4- Second and Higher Order Linear Ordinary Differential Equations

Introduction

The general form of linear DE of order *n* may be written as:

$$a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + ...+ a_1 \frac{dy}{dx} + a_o y = g(x), \quad (a_n \neq 0) \quad .... (1)$$

where;

$a_n, a_{n-1}, ..., a_o$ are called the coefficients for the DE and they are, in general, as functions of *x*,

$g(x)$ is a function of *x*.

* If the coefficients $a_n, a_{n-1}, ..., a_o$ are constants, then Eq.(1) is called linear DE with constant coefficients and if they are functions of *x*, then Eq.(1) is called linear DE with variable coefficients.

* If $g(x) = 0$, then Eq.(1) is called homogeneous linear DE, and if $g(x) \neq 0$, then Eq.(1) is called non-homogeneous linear DE.

Differential operator (D-operator)

A second standard form of Eq.(1) is based on the following notations:

$$\frac{dy}{dx} = Dy, \quad \frac{d^2 y}{dx^2} = D^2 y, \quad \text{in general} \quad \frac{d^n y}{dx^n} = D^n y,$$

where *D* is called the differential operator.

Thus, Eq.(1) can now be written as:

$$a_n D^n y + a_{n-1} D^{n-1} y + ...+ a_1 Dy + a_o y = g(x),$$

or $$(a_n D^n + a_{n-1} D^{n-1} + ...+ a_1 D + a_o) y = g(x).$$

Superposition principle

Let $y_1, y_2, \ldots\ldots, y_n$ be solutions of a linear DE of order *n*, then the linear combination:

$$y = C_1 y_1 + C_2 y_2 + \ldots\ldots + C_k y_k, \qquad k \le n$$

is also a solution, where $C_1, C_2, \ldots\ldots, C_k$ are arbitrary constants.

Linear dependence and independence

A set of n functions $y_1, y_2, \ldots\ldots, y_n$ is said to be linearly dependent if there exist n constants $C_1, C_2, \ldots\ldots, C_n$ (not all zero) such that:

$$C_1 y_1 + C_2 y_2 + \ldots\ldots + C_n y_n = 0, \qquad \text{or} \qquad \sum_{i=1}^{n} C_i y_i = 0.$$

If no such constants can be found (i.e. do not exist), then the set of functions is said to be linearly independent. For example:

* For the functions $y_1 = 3e^{2x}$, $y_2 = 2e^{2x}$, and $y_3 = e^{-x}$,
  if we put $C_1 = 2$, $C_2 = -3$, and $C_3 = 0$, then

$$C_1 y_1 + C_2 y_2 + C_3 y_3 = 2(3e^{2x}) + (-3)(2e^{2x}) + (0)(e^{-x}) = 0.$$

Thus, y_1, y_2, and y_3 are linearly dependent.

* For the functions $y_1 = e^{-x}$, $y_2 = e^{x}$, and $y_3 = e^{3x}$,

$$C_1 y_1 + C_2 y_2 + C_3 y_3 = C_1 e^{-x} + C_2 e^{x} + C_3 e^{3x} \ne 0.$$

Thus, y_1, y_2, and y_3 are linearly independent.

Wronskian determinants

It is not always easy to check the linear dependence of a given set of functions by searching for the value of the constants C_i which make $\sum_{i=1}^{n} C_i y_i = 0$. For this purpose, Wronskian determinant may be used as an alternative method.

Let $y_1, y_2, \ldots\ldots, y_n$ are given functions to be checked for linear dependence, then the Wronskian determinant is defined as,

$$w(y_1, y_2, \ldots\ldots, y_n) = \begin{vmatrix} y_1 & y_2 & . & . & y_n \\ y'_1 & y'_2 & . & . & y'_n \\ . & . & . & . & . \\ . & . & . & . & . \\ y_1^{n-1} & y_2^{n-1} & . & . & y_n^{n-1} \end{vmatrix}.$$

If $w(y_1, y_2, \ldots\ldots, y_n) = 0$, then $y_1, y_2, \ldots\ldots, y_n$ are linearly dependent.

If $w(y_1, y_2, \ldots\ldots, y_n) \neq 0$, then $y_1, y_2, \ldots\ldots, y_n$ are linearly independent.

For example, the Wronskian determinant for the functions $y_1 = 2x^2$ and $y_2 = -3x^3$ is,

$$w(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix} = \begin{vmatrix} 2x^2 & -3x^3 \\ 4x & -9x^2 \end{vmatrix} = [(2x^2).(-9x^2)] - [(-3x^3).(4x)] = -6x^4.$$

Since $w(y_1, y_2) \neq 0$, then y_1 and y_2 are linearly independent.

General solution of homogeneous linear DE

The general solution (complete solution) of any homogeneous linear DE of n^{th} order will be the linear combination of n linearly independent solutions $(y_1, y_2, \ldots\ldots, y_n)$ for which $w(y_1, y_2, \ldots\ldots, y_n) \neq 0$. Each linearly independent solution contains one constant, therefore the general solution will be contain n constants.

Solution of homogeneous linear DE with constant coefficients

A second order homogeneous linear DE with constant coefficients can be written as:

$$a.\frac{d^2y}{dx^2} + b.\frac{dy}{dx} + c.y = 0 \qquad \text{or} \qquad (a.D^2 + b.D + c)y = 0.$$

Let the solution is $y = e^{mx} \quad \Rightarrow \quad y' = me^{mx} \quad \Rightarrow \quad y'' = m^2e^{mx}$,

Substituting in the DE gives:

$$a.(m^2e^{mx}) + b.(me^{mx}) + c(e^{mx}) = 0 \quad \Rightarrow \quad e^{mx}(am^2 + b.m + c) = 0,$$

but $e^{mx} \neq 0 \quad \Rightarrow \quad \therefore am^2 + b.m + c = 0$. (Auxiliary or characteristic equation)

In practice it is obtained not by substituting $y = e^{mx}$ into the given DE and then simplifying, but rather by equating to zero the operational coefficient of y and then letting the symbol D plays the role of m, i.e. $a.D^2 + b.D + c = 0$.

$$\therefore \quad m_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

The roots m_i may be:

1- Real and unequal roots when $(b^2 - 4ac > 0)$,

$m_1 \neq m_2$

$\therefore y_1 = e^{m_1 x}$ and $y_2 = e^{m_2 x}$

$y = C_1 y_1 + C_2 y_2 \quad \Rightarrow \quad y = C_1 e^{m_1 x} + C_2 e^{m_2 x}$.

2- Real and equal roots when $(b^2 - 4ac = 0)$,

$m_1 = m_2 = m = -\dfrac{b}{2a}$

$\therefore y_1 = y_2 = e^{mx}$,

$y = C_1 e^{mx} + C_2 e^{mx} = Ce^{mx}$.

This could not be total solution because the DE is of the 2nd order and there must be two constants of integration. Thus, the solution $y = Ce^{mx}$ is considered as a part of the solution and the total solution will be assumed as,

$y = u(x).y_1$, where $y_1 = e^{mx}$.

$y' = u.y_1' + y_1.u'$,

$y'' = u.y_1'' + y_1'.u' + y_1.u'' + u'.y_1' = u.y_1'' + 2u'.y_1' + u''.y_1$

Substituting in the DE gives

$a(u.y_1'' + 2u'.y_1' + u''.y_1) + b(u.y_1' + u'.y_1) + c(u.y_1) = 0$,

$(ay_1'' + by_1' + cy_1)u + (2ay_1' + by_1)u' + ay_1 u'' = 0$.

But, $ay_1'' + by_1' + cy_1 \neq 0$, (since y_1 is a solution) and $2ay_1' + by_1 = 2a(\frac{-b}{2a} e^{\frac{-b}{2a}x}) + b(e^{\frac{-b}{2a}x}) = 0$,

$\therefore ay_1 u'' = 0$. But, $a \neq 0$ and $y_1 \neq 0$,

$\therefore u'' = 0 \quad \Rightarrow \quad u' = C_1 \quad \Rightarrow \quad u = C_1 x + C_2$.

$\therefore y = u.y_1 = (C_1 x + C_2) e^{mx}$.

3- Complex roots when $(b^2 - 4ac < 0)$,

$$m_{1,2} = -\frac{b}{2a} \pm \left(\frac{1}{2a}\sqrt{4ac - b^2}\right) i = \alpha \pm \beta i,$$

$\therefore y_1 = e^{(\alpha + \beta i)x}$ and $y_2 = e^{(\alpha - \beta i)x}$,

$\therefore y = Ae^{(\alpha + \beta i)x} + Be^{(\alpha - \beta i)x} \quad \Rightarrow \quad y = e^{\alpha x}(Ae^{\beta i x} + Be^{-\beta i x})$.

But, $e^{\pm \beta i x} = \cos \beta x \pm (\sin \beta x) i$, (Euler formula)

$\therefore y = e^{\alpha x}[A\{\cos \beta x + (\sin \beta x) i\} + B\{\cos \beta x - (\sin \beta x) i\}]$,

$\Rightarrow \quad y = e^{\alpha x}[(A + B)\cos \beta x + (A - B)(\sin \beta x) i]$,

$\Rightarrow \quad y = e^{\alpha x}(C_1 \cos \beta x + C_2 \sin \beta x)$.

Example 1: Solve $y'' - 3y' + 2y = 0$.

Solution :

Using D-operator gives $D^2 y - 3Dy + 2y = 0$ or $(D^2 - 3D + 2)y = 0$,

$\therefore \quad m^2 - 3m + 2 = 0$, (Auxiliary or characteristic equation)

$(m-1)(m-2) = 0 \quad \Rightarrow \quad$ Either $m - 1 = 0 \quad \Rightarrow \quad m_1 = 1$,

or $m - 2 = 0 \quad \Rightarrow \quad m_2 = 2$,

$\therefore y_1 = e^{m_1 x} = e^x$ and $y_2 = e^{m_2 x} = e^{2x}$,

$y = C_1 y_1 + C_2 y_2 \quad \Rightarrow \quad y = C_1 e^x + C_2 e^{2x}$. (G.S)

Example 2: Solve $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = 0$.

Solution :

$$D^2y + 6Dy + 9y = 0 \quad \text{or} \quad (D^2 + 6D + 9)y = 0,$$

$$\therefore \quad m^2 + 6m + 9 = 0, \qquad \text{(Auxiliary equation)}$$

$$(m+3)(m+3) = 0 \quad \Rightarrow \quad m_1 = m_2 = -3 \quad \Rightarrow \quad y_1 = y_2 = e^{-3x},$$

$$\therefore \quad y = C_1e^{-3x} + C_2xe^{-3x} \quad \text{or} \quad y = (C_1 + C_2x)e^{-3x}. \qquad \text{(G.S)}$$

Example 3: Solve $y'' - 4y' + 7y = 0$.

Solution :

$$(D^2 - 4D + 7)y = 0 \quad \Rightarrow \quad \therefore \quad m^2 - 4m + 7 = 0, \qquad \text{(Auxiliary equation)}$$

$$m_{1,2} = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(7)}}{2(1)} = \frac{4 \pm \sqrt{-12}}{2},$$

$$= \frac{4 \pm \sqrt{12}i}{2} = 2 \pm \sqrt{3}i, \qquad (\alpha = 2 \text{ and } \beta = \sqrt{3})$$

$$y = e^{\alpha x}(C_1 \cos\beta x + C_2 \sin\beta x) \quad \Rightarrow \quad y = e^{2x}(C_1 \cos\sqrt{3}x + C_2 \sin\sqrt{3}x). \qquad \text{(G.S)}$$