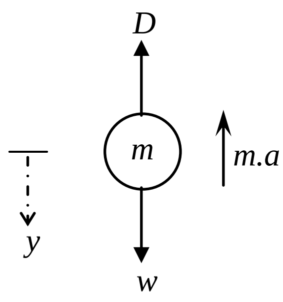
4- Motion of bodies

Example: A body, of mass *m*, falls from rest. If the drag (resisting) force of air is assumed to be proportional to the instantaneous velocity of the body, find the equation of motion of this body.

Solution:



Since the drag force D is proportional to the instantaneous velocity v,

$\therefore\ D \propto v \quad \Rightarrow \quad D = R.v$. (*R* is the proportion constant)

$\sum F_y = m.a \quad \Rightarrow \quad w - D = m.a \quad \Rightarrow \quad m.g - R.v = m.a$,

$\Rightarrow \quad m.g - R.\dfrac{dy}{dt} = m.\dfrac{d^2y}{dt^2} \quad \Rightarrow \quad \dfrac{d^2y}{dt^2} + \dfrac{R}{m}.\dfrac{dy}{dt} = g$. (reducible to 1st order DE)

Since y does not appear in the above DE, then this equation can be reduced to a first order DE by letting $z = f(t) = \dfrac{dy}{dt} \quad \Rightarrow \quad \dfrac{dz}{dt} = \dfrac{d^2y}{dt^2}$.

$\therefore \quad \frac{dz}{dt}+\frac{R}{m}.z=g$ (Linear DE) $\Rightarrow \quad \mu=e^{\int P(t)dt} \quad \Rightarrow \quad \mu=e^{\int \frac{R}{m}dt}=e^{Rt/m}$.

$\mu.z=\int \mu.Qdt+C \qquad \Rightarrow \qquad e^{Rt/m}.z=\int e^{Rt/m}.g.dt+C_1$,

$\Rightarrow \qquad z.e^{Rt/m}=\frac{mg}{R}.e^{Rt/m}+C_1 \qquad \Rightarrow \quad z=\frac{mg}{R}+C_1e^{-Rt/m}$.

But $\quad z=\frac{dy}{dt} \qquad \Rightarrow \qquad \frac{dy}{dt}=\frac{mg}{R}+C_1e^{-Rt/m}$,

$$\therefore \qquad y=\frac{mg}{R}.t-\frac{mC_1}{R}.e^{-Rt/m}+C_2. \qquad \text{(G.S)}$$

Initial conditions (I.C);

1- $v(0)=y'(0)=0 \quad \Rightarrow \quad 0=\frac{mg}{R}+C_1e^0 \quad \Rightarrow \quad C_1=\frac{-mg}{R}$.

2- $y(0)=0 \;\Rightarrow\; 0=0-\frac{mC_1}{R}.e^0+C_2 \;\Rightarrow\; C_2=\frac{mC_1}{R} \;\Rightarrow\; C_2=\frac{-m^2g}{R^2}$.

$$\therefore \qquad y=\frac{mg}{R}.t-\frac{m}{R}.\frac{-mg}{R}.e^{-Rt/m}+\frac{-m^2g}{R^2},$$

or $$y=\frac{mg}{R}.t-\frac{m^2g}{R^2}(1-e^{-Rt/m}). \qquad \text{(P.S)}$$

5- General Applications

Example: the population growth (P) at any time in a city is governed by the equation $\frac{dP}{dt}=(B-D.P)P$, where B and D are the birth and death rate, respectively. If $B=0.1$, $D=1\times10^{-7}$ (t is in year), and $P_o=5000$ person, find P as a function of time. What is the limiting (maximum) value of population? At what time will be the population equal to one half of the limiting value?

Solution:

$$\frac{dP}{dt}=(B-D.P)P \qquad \Rightarrow \qquad \frac{dP}{dt}-BP=-DP^2. \qquad \text{(Bernoulli's equation)}$$

Division by P^2 gives $\quad P^{-2}.\frac{dP}{dt}-BP^{-1}=-D$.

Let $\quad z=P^{-1} \quad \Rightarrow \quad \frac{dz}{dt}=-P^{-2}\frac{dP}{dt} \quad \Rightarrow \quad P^{-2}\frac{dP}{dt}=-\frac{dz}{dt}$,

$$\therefore \quad -\frac{dz}{dt} - B.z = -D \quad \Rightarrow \quad \frac{dz}{dt} + B.z = D, \quad \text{(Linear DE with respect to } z\text{)}$$

$$\mu = e^{\int B dt} = e^{Bt}.$$

$$\mu.z = \int \mu.Q dt + C \quad \Rightarrow \quad e^{Bt}.z = \int e^{Bt}.D dt + C_1,$$

$$\Rightarrow \quad z.e^{Bt} = \frac{D}{B}.e^{Bt} + C_1 \quad \Rightarrow \quad z = \frac{D}{B} + \frac{C_1}{e^{Bt}},$$

$$\text{but} \quad z = P^{-1} = \frac{1}{P} \quad \Rightarrow \quad \frac{1}{P} = \frac{D}{B} + \frac{C_1}{e^{Bt}} \quad \Rightarrow \quad \frac{1}{P} = \frac{De^{Bt} + C_1 B}{Be^{Bt}},$$

$$\Rightarrow \quad P = \frac{Be^{Bt}}{De^{Bt} + C} \quad [C = C_1 B] \quad \text{or} \quad P = \frac{B}{D + Ce^{-Bt}}. \qquad \text{(G.S)}$$

Initial conditions (I.C);

$$\text{At} \quad t = 0, \quad P = 5000 \quad \Rightarrow \quad 5000 = \frac{0.1}{1\times 10^{-7} + Ce^{-0.1(0)}},$$

$$\Rightarrow \quad C = \frac{0.1}{5000} - 1\times 10^{-7} = 2\times 10^{-5}.$$

$$\therefore \quad P = \frac{0.1}{1\times 10^{-7} + 2\times 10^{-5} e^{-0.1.t}} \quad \text{or} \quad P = \frac{1000000}{1 + 200e^{-0.1.t}}. \qquad \text{(P.S)}$$

Max. population $P_{\max}$ occurs after a long period of time (i.e when $t \to \infty$),

$$\therefore \quad P_{\max} = \frac{1000000}{1 + 200e^{-0.1(\infty)}} \quad \Rightarrow \quad P_{\max} = \frac{1000000}{1 + 200\times 0} = 1000000.$$

$$(\textit{Note;} \quad e^{-0.1(\infty)} = e^{-\infty} = \frac{1}{e^{\infty}} = \frac{1}{\infty} \approx 0)$$

$$\text{When} \quad P = \frac{1}{2}P_{\max} \quad \Rightarrow \quad P = \frac{1}{2}\times 1000000 = 500000 \text{ Person},$$

$$\therefore \quad 500000 = \frac{1000000}{1 + 200e^{-0.1.t}} \quad \Rightarrow \quad 1 + 200e^{-0.1.t} = \frac{1000000}{500000},$$

$$\Rightarrow \quad e^{-0.1.t} = \frac{2-1}{200} \quad \Rightarrow \quad -0.1.t = \ln(5\times 10^{-3}) \quad \Rightarrow \quad -0.1.t \approx -5.3,$$

$$\therefore \quad t \approx 53 \text{ year.}$$