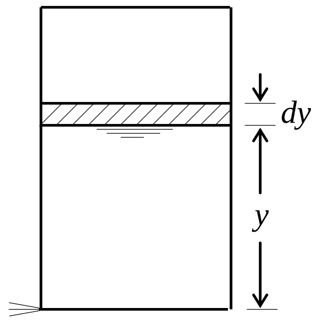
3- Flow through orifices

Consider a tank which contains any liquid and there is an orifice (hole) at its bottom through which the liquid drains under the influence of gravity. Thus, the depth of water is changed through time. In an interval of time dt, the water level will fall by the amount dy, and the change of volume of liquid inside the tank is equal to the volume of liquid drained outside the tank, i.e,

$$(dV)_{in} = (dV)_{out} \qquad \Rightarrow \qquad A.dy = -Q.dt\,,$$



where,

A is the cross sectional area of the tank.

Q is the discharge of liquid through the orifice $= C_d.a.v$.

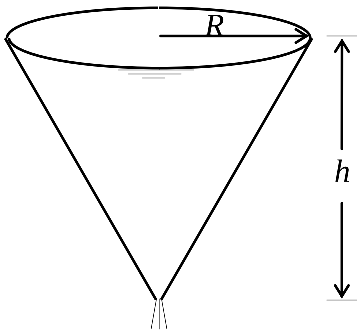
C_d is the coefficient of discharge.

a is the area of the orifice (hole).

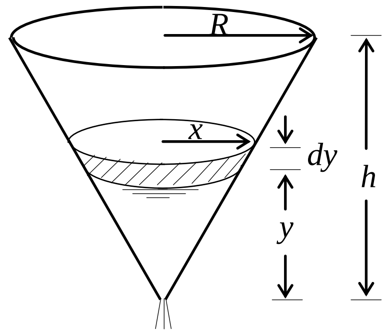
v is the velocity of liquid leaving the orifice $= \sqrt{2gy}$.

$(-ve)$ the negative sign indicates that as t increases, y decreases.

Example 1: An inverted right circular conical tank, as shown in the figure, is initially filled with water. The water drains, due to gravity, through a small hole of radius r at the bottom. Find the height of water as a function of time and the time required for the tank to drain completely.



Solution:



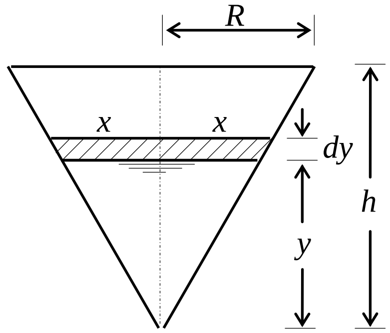
$$(dV)_{in} = (dV)_{out} \quad \Rightarrow \quad A.dy = -Q.dt\,,$$

$$\Rightarrow \quad A.dy = -C_d.a.v.dt\,,$$

$$\Rightarrow \quad \pi x^2.dy = -C_d.\pi r^2.\sqrt{2gy}.dt\,,$$

$$\Rightarrow \quad x^2.dy = -C_d r^2\sqrt{2gy}.dt\,.$$

But $x = \dfrac{Ry}{h} \quad \Rightarrow \quad \dfrac{R^2y^2}{h^2}.dy = -C_d r^2\sqrt{2g}.\sqrt{y}.dt\,.$ (Separable variables DE)



$$\therefore \quad \frac{y^2}{\sqrt{y}}.dy = -\frac{C_d r^2 h^2\sqrt{2g}}{R^2}.dt\,,$$

$$\Rightarrow \quad y^{3/2}.dy = -\frac{C_d r^2 h^2\sqrt{2g}}{R^2}.dt\,,$$

$$\Rightarrow \quad \frac{2}{5}y^{5/2} = -\frac{C_d r^2 h^2\sqrt{2g}}{R^2}.t + C\,. \qquad \text{(G.S)}$$

$$\frac{x}{y} = \frac{R}{h} \quad \Rightarrow \quad x = \frac{Ry}{h}$$

Appling the initial condition (I.C);

Initially, at $t = 0$, the tank is filled with water, $y = h$,

$$\therefore \quad y(0) = h \quad \Rightarrow \quad \frac{2}{5}h^{5/2} = 0 + C \quad \Rightarrow \quad C = \frac{2}{5}h^{5/2}\,.$$

$$\therefore \quad \frac{2}{5}y^{5/2} = -\frac{C_d r^2 h^2\sqrt{2g}}{R^2}.t + \frac{2}{5}h^{5/2} \quad \text{or} \quad y^{5/2} = -\frac{5C_d r^2 h^2\sqrt{2g}}{2R^2}.t + h^{5/2}\,. \qquad \text{(P.S)}$$

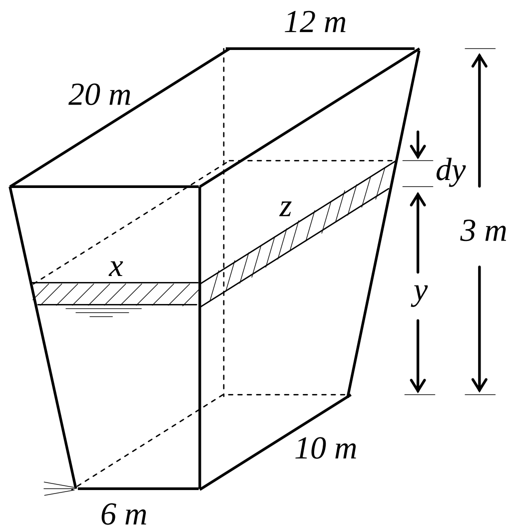
The tank will be empty when $y = 0$,

$$\therefore \quad 0 = -\frac{5C_d r^2 h^2\sqrt{2g}}{2R^2}.t + h^{5/2} \quad \Rightarrow \quad t = \frac{2R^2h^{5/2}}{5C_d r^2 h^2\sqrt{2g}}\,,$$

$$\text{or} \quad t = \frac{2}{5C_d}.\left(\frac{R}{r}\right)^2.\sqrt{\frac{h}{2g}}\,.$$

Example 2: A water tank, rectangular in cross section, has the dimensions $20\times 12m$ at the top and $6\times 10m$ at the bottom and is $3m$ in height. It is filled with water and has a circular orifice of $5cm$ diameter at its bottom. Assuming $C_d = 0.6$ for the orifice, find the equation of the height of water in the tank with time, then compute the time required for empting the tank.

Solution:



$$(dV)_{in} = (dV)_{out} \quad \Rightarrow \quad A.dy = -Q.dt\,,$$

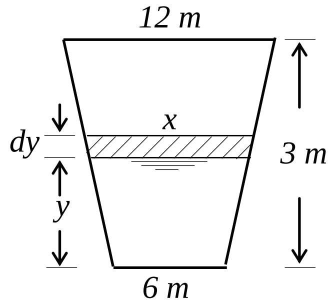
$$\Rightarrow \quad A.dy = -C_d\,.a.v.dt\,,$$

$$\Rightarrow \quad x.z.dy = -C_d\,.\pi r^2.\sqrt{2gy}.dt\,,$$

$$(2y+6)(\frac{10}{3}y+10)dy = -0.6\pi(\frac{2.5}{100})^2\sqrt{2\times 9.81y}.dt\,,$$

$$\Rightarrow \; 20(\frac{y^2}{3}+2y+3)dy = -5.218\times 10^{-3}\sqrt{y}.dt\,,$$

(Separable variables DE)



$$\therefore \quad (\frac{y^2}{3\sqrt{y}}+\frac{2y}{\sqrt{y}}+\frac{3}{\sqrt{y}})dy = -2.61\times 10^{-4}\,dt\,,$$

$$\Rightarrow \quad (\frac{1}{3}y^{3/2}+2y^{1/2}+3y^{-1/2})dy = -2.61\times 10^{-4}\,dt\,,$$

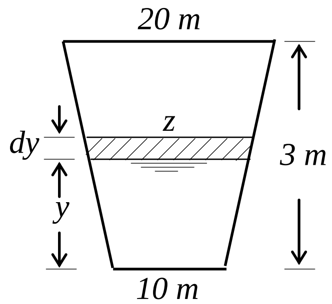
$$\therefore \quad \frac{2}{15}y^{5/2}+\frac{4}{3}y^{3/2}+6y^{1/2} = -2.61\times 10^{-4}.t + C\,. \qquad \text{(G.S)}$$

$$\frac{x-6}{y} = \frac{12-6}{3}$$

$$\therefore x = 2y+6$$

Appling the initial condition (I.C);

Initially, at $t=0$, the tank is filled with water, $y=3m$,



$$\therefore \quad y(0)=3 \quad \Rightarrow \quad \frac{2}{15}\times 3^{5/2}+\frac{4}{3}\times 3^{3/2}+6\times 3^{1/2} = 0 + C\,,$$

$$\Rightarrow \quad C = 19.4\,.$$

$$\therefore \quad \frac{2}{15}y^{5/2}+\frac{4}{3}y^{3/2}+6y^{1/2} = -2.61\times 10^{-4}.t + 19.4\,. \qquad \text{(P.S)}$$

The tank will be empty when $y=0$,

$$\therefore \quad 0 = -2.61\times 10^{-4}.t + 19.4\,,$$

$$\Rightarrow \quad t = 74329.5\sec$$

$$\approx 20.65hr$$

$$\frac{z-10}{y} = \frac{20-10}{3}$$

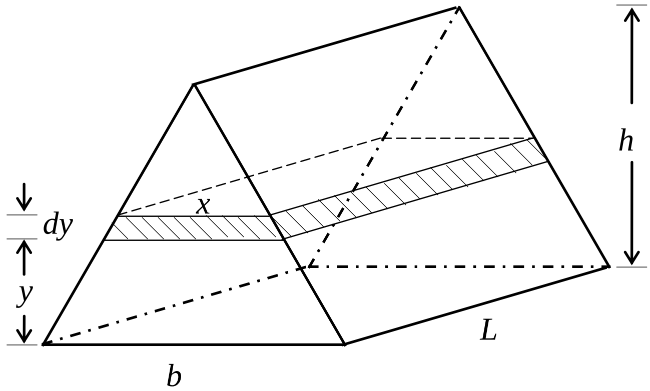
$$\therefore z = \frac{10}{3}y + 10$$

Useful expressions,

$$A = x.L\,.$$

$$\frac{x}{b} = \frac{h-y}{h},$$

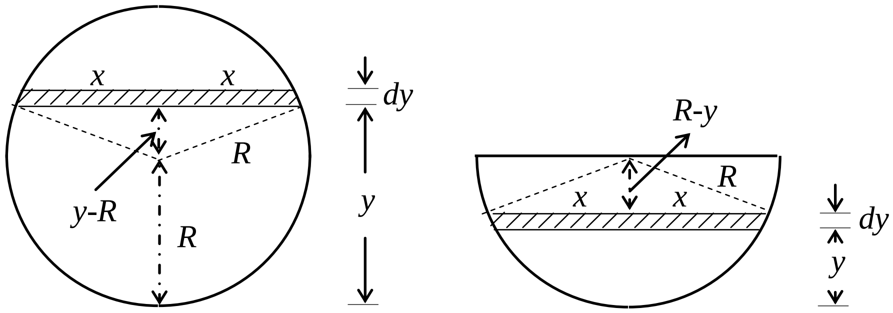
$$\therefore \quad x = \frac{b}{h}.(h-y)\,.$$



$$A = \pi x^2\,.$$

$$x^2 + (y-R)^2 = R^2,$$

$$\therefore \quad x^2 = 2Ry - y^2\,.$$



$$A = 2x.L\,.$$

$$x^2 + y^2 = R^2,$$

$$\therefore \quad x = \sqrt{R^2 - y^2}\,.$$