore \quad y = C_1 e^{-x} + C_2 x e^{-x} + C_3 e^{-5x}$,

or $y = (C_1 + C_2 x).e^{-x} + C_3 e^{-5x}$. (G.S)

Example 6: Solve $\dfrac{d^4y}{dx^4} + 18\dfrac{d^2y}{dx^2} + 81y = 0$.

Solution :

$$(D^4 + 18D^2 + 81)y = 0 \quad \Rightarrow \quad m^4 + 18m^2 + 81 = 0.$$

$(m^2 + 9)(m^2 + 9) = 0 \quad \Rightarrow$ Either $m^2 + 9 = 0 \quad \Rightarrow \quad m_{1,2} = \pm\sqrt{-9} = \pm 3i$,

or $m^2 + 9 = 0 \quad \Rightarrow \quad m_{3,4} = \pm\sqrt{-9} = \pm 3i$,

$\therefore \quad y = e^{(0)x}(C_1 \cos 3x + C_2 \sin 3x) + x e^{(0)x}(C_3 \cos 3x + C_4 \sin 3x)$,

or $y = (C_1 + C_3 x)\cos 3x + (C_2 + C_4 x)\sin 3x$. (G.S)

Example 7: Solve $\dfrac{d^4y}{dx^4} + 4y = 0$.

Solution :

$$(D^4 + 4)y = 0 \quad \Rightarrow \quad m^4 + 4 = 0 \quad \Rightarrow \quad m^4 = -4 \quad \Rightarrow \quad m^2 = \pm\sqrt{-4} = \pm 2i.$$

Either $m^2 = 2i \Rightarrow m_{1,2} = \pm\sqrt{2i} \Rightarrow m_{1,2} = \pm\sqrt{(1+i)^2} \Rightarrow m_{1,2} = \pm(1+i)$,

or $m^2 = -2i \Rightarrow m_{3,4} = \pm\sqrt{-2i} \Rightarrow m_{3,4} = \pm\sqrt{(1-i)^2} \Rightarrow m_{3,4} = \pm(1-i)$.

We can rearrange the roots as: $m_{1,2} = 1 \pm i$ and $m_{3,4} = -1 \pm i$,

$\therefore \quad y = e^{x}(C_1 \cos x + C_2 \sin x) + e^{-x}(C_3 \cos x + C_4 \sin x)$. (G.S)

Note;

$$\pm ai = \left(\sqrt{\frac{a}{2}} \pm \sqrt{\frac{a}{2}}i\right)^2.$$

Example 8: Solve $\dfrac{d^5 z}{dt^5} - 4\dfrac{d^3 z}{dt^3} = 0$.

Solution :

$(D^5 - 4D^3)z = 0 \Rightarrow m^5 - 4m^3 = 0 \Rightarrow m^3(m^2 - 4) = 0$,

either $m^3 = 0 \Rightarrow m_{1,2,3} = 0$,

or $m^2 - 4 = 0 \Rightarrow m^2 = 4 \Rightarrow m_{4,5} = \pm 2$,

$\therefore \quad z = C_1 e^{(0)t} + C_2 te^{(0)t} + C_3 t^2 e^{(0)t} + C_4 e^{(2)t} + C_5 e^{(-2)t}$,

or $z = C_1 + C_2 t + C_3 t^2 + C_4 e^{2t} + C_5 e^{-2t}$. (G.S)

Solution of non-homogeneous linear DE with constant coefficients

To find the general solution (complete solution) of a given non-homogeneous linear DE, the following steps are followed:

1- We find a general solution for the corresponding homogeneous linear DE (i.e. we put $g(x) = 0$). This solution is called the homogeneous or complementary solution, usually denoted by y_c, which will contain *n* constants (where *n* is the order of the given DE).

2- We find a particular solution for the given homogeneous linear DE. This solution is called the particular solution, usually denoted by y_p, which will be free from constants.

3- The complete solution will be:

$$y = y_c + y_p .$$

There are different methods to find the particular solution.

1- Undetermined coefficients method

In this method we assume a trial solution containing unknown constants which are to be determined by substitution in the given DE. The trial solution to be assumed in each case depends on the special form of $g(x)$.

$g(x)$	**Assumed trial solution y_p**
a	A
ax^n (n a positive integer)	$A_o + A_1x + A_2x^2 + \ldots + A_nx^n$
ae^{mx} (m either real or complex)	Ae^{mx}
$a\cos\alpha x$ or $a\sin\alpha x$	$A\cos\alpha x + B\sin\alpha x$
$a\cosh\alpha x$ or $a\sinh\alpha x$	$A\cosh\alpha x + B\sinh\alpha x$
ax^ne^{mx}	$(A_o + A_1x + \ldots + A_nx^n)e^{mx}$
$ax^n\cos\alpha$ or $ax^n\sin\alpha$	$(A_o + A_1x + \ldots + A_nx^n)\cos\alpha x$ $+ (B_o + B_1x + \ldots + B_nx^n)\sin\alpha x$
$ae^{mx}\cos\alpha$ or $ae^{mx}\sin\alpha$	$(A\cos\alpha x + B\sin\alpha x)e^{mx}$
$ax^ne^{mx}\cos\alpha$ or $ax^ne^{mx}\sin\alpha$	$[(A_o + A_1x + \ldots + A_nx^n)\cos\alpha x$ $+ (B_o + B_1x + \ldots + B_nx^n)\sin\alpha x]e^{mx}$

Note,

If any term of the assumed trial solution does appear in the complementary solution (linearly dependent), we must multiply the trial solution by the smallest positive integer power of x which is large enough so that none of the terms, which are then present, appear in the complementary solution.

Example 1: Solve $y'' + 2y' + 10y = 25x^2$.

Solution :

Step 1: Find the complementary solution y_c,

$$(D^2 + 2D + 10)y = 0 \quad \Rightarrow \quad m^2 + 2m + 10 = 0,$$

$$m_{1,2} = \frac{-2 \pm \sqrt{2^2 - 4(1)(10)}}{2(1)} = \frac{-2 \pm \sqrt{-36}}{2} = \frac{-2 \pm 6i}{2} = -1 \pm 3i,$$

$$\therefore \quad y_c = e^{-x}(C_1 \cos 3x + C_2 \sin 3x).$$

Step 2: Find the particular solution y_p,

Let $y_p = A_o + A_1 x + A_2 x^2 \quad \Rightarrow \quad y'_p = A_1 + 2A_2 x \quad \Rightarrow \quad y''_p = 2A_2$,

Substituting y_p and its derivatives in the given DE, yields

$$2A_2 + 2(A_1 + 2A_2 x) + 10(A_o + A_1 x + A_2 x^2) = 25x^2,$$

$$(2A_2 + 2A_1 + 10A_o) + (4A_2 + 10A_1)x + (10A_2)x^2 = 25x^2$$

$$\therefore \quad 10A_2 x^2 = 25x^2 \quad \Rightarrow \quad 10A_2 = 25 \quad \Rightarrow \quad A_2 = \frac{5}{2},$$

$$(4A_2 + 10A_1)x = 0 \quad \Rightarrow \quad 4A_2 + 10A_1 = 0 \quad \Rightarrow \quad A_1 = -\frac{4A_2}{10} = -\frac{4}{10} \times \frac{5}{2} = -1,$$

$$2A_2 + 2A_1 + 10A_o = 0 \quad \Rightarrow \quad A_o = \frac{-2A_2 - 2A_1}{10} = \frac{-2\left(\frac{5}{2}\right) - 2(-1)}{10} = -\frac{3}{10}.$$

$$\therefore \quad y_p = -\frac{3}{10} - x + \frac{5}{2}x^2.$$

Step 3: Find the complete solution y,

$$y = y_c + y_p \quad \Rightarrow \quad y = e^{-x}(C_1 \cos 3x + C_2 \sin 3x) - \frac{3}{10}A_o - x + \frac{5}{2}x^2. \quad \text{(G.S)}$$

Example 2: Solve $y'' - 2y' - 3y = 5\cos 2x - 9$.

Solution :

To find the complementary solution y_c,

$$(D^2 - 2D - 3)y = 0 \quad \Rightarrow \quad \therefore \quad m^2 - 2m - 3 = 0,$$

$$(m+1)(m-3) = 0 \quad \Rightarrow \quad m_1 = -1 \quad \text{and} \quad m_2 = 3,$$

$$\therefore \quad y_c = C_1 e^{-x} + C_2 e^{3x}.$$

To find the particular solution y_p,

Let $y_p = A\cos 2x + B\sin 2x + C \quad \Rightarrow \quad y'_p = -2A\sin 2x + 2B\cos 2x,$

$$\Rightarrow \quad y''_p = -4A\cos 2x - 4B\sin 2x,$$

Substituting,

$$(-4A\cos 2x - 4B\sin 2x) - 2(-2A\sin 2x + 2B\cos 2x) - 3(A\cos 2x + B\sin 2x + C) = 5\cos 2x - 9$$

$$(-7A - 4B)\cos 2x + (4A - 7B)\sin 2x - 3C = 5\cos 2x - 9,$$

$$\therefore \quad (-7A - 4B)\cos 2x = 5\cos 2x \quad \Rightarrow \quad -7A - 4B = 5 \ \ldots\ldots (1),$$

$$(4A - 7B)\sin 2x = 0 \quad \Rightarrow \quad (4A - 7B)\sin 2x = 0 \ \ldots\ldots(2) \quad \Rightarrow \quad A = -\frac{7}{13} \ \& \ B = -\frac{4}{13}.$$

$$-3C = -9 \quad \Rightarrow \quad C = 3.$$

$$\therefore \quad y_p = -\frac{7}{13}\cos 2x - \frac{4}{13}\sin 2x + 3$$

To find the complete solution y,

$$y = y_c + y_p \quad \Rightarrow \quad y = C_1 e^{-x} + C_2 e^{3x} - \frac{7}{13}\cos 2x - \frac{4}{13}\sin 2x + 3. \qquad \text{(G.S)}$$

Example 3: Solve $\quad y'' + 2y' + y = e^x \sin x.$

Solution :

$$(D^2 + 2D + 1)y = 0 \quad \Rightarrow \quad m^2 + 2m + 1 = 0,$$

$$(m+1)(m+1) = 0 \quad \Rightarrow \quad m_1 = m_2 = -1,$$

$$\therefore \quad y_c = C_1 e^{-x} + C_2 x e^{-x} \quad \text{or} \quad y_c = (C_1 + C_2 x)e^{-x}.$$

Let $y_p = (A\cos x + B\sin x)e^x$,

$$y'_p = (A\cos x + B\sin x)e^x + (-A\sin x + B\cos x)e^x,$$

$$= (A+B)e^x \cos x + (B-A)e^x \sin x,$$

$$y''_p = (A+B)(-e^x \sin x + e^x \cos x) + (B-A)(e^x \cos x + e^x \sin x),$$

$$= 2Be^x \cos x - 2Ae^x \sin x$$

Substituting,

$$(2Be^x\cos x-2Ae^x\sin x)+2[(A+B)e^x\cos x+(B-A)e^x\sin x]+(A\cos x+B\sin x)e^x=e^x\sin x$$

$$(4B+3A)e^x\cos x+(-4A+3B)e^x\sin x=e^x\sin x,$$

$$\therefore\quad (4B+3A)e^x\cos x=0 \quad\Rightarrow\quad 4B+3A=0 \ \ldots.(1),$$

$$(-4A+3B)e^x\sin x=e^x\sin x \quad\Rightarrow\quad 3B-4A=1 \ \ldots.(2) \quad\Rightarrow\quad A=-\frac{4}{25}\ \&\ B=\frac{3}{25}$$

$$\therefore\quad y_p=(-\frac{4}{25}\cos x+\frac{3}{25}\sin x)e^x.$$

$$y=y_c+y_p \quad\Rightarrow\quad y=(C_1+C_2x)e^{-x}+(-\frac{4}{25}\cos x+\frac{3}{25}\sin x)e^x. \qquad \text{(G.S)}$$

Example 4: Solve $(D^3-5D^2-2D+24)y=xe^{3x}$.

Solution :

$$m^3-5m^2-2m+24=0,$$

By trial & error, if $m=-2$, then $(-2)^3-5(-2)^2-2(-2)+24=0$,

$\therefore\quad m=-2$ is a root $\Rightarrow$ $(m+2)$ is a factor.

Use long division to find the other factor,

$$\begin{array}{r|l} & m^2-7m+12 \\ \hline m+2 & m^3-5m^2-2m+24 \\ & m^3+2m^2 \\ \hline & -7m^2-2m+24 \\ & -7m^2-14m \\ \hline & 12m+24 \\ & 12m+24 \\ \hline & 0 \end{array}$$

$$\therefore\quad (m+2)(m^2-7m+12)=0,$$

$$\Rightarrow\quad (m+2)(m-3)(m-4)=0\ ,$$

$$\Rightarrow\quad m_1=-2,\ m_2=3,\ \text{and}\ m_3=4,$$

$$\therefore\quad y_c=C_1e^{-2x}+C_2e^{3x}+C_3e^{4x}.$$

Let $y_p=(A_o+A_1x)e^{3x}.\mathbf{x}=(A_ox+A_1x^2)e^{3x}$,

$$y'_p=3(A_ox+A_1x^2)e^{3x}+(A_o+2A_1x)e^{3x},$$

$$=[A_o+(3A_o+2A_1)x+3A_1x^2]e^{3x},$$

$$y''_p=3[A_o+(3A_o+2A_1)x+3A_1x^2]e^{3x}+[3A_o+2A_1+6A_1x]e^{3x},$$

$$=[(6A_o+2A_1)+(9A_o+12A_1)x+9A_1x^2]e^{3x},$$

$$y'''_p=3[(6A_o+2A_1)+(9A_o+12A_1)x+9A_1x^2]e^{3x}+(9A_o+12A_1+18A_1x)e^{3x},$$

$$=[(27A_o+18A_1)+(27A_o+54A_1)x+27A_1x^2]e^{3x}$$

Substituting,

$$[(27A_o+18A_1)+(27A_o+54A_1)x+27A_1x^2]e^{3x}-5[(6A_o+2A_1)+(9A_o+12A_1)x+9A_1x^2]e^{3x}$$
$$-2[A_o+(3A_o+2A_1)x+3A_1x^2]e^{3x}+24(xA_o+A_1x^2)e^{3x}=xe^{3x},$$

$$[(-5A_o+8A_1)+(-10A_1)x]e^{3x}=xe^{3x},$$

$$\therefore\quad -10A_1xe^{3x}=xe^{3x}\quad\Rightarrow\quad -10A_1=1\quad\Rightarrow\quad A_1=-\frac{1}{10},$$

$$-5A_o+8A_1=0\quad\Rightarrow\quad A_o=\frac{-8A_1}{-5}=\frac{8}{5}\times\frac{-1}{10}=-\frac{4}{25}$$

$$\therefore\quad y_p=(-\frac{4}{25}x-\frac{1}{10}x^2)e^{3x}.$$

$$y=y_c+y_p\quad\Rightarrow\quad y=C_1e^{-2x}+C_2e^{3x}+C_3e^{4x}+(-\frac{4}{25}x-\frac{1}{10}x^2)e^{3x},$$

or $$y=C_1e^{-2x}+(C_2-\frac{4}{25}x-\frac{1}{10}x^2)e^{3x}\frac{1}{10}x-\frac{1}{16}x^2)e^{3x}+C_3e^{4x}.\qquad\text{(G.S)}$$

Example 5: Solve $y''+y=x\sin x+\cos x$.

Solution :

$$(D^2+1)y=0\quad\Rightarrow\quad m^2+1=0\quad\Rightarrow\quad m^2=-1\quad\Rightarrow\quad m_{1,2}=\pm i,$$

$$\therefore\quad y_c=e^{(0)x}(C_1\cos x+C_2\sin x)=C_1\cos x+C_2\sin x.$$

Let $y_p=[(A_o+A_1x)\cos x].\boldsymbol{x}+[(B_o+B_1x)\sin x].\boldsymbol{x}+\cancel{D_1\cos x}+\cancel{D_2\sin x},$

$$=A_ox\cos x+B_ox\sin x+A_1x^2\cos x+B_1x^2\sin x,$$

$$y'_p=-A_ox\sin x+A_o\cos x+B_ox\cos x+B_o\sin x-A_1x^2\sin x+2A_1x\cos x+B_1x^2\cos x+2B_1x\sin x$$

$$=A_o\cos x+B_o\sin x+(2A_1+B_o)x\cos x+(-A_o+2B_1)x\sin x+B_1x^2\cos x-A_1x^2\sin x,$$

$$y''_p=-A_o\sin x+B_o\cos x-(2A_1+B_o)x\sin x+(2A_1+B_o)\cos x+(-A_o+2B_1)x\cos x$$
$$+(-A_o+2B_1)\sin x-B_1x^2\sin x+2B_1x\cos x-A_1x^2\cos x-2A_1x\sin x,$$

$$=(2A_1+2B_o)\cos x+(-2A_o+2B_1)\sin x+(-A_o+4B_1)x\cos x+(-4A_1-B_o)x\sin x-A_1x^2\cos x-B_1x^2\sin x$$

Substituting,

$$(2A_1+2B_o)\cos x+(-2A_o+2B_1)\sin x+(-A_o+4B_1)x\cos x+(-4A_1-B_o)x\sin x-A_1x^2\cos x-B_1x^2\sin x$$
$$+A_ox\cos x+B_ox\sin x+A_1x^2\cos x+B_1x^2\sin x=x\sin x+\cos x,$$

$$(2A_1+2B_o)\cos x+(-2A_o+2B_1)\sin x+4B_1x\cos x-4A_1x\sin x=x\sin x+\cos x,$$

$$\therefore\quad 4B_1x\cos x=0\quad\Rightarrow\quad B_1=0,$$

$$(-2A_o + 2B_1)\sin x = 0 \quad \Rightarrow \quad A_o = B_1 \quad \Rightarrow \quad A_o = 0,$$

$$-4A_1 x\sin x = x\sin x \quad \Rightarrow \quad -4A_1 = 1 \quad \Rightarrow \quad A_1 = -\frac{1}{4},$$

$$(2A_1 + 2B_o)\cos x = \cos x \;\Rightarrow\; 2A_1 + 2B_o = 1 \;\Rightarrow\; B_o = \frac{1-2A_1}{2} = \frac{1-2(-1/4)}{2} = \frac{3}{4},$$

$$\therefore \quad y_p = \frac{3}{4}x\sin x - \frac{1}{4}x^2\cos x.$$

$$y = y_c + y_p \quad \Rightarrow \quad y = C_1\cos x + C_2\sin x + \frac{3}{4}x\sin x - \frac{1}{4}x^2\cos x. \qquad \text{(G.S)}$$