3- Applications on First-Order ODE

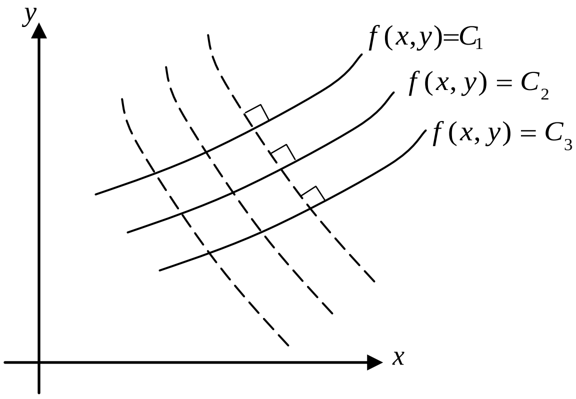
Introduction

The mathematical formulation of physical problems involving continuously changing quantities, often leads to differential equations of the first-order.

1- Orthogonal Trajectories

In many engineering problems, a family (set) of curves is given and it is required to find another family whose curves intersect each of the given curves at right angle.

Consider the function $f(x, y) = C$ where *C* is a constant. By changing the value of the constant *C*, a family (set) of curves are obtained for $f(x, y)$, where each curve has one value of the constant. It is required to find another set of curves which are orthogonal to the first set. This is done by eliminating the constant *C* from $f(x, y) = C$ by differentiation, and then replacing $\frac{dy}{dx}$ of these curves by $[-1/\frac{dy}{dx}]$ to get the required orthogonal set.



Example 1: Find the orthogonal trajectories of $y = Cx^2$.

Solution:

*Step **1**;* Find the slope of the given set,

Method I,

$y = Cx^2 \quad \Rightarrow \quad \dfrac{y}{x^2} = C$. By differentiation $\dfrac{x^2.dy - y.(2xdx)}{x^4} = 0$,

$\Rightarrow \quad x^2dy - 2xydx = 0 \quad \Rightarrow \quad \left(\dfrac{dy}{dx}\right)_1 = \dfrac{2y}{x}$. (The slope of the given set)

Method II,

$y = Cx^2$. By differentiation $dy = 2Cxdx \quad \Rightarrow \quad \left(\dfrac{dy}{dx}\right)_1 = 2Cx$.

From the given set $C = \dfrac{y}{x^2} \quad \Rightarrow \quad \therefore \left(\dfrac{dy}{dx}\right)_1 = 2\left(\dfrac{y}{x^2}\right)x \quad \Rightarrow \quad \left(\dfrac{dy}{dx}\right)_1 = \dfrac{2y}{x}$.

*Step **2**;* Find the slope of the required set,

Since the required and given sets are orthogonal, then $\left(\dfrac{dy}{dx}\right)_2 = -1/\left(\dfrac{dy}{dx}\right)_1$.

$\therefore \left(\dfrac{dy}{dx}\right)_2 = -1/\left(\dfrac{2y}{x}\right) \quad \Rightarrow \quad \left(\dfrac{dy}{dx}\right)_2 = \dfrac{-x}{2y}$.

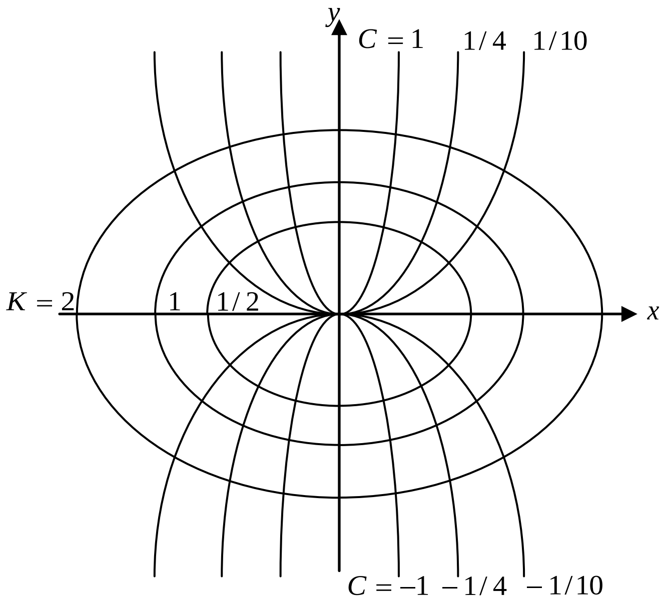
*Step **3**;* Find the required set,

To find the required set we must solve the above differential equation,

$\dfrac{dy}{dx} = \dfrac{-x}{2y}$ (separable variables DE)

$\Rightarrow \quad 2ydy = -x.dx$,

$\therefore \; y^2 = \dfrac{-x^2}{2} + K$ or $y^2 + \dfrac{x^2}{2} = K$.



Notes,

* K must be positive since it is the sum of two squares.

* $y = Cx^2$ is a family of parabolas.

* $y^2 + \dfrac{x^2}{2} = K$ is a family of ellipses.

Example 2: Find the orthogonal trajectories of $xy = C$.

Solution:

*Step **1**;* Find the slope of the given set,

By differentiation $\quad xdy + ydx = 0 \quad \Rightarrow \quad \left(\dfrac{dy}{dx}\right)_1 = \dfrac{-y}{x}.$

*Step **2**;* Find the slope of the required set,

$$\left(\frac{dy}{dx}\right)_2 = -1/\left(\frac{dy}{dx}\right)_1 \quad \Rightarrow \quad \left(\frac{dy}{dx}\right)_2 = -1/\left(\frac{-y}{x}\right) \quad \Rightarrow \quad \left(\frac{dy}{dx}\right)_2 = \frac{x}{y}.$$

*Step **3**;* Find the required set,

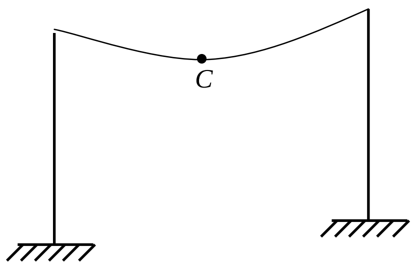
$\dfrac{dy}{dx} = \dfrac{x}{y}$ (separable variables DE) $\quad \Rightarrow \quad ydy = x.dx$,

$\therefore \quad \dfrac{y^2}{2} = \dfrac{x^2}{2} + K_1 \quad$ or $\quad y^2 - x^2 = K. \quad [K = 2K_1]$

2- Suspended Cables

Example 1: Derive the differential equation of the curve of a perfectly flexible cable, of uniform weight per unit length w, suspended between two points.

Solution:



Start from the lowest point C and consider a cable segment of length x from point C,

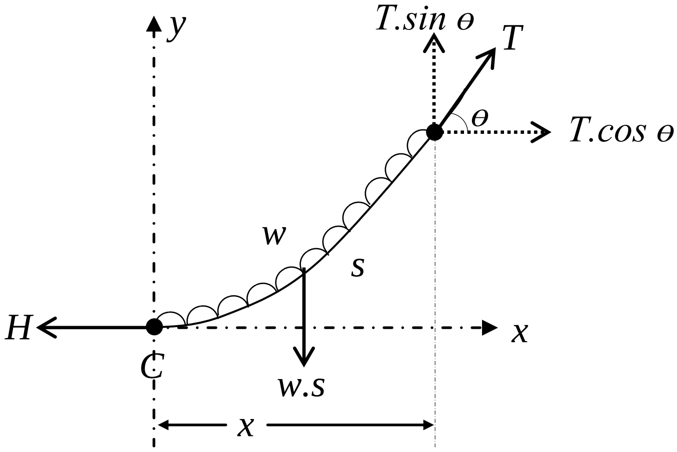
$\sum F_y = 0 \quad \Rightarrow \quad T\sin\theta = w.s \quad \ldots\ldots. \quad (1)$

$\sum F_x = 0 \quad \Rightarrow \quad T\cos\theta = H \quad \ldots\ldots.. \quad (2)$

Dividing Eq. (1) by (2), gives

$\tan\theta = \dfrac{w.s}{H}. \quad$ But $\tan\theta = \dfrac{dy}{dx}$,

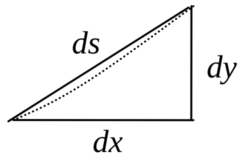
$\therefore \quad \dfrac{dy}{dx} = \dfrac{w.s}{H}.$



Differentiating the last equation with respect to x, yields

$$\frac{d^2y}{dx^2} = \frac{w}{H}.\frac{ds}{dx}.$$

But $ds = \sqrt{(dx)^2 + (dy)^2}$ or $\dfrac{ds}{dx} = \sqrt{1 + (\dfrac{dy}{dx})^2}$,



$$\therefore \frac{d^2y}{dx^2} = \frac{w}{H}.\sqrt{1 + (\frac{dy}{dx})^2} \;.$$ (2nd order reducible to 1st order DE)

Since y does not appear in the above DE, then this equation can be reduced to a first order DE by letting

$$z = f(x) = \frac{dy}{dx} \quad \Rightarrow \quad \frac{dz}{dx} = \frac{d^2y}{dx^2}.$$

$$\therefore \frac{dz}{dx} = \frac{w}{H}.\sqrt{1+z^2} \quad \text{(Separable DE)} \quad \Rightarrow \quad \frac{dz}{\sqrt{1+z^2}} = \frac{w}{H}.dx,$$

$$\Rightarrow \quad \sinh^{-1} z = \frac{w}{H}.x + C_1 \quad \Rightarrow \quad z = \sinh(\frac{w}{H}.x + C_1).$$

But $z = \dfrac{dy}{dx} \quad \Rightarrow \quad \dfrac{dy}{dx} = \sinh(\dfrac{w}{H}.x + C_1)$,

$$\therefore \quad y = \frac{H}{w}\cosh(\frac{w}{H}.x + C_1) + C_2. \qquad \text{(G.S)}$$

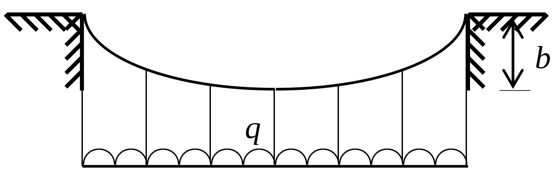
Appling the boundary conditions (B.C);

1- At the lowest point; $y'(0) = 0 \quad \Rightarrow \quad 0 = \sinh(0 + C_1) \quad \Rightarrow \quad C_1 = 0$.

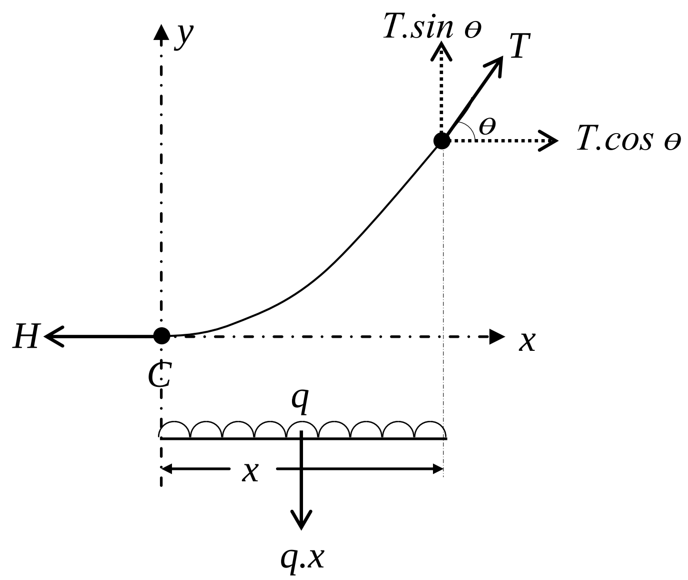
2- At the lowest point; $y(0) = 0 \quad \Rightarrow \quad 0 = \dfrac{H}{w}\cosh(0) + C_2 \quad \Rightarrow \quad C_2 = -\dfrac{H}{w}$.

$$\therefore \quad y = \frac{H}{w}\cosh(\frac{w}{H}.x + 0) - \frac{H}{w} \quad \Rightarrow \quad y = \frac{H}{w}(\cosh\frac{wx}{H} - 1). \qquad \text{(P.S)}$$

Example 2: A suspended cable is hung between two points, of the same level, and is subjected to a horizontal uniformly distributed load, attached to the cable by vertical hangars, as shown in the figure. What is the shape of the cable at equilibrium? and how does the tension vary along the cable? (Neglect self weight of the cable).



Solution:



Start from the lowest point *C* (at midspan due to symmetry) and consider a cable segment of length *x* from point *C*,

$$\sum F_y = 0 \;\Rightarrow\; T\sin\theta = q.x \quad \ldots\ldots. \quad (1)$$

$$\sum F_x = 0 \;\Rightarrow\; T\cos\theta = H \quad \ldots\ldots. \quad (2)$$

Dividing Eq. (1) by (2), gives

$$\tan\theta = \frac{q.x}{H}. \qquad \text{But} \quad \tan\theta = \frac{dy}{dx},$$

$$\therefore \frac{dy}{dx} = \frac{q.x}{H} \;\Rightarrow\; y = \frac{qx^2}{2H} + C. \qquad \text{(G.S)}$$

Boundary conditions (B.C);

1- At the lowest point; $y(0) = 0 \;\Rightarrow\; 0 = \dfrac{q(0)^2}{2H} + C \;\Rightarrow\; C = 0 \;\Rightarrow\; y = \dfrac{qx^2}{2H}.$

2- At the right support; $y(L/2) = b \;\Rightarrow\; b = \dfrac{q(L/2)^2}{2H} \;\Rightarrow\; H = \dfrac{qL^2}{8b}.$

$$\therefore \quad y = \frac{qx^2}{2(qL^2/8b)} \;\Rightarrow\; y = \frac{4bx^2}{L^2}. \qquad \text{(P.S)}$$

Consider the tension in the cable:

At any point along the cable $T = \sqrt{T_x{}^2 + T_y{}^2} \;\Rightarrow\; T = \sqrt{(T.\cos\theta)^2 + (T.\sin\theta)^2}$,

$$\Rightarrow \; T = \sqrt{(H)^2 + (q.x)^2} \;\Rightarrow\; T = \sqrt{H^2 + q^2.x^2}.$$

Thus, the tension is minimum when $x = 0$ (i.e, at the lowest point where it is equal to *H*) and it increases towards the ends.

Note;

If we consider the self weight of the cable, then

$$\sum F_y = 0 \;\Rightarrow\; T\sin\theta = q.x + w.s \quad \ldots\ldots\ldots\ldots\ldots\ldots. \quad (1)$$

$$\sum F_x = 0 \;\Rightarrow\; T\cos\theta = H \quad \ldots\ldots\ldots\ldots\ldots\ldots. \quad (2)$$

Dividing Eq. (1) by (2), gives

$$\tan\theta = \frac{q.x + w.s}{H}. \qquad \text{But} \quad \tan\theta = \frac{dy}{dx} \;\Rightarrow\; \frac{dy}{dx} = \frac{q.x + w.s}{H}.$$

Differentiating the last equation with respect to *x*, yields

$$\frac{d^2y}{dx^2}=\frac{q}{H}+\frac{w}{H}.\frac{ds}{dx}. \qquad \text{But } ds=\sqrt{(dx)^2+(dy)^2} \qquad \text{or} \qquad \frac{ds}{dx}=\sqrt{1+(\frac{dy}{dx})^2},$$

$$\frac{d^2y}{dx^2}=\frac{q}{H}+\frac{w}{H}.\sqrt{1+(\frac{dy}{dx})^2}.$$

The above resulted DE is so difficult to be solved analytically.